**FITZHUGH-NAGUMO MODEL AND SIGNAL
PROCESSING IN THE VISUAL CORTEX OF FLY**

by

BAILI CHEN, B.S., M.S.

A THESIS

IN

MATHEMATICS

**Submitted to the Graduate Faculty
of Texas Tech University in
Partial Fulfillment of
the Requirements for
the Degree of**

MASTER OF SCIENCE

Approved

Chairperson of the Committee

Accepted

Dean of the Graduate School

May, 2004

ACKNOWLEDGEMENTS

I would like to express my utmost gratitude to my advisor Dr. Clyde F. Martin for his professional guidance and direction. I deeply appreciate his dedication of his time and patience and his constant encouragement during my research work. I would like to thank Dr. P. Seshaiyer and Dr. W. P. Dayawansa for their helpful advice as my committee members. I also would like to thank Menaka and Rachel for their help during the preparing for the defense.

I feel a deep gratitude for my parents who always encourage me and give me great support. Also I would like to thank my husband for his enormous support during my research work.

CONTENTS

ACKNOWLEDGEMENTS	ii
ABSTRACT	iv
LIST OF FIGURES	v
CHAPTER	
I INTRODUCTION	1
II FITZHUGH-NAGUMO MODEL AND OSCILLATIONS OF TANGENTIAL CELLS	3
Introduction	3
FitzHugh-Nagumo Dynamics of the Tangential Cells	4
III AYMPTOTIC APPROXIMATION OF LIMIT CYCLE IN FITZHUGH-NAGUMO MODEL	8
Introduction to Asymptotic Approximation Method	8
Asymptotic Approximation of Limit Cycle	10
Conclusion and Future Work	19
REFERENCES	20

ABSTRACT

In this thesis, FitzHugh-Nagumo equations are used to model the dynamics of tangential cells in fly's visual cortex. Membrane current produced by the visual stimuli is modeled as the external input. An asymptotic method is used to show that the amplitude of the oscillation depends critically on the magnitude of the external input. Furthermore, in a neighborhood of a bifurcation, the amplitude increases monotonically as a function of the external input, this result helps to explain how the visual cortex of fly encodes visual stimulus and detects motion.

LIST OF FIGURES

1	The graph of $I = \frac{1}{3}x_0^3 + (b-1)x_0 + a$	6
2	x-nullcline and y-nullcline of the FitzHugh-Nagumo system	12
3	The graph of $b_1 = \frac{3b\epsilon}{a_1} + \hat{y} - \frac{3b\epsilon}{a_0}$	18

CHAPTER I

INTRODUCTION

The work starts from studying the processing of visual motion in the eyes of flies. In flies the eye is built from many single elements called facets or ommatidia. Each ommatidium possesses its own lens and its own set of photoreceptors. The latter send their axons into a part of the brain devoted to image processing called the visual ganglia. Within these ganglia, images become processed by an array of local motion detectors. (The correlation-type of motion detectors was proposed by Reichardt and Hassenstein 1961, 1987, their model is called elementary motion detector, EMD.). The output of such local motion detectors become spatially integrated by various large field elements and this happens on the dendrites of tangential cells located in the posterior part of the third visual ganglion called the lobula plate. These neurons don't integrate the output signals of local motion detectors independently but interact with each other. One of the goal of this research is to use a network of oscillators to model the detection of motion signals in visual cortex of flies.

There is accumulated experimental evidence that the correlation-type of the motion detector provides input to the tangential cells in the lobula plate. The knowledge about the dynamics of the intrinsic membrane properties of the lobula plate tangential cells is important because it allows us to dissect out the input signal which come from the motion signals through flies' compound eyes.

Various types of electrical responses are produced by different tangential cells when stimulated by visual motion along their preferred direction, i.e. graded, spiking and mixed. Pronounced differences must exist between the intrinsic membrane properties of the neurons. Responding in a purely graded way they might not be endowed with voltage-gated ion channels at all. Thus the response is regarded as totally passive, while tangential cells producing spiking activities have to possess voltage-gated ionic currents in their axons. In addition, tangential cells responding in a mixed mode must also have voltage-activated

membrane currents.

This work focuses on the tangential cells with voltage-gated ion channels which can produce spiking activities in response to visual stimuli. FitzHugh-Nagumo model will be used to reveal the relation between the dynamics of tangential cells and the magnitude of membrane current produced by the visual stimuli.

CHAPTER II
FITZHUGH-NAGUMO MODEL AND
OSCILLATIONS OF TANGENTIAL CELLS

Introduction

Fifty years ago Hodgkin and Huxley reported their comprehensive quantitative studies on electrical excitation and propagation along a nerve axon. Under experimental conditions which justify the assumption that the axon is spatially homogeneous in the longitudinal direction of the cell, a current produces relaxation oscillations of the action potential whose amplitude and period depend crucially on the magnitude of the current. They use a system of four couple nonlinear differential equations to capture the key features of the action potential. Because of the complexity of the equations, FitzHugn and Nagumo introduced a formal reduction of these equations to a two dimentional system that can be expressed in the form:

$$\dot{x} = x - \frac{1}{3}x^3 - y + I, \tag{1}$$

$$\dot{y} = \epsilon(bx - y + a) \quad \epsilon \ll 1, 0 < a < 1, b > 1 \tag{2}$$

where $x(t)$ is the potential (the potential difference across the membrane), $y(t)$ is the recovery variable which measures the state of excitability of the cell and I denotes the input current and represent the control parameter. The other parameters a, b, ϵ are constants related to the physiological state of the neuron.

FitzHugh-Nagumo Dynamics of the Tangential Cells

Fitzhugh has showed that for each $I \in (-\infty, \infty)$, equations (1) and (2) has a unique steady-state solution (x_0, y_0) . To determine the stability of the steady state, we consider the Jacobian matrix of (1) and (2) evaluated at (x_0, y_0) :

$$A = \begin{pmatrix} 1 - x_0^2 & -1 \\ b\epsilon & -\epsilon \end{pmatrix}$$

Let the eigenvalue of this matrix be $\lambda = u + iw$ hence

$$\det \begin{pmatrix} \lambda - 1 + x_0^2 & 1 \\ -b\epsilon & \lambda + \epsilon \end{pmatrix} = 0$$

$$\lambda^2 + (\epsilon - 1 + x_0^2)\lambda + \epsilon x_0^2 + \epsilon b - \epsilon = 0.$$

then the real part of eigenvalue is

$$u = \frac{(1 - \epsilon) - x_0^2}{2}.$$

From (1), we have

$$I = \frac{1}{3}x_0^3 + (b - 1)x_0 + a. \tag{3}$$

This cubic equation (3) has three roots of which only one is real .

Taking derivatives with respect to x_0 from both sides of (3), we get

$$\frac{dI}{dx_0} = x_0^2 + b - 1 > 0.$$

Let x_0 be a function of I , then $x_0 = x_0(I)$ is monotonically increasing function of I (see Figure 1).

For $x_0 = -\sqrt{1 - \epsilon}$,

The corresponding value of I is:

$$I_1 = -\frac{1}{3}(1 - \epsilon)\sqrt{1 - \epsilon} - (b - 1)\sqrt{1 - \epsilon} + a.$$

For $x_0 = \sqrt{1 - \epsilon}$,

The corresponding value of I is:

$$I_2 = \frac{1}{3}(1 - \epsilon)\sqrt{1 - \epsilon} + (b - 1)\sqrt{1 - \epsilon} + a.$$

Figure 1: The graph of $I = \frac{1}{3}x_0^3 + (b-1)x_0 + a$.

Table 1: Parameter I and the dynamics of the system

$I < I_1$	damped oscillation
$I = I_1 + \delta$	low amplitude oscillation
$I_1 < I < I_2$	relaxtion oscillation
$I = I_2 - \delta$	low amplitude oscillation
$I > I_2$	damped oscillation
$I \gg I_2$	steady state

Investigate the eigenvalues of the matrix A , we obtain the above table(Table 1).

Therefore, the dynamics of FitzHugh-Nagumo model strongly depends on the control parameter I , furthermore, we can show that for a certain range of I within (I_1, I_2) the amplitude of the oscillations depend critially on the magnitude of the current I .

Let I_0 be the input current corresponds to $x_0 = 0$. In the following discussion, we only consider the range (I_1, I_0) . Within this range, when I increase from I_1 to I_0 , x_0 increase from $-\sqrt{1-\epsilon}$ to 0. Asymptotic method is used to show that in this range of I , the amplitude of the oscillation increase with the magnitude of input current I .

CHAPTER III

ASYMPTOTIC APPROXIMATION OF LIMIT CYCLE IN FITZHUGH-NAGUMO MODEL

Introduction to Asymptotic Approximation Method

Asymptotic approximation method is used frequently in the non-linear oscillations with small parameter. Here we use a example to illustrate the method of asymptotic approximations.

Consider the oscillations defined by a differential equation of the form:

$$\frac{d^2x}{dt^2} + w^2x = \epsilon f(x, \frac{dx}{dt}), \quad (4)$$

where ϵ is a small positive parameter.

When perturbation is absent, i.e., when $\epsilon = 0$, the oscillations will, evidently be purely harmonic,

$$x = a \cos \phi.$$

With a constant amplitude and a uniformly rotating phase angle:

$$\frac{da}{dt} = 0, \quad \frac{d\phi}{dt} = w \quad (\phi = wt + \theta).$$

The existence of non-linear perturbation ($\epsilon \neq 0$) results in the appearance of overtones in the solution of equation (4), a factor that establish dependence between the instantaneous frequency $\frac{d\phi}{dt}$ and the amplitude, and finally gives rise to a systematic increase or decrease in the amplitude of the oscillations.

With all this in view we shall seek a general solution of equation (4) in the form,

$$x = a \cos \phi + \epsilon u_1(a, \phi) + \epsilon^2 u_2(a, \phi) + \epsilon^3 u_3(a, \phi) + \dots \quad (5)$$

Here $u_1(a, \phi), u_2(a, \phi), \dots$ are periodic functions of the angle ϕ with a period 2π and the quantities a, ϕ are functions of time defined by the differential equations:

$$\frac{da}{dt} = \epsilon A_1(a) + \epsilon^2 A_2(a) + \dots \quad (6)$$

$$\frac{d\phi}{dt} = w + \epsilon B_1(a) + \epsilon^2 B_2(a) + \dots \quad (7)$$

We are to choose suitable expressions for the functions $u_1(a, \phi), u_2(a, \phi), \dots, A_1(a), B_1(a), A_2(a), B_2(a), \dots$ in such a way that the equation (5), after replacing a and ϕ , by the functions defined in equations (6) and (7), would serve as a solution of equation (4).

As soon as this problem is solved and explicit expressions for the coefficients of expansions occurring in the right-hand-sides of equations (5) . (6) and (7) are obtained, the problem of integrating equation (4) is reduced to that of integrating equations (6) and (7) which have separable variables, thus making the investigation possible with the help of well known elementary methods.

Here we will not go in details to solve the equation (4) using this asymptotic approximation method. We just use this example to illustrate the idea of asymptotic approximation. Later, this method will be used to study FitzHugh-Nagumo model with external input.

Asympototic Approximation of Limit Cycle

Theorem (Poincare-Bendixson). *Let $\mathbb{R}$ be a closed, bounded region consisting of non-singular points of a plane system $\dot{x} = X(x)$ such that some positive half-path $\mathbb{H}$ of the system lies entirely within $\mathbb{R}$. Then either $\mathbb{H}$ is itself a closed path, or it approaches a closed path, or it terminates at an equilibrium point.*

This theorem tells us that if we can construct a bounding surface around a fixed point so that all flux arrows on the surface are pointing towards the interior, and the fixed point in the interior is repulsive, then there must exist a stable limit cycle around that fixed point.

From the discussion in Chapter II about the FitzHugh-Nagumo model of the tangential cell, we know that when the input current is within the specified range, there is one unstable fixed point, if the flux is bounded, then by theorem of Poincare-Bendixson there must be a limit cycle.

In the F-N equation (1) and (2) we make the change of variable $\tau = \epsilon t$, then (1) and (2) becomes

$$\epsilon \dot{x} = x - \frac{1}{3}x^3 - y + I \triangleq f(x, y), \quad (8)$$

$$\dot{y} = bx - y + a \triangleq g(x, y). \quad (9)$$

then

$$\epsilon \frac{dx}{dy} = \frac{f(x, y)}{g(x, y)} \triangleq h(x, y).$$

so we have

$$\frac{dx}{dy} = \frac{h(x, y)}{\epsilon}.$$

ϵ is assumed to be small, so $\frac{dx}{dy}$ is very large, that is the integral curve $x = x(y)$ is practically horizontal everywhere in the (x, y) plane except on the curve $f(x, y) = 0$.

The curve $f(x, y) = 0$ is the x-nullcline consisting of 3 branches, separated by points A and B , at these points the curve $f(x, y) = 0$ has horizontal tangents (see Figure 2).

On the curve $f(x, y) = 0$ we have $\dot{x} = 0$, so trajectory cannot follow this curve being exactly on it, but very close to it on both side (branch 1 and 3), they are nearer to branches 1 and 3 the smaller the value of ϵ .

At the junction point A , the trajectory will goes horizontally to the left until it meets the curve $f(x, y) = 0$ on which it follows close to this curve in the prescribed direction until it reaches B . At this point the trajectory cannot follow branch 2 because the allowed direction on this branch is from O to B , hence it follow the horizontal direction from B to C , arrive at branch 1 again.

This determine the motion on the piecewise analytic cycle $ADBCA$ consisting of two analytic branches CA and DB closed by discontinuous stretches AD and BC .

In the following discussion, the idea of introducing certain connecting regions is used to approximate the limit cycle of system (8,9) piecewise. This idea comes from A.A.Dorodnitsin, who used this method to study the Van-der-Pol equation (see reference [3] p.187-195). For sufficiently small ϵ , the limit cycle is close to the trajectory of

$$f(x, y) = 0$$

$$\dot{y} = g(x, y).$$

From $f(x, y) = x - \frac{1}{3}x^3 - y + I$, we get $f_x = 1 - x^2$ and it is seen that when $x \neq 1$ and $x \neq -1$, we have $f_x \neq 0$.

Let P be a point on the right branch of the trajectory and P not within a small neighborhood of junction point A but close to it, then $f_x \neq 0$ on this part of the curve CP , so by implicit function theorem $f(x, y) = 0$ has a unique solution x , $x = x_0(y)$

Then $x_0(y)$ is an approximation to the trajectory of (8,9), in the region R1, we use an asymptotic expansion on this part of the trajectory to write

$$x = x(y) = x_0(y) + \epsilon x_1(y) + \epsilon^2 x_2(y) + \cdots + \epsilon^n x_n(y) + \cdots$$

then

$$\epsilon \frac{dx}{dy} = \epsilon(x_0'(y) + \epsilon x_1'(y) + \epsilon^2 x_2'(y) + \cdots + \epsilon^n x_n'(y) + \cdots),$$

Likewise,

$$\begin{aligned} h(x, y) &= h(x_0(y) + \epsilon x_1(y) + \epsilon^2 x_2(y) + \cdots + \epsilon^n x_n(y) + \cdots, y) \\ &= h(x_0(y), y) + h_x'(x_0(y), y)(\epsilon x_1(y) + \epsilon^2 x_2(y) + \cdots + \epsilon^n x_n(y) \\ &+ \cdots) + \frac{h_{xx}(x_0(y), y)}{2!}(\epsilon x_1(y) + \epsilon^2 x_2(y) + \cdots + \epsilon^n x_n(y) + \cdots)^2 + \cdots \end{aligned}$$

since

$$\epsilon \frac{dx}{dy} = h(x, y).$$

We compare the coefficients of powers of ϵ to see that:

$$h(x_0(y), y) = 0 \tag{10}$$

$$h_x'(x_0(y), y) \times x_1(y) = x_0'(y) \tag{11}$$

$$\vdots$$

$$x_{n-1}'(y) = x_n(y)h_x'(x_0(y), y) + \text{ terms in } (x_0(y), x_1(y), \dots, x_{n-1}(y)).$$

From (10), we have $f(x_0(y), y) = 0$ and so

$$h_x'(x_0(y), y) = \left(\frac{f}{g}\right)'_x = \frac{f_x'g - fg_x'}{g^2} = \frac{f_x'}{g}.$$

From $f(x_0(y), y) = 0$, we have

$$f_x'x_0'(y) + f_y' = 0.$$

and then

$$x_0'(y) = -\frac{f_y'}{f_x'}.$$

Therefore from (11), we have

$$\begin{aligned} x_1(y) &= \frac{x_0'(y)}{h_x'(x_0(y), y)} \\ &= -\frac{f_y'g}{f_x'^2} \\ &= -\frac{f_y'(x_0(y), y)g(x_0(y), y)}{f_x'^2(x_0(y), y)}. \end{aligned}$$

We use $x(y) = x_0(y) + \epsilon x_1(y) + O(\epsilon^2)$ to approximate $x(y)$.

Since f_x' approaches 0 when the trajectory approach the junction point A , so we cannot use this $x(y)$ to approximate the trajectory in a small neighborhood of junction point A .

In neighborhood of $A(a_1, b_1)$, $f_y = -1$ and so by the implicit function theorem, $f(x, y) = 0$ has a unique solution $y = y(x)$, for $x \in \text{neighborhood of } A$.

We seek an asymptotic approximation of $y(x)$ in the form $y(x) = \sum_{n=0}^{\infty} y_n(x)\epsilon^n$

$$\begin{aligned}\epsilon \frac{dx}{dy} &= \frac{f(x, y)}{g(x, y)} \\ \frac{dy}{\epsilon dx} &= \frac{g(x, y)}{f(x, y)}.\end{aligned}$$

Let $z = \epsilon x$, then we have

$$\begin{aligned}\frac{dy}{dz} &= \frac{g(\frac{z}{\epsilon}, y)}{f(\frac{z}{\epsilon}, y)} \\ &= \frac{b\frac{z}{\epsilon} - y + a}{\frac{z}{\epsilon} - \frac{1}{3}(\frac{z}{\epsilon})^3 - y + I} \\ &= \frac{b\frac{z}{\epsilon} - (y_0(z) + \epsilon y_1(z) + \epsilon^2 y_2(z) + \dots + \epsilon^n y_n(z) + \dots) + a}{\frac{z}{\epsilon} - \frac{1}{3}(\frac{z}{\epsilon})^3 - (y_0(z) + \epsilon y_1(z) + \epsilon^2 y_2(z) + \dots + \epsilon^n y_n(z) + \dots) + I} \\ &= (3bz\epsilon^2 + (3a - 3y_0(z))\epsilon^3 - 3y_1(z)\epsilon^4 - 3y_2(z)\epsilon^5 - \dots - 3\epsilon^{n+3}y_n(z)) \\ &\quad \left(-\frac{1}{z^3} + (-3z^{-5})\epsilon^2 + (-z^{-6})(3I - 3y_0(z))\right. \\ &\quad \left.\epsilon^3 + \dots\right).\end{aligned}\tag{12}$$

From $y(x) = \sum_{n=0}^{\infty} y_n(x)\epsilon^n$ we have

$$\frac{dy}{dz} = y_0'(z) + \epsilon y_1'(z) + \epsilon^2 y_2'(z) + \dots + \epsilon^n y_n'(z) + \dots.\tag{13}$$

Comparing the coefficients of powers of ϵ in right-hand-sides of (12) and (13), we obtain that

$$\begin{aligned}y_0'(z) &= 0 \\ y_1'(z) &= 0 \\ y_2'(z) &= -\frac{3b}{z^2} \\ y_3'(z) &= -z^{-3}(3a - 3y_0(z)).\end{aligned}$$

Now let

$$y_0(z) = C_0 \text{ (constant)}$$

$$y_1(z) = C_1 \text{ (constant)}$$

$$y_2(z) = \frac{3b}{z}.$$

We use $y(z) = C_0 + C_1\epsilon + \frac{3b}{z}\epsilon^2 + O(\epsilon^3)$ to approximate $y(z)$

$$y(z) = y(\epsilon x) \approx C_0 + C_1\epsilon + \frac{3b}{x}\epsilon.$$

In summary, in region R_1 we use $x = x(y) = x_0(y) + \epsilon x_1(y) + O(\epsilon^2)$ to approximate the trajectory and in region R_2 , we use $y(z) = C_0 + C_1\epsilon + \frac{3b}{z}\epsilon^2 + O(\epsilon^3)$ to approximate the trajectory.

We pick a point on the trajectory which lies in the overlapping area of region R_1 and region R_2 , say point (a_0, b_0) . We have $a_0 = x(\hat{y}) = x_0(\hat{y}) + \epsilon x_1(\hat{y}) + O(\epsilon^2)$, so

$$a_0 \approx x_0(\hat{y}) + \epsilon \frac{bx_0(\hat{y}) - \hat{y} + a}{(1 - x_0^2(\hat{y}))^2}. \quad (14)$$

Since the junction point (a_1, b_1) is on the trajectory in region R_2 then $C_0 + C_1\epsilon + \frac{3b}{a_1}\epsilon = b_1$. let $C_0 = b_1, C_1 = -\frac{3b}{a_1}$

so that

$$y(\epsilon x) \approx b_1 - \frac{3b}{a_1}\epsilon + \frac{3b}{x}\epsilon.$$

Therefore, $\hat{y}$ can be approximated by $b_1 - \frac{3b}{a_1}\epsilon + \frac{3b}{a_0}\epsilon$.

So we have

$$\begin{aligned} \hat{y} &= b_1 - \frac{3b\epsilon}{a_1} + \frac{3b\epsilon}{a_0} \\ b_1 &= \frac{3b\epsilon}{a_1} + \hat{y} - \frac{3b\epsilon}{a_0}. \end{aligned} \quad (15)$$

We can express $x_0(\hat{y})$ in terms of $\hat{y}$ and I , by solving the cubic equation $x_0 - \frac{1}{3}x_0^3 - \hat{y} + I = 0$ for real root.

$$x_0(\hat{y}) = \sqrt[3]{\frac{3(I - \hat{y})}{2} + \sqrt{\frac{9}{4}(I - \hat{y})^2 - 1}} + \sqrt[3]{\frac{3(I - \hat{y})}{2} - \sqrt{\frac{9}{4}(I - \hat{y})^2 - 1}}. \quad (16)$$

Then in (14), we replace $x_0(\hat{y})$ with right-hand-side of (16) to obtain

$$a_0 = K_1 + K_2 + \epsilon \frac{b(K_1 + K_2) - \hat{y} + a}{(1 - (K_1 + K_2)^2)^2}. \quad (17)$$

where $K_1 = \sqrt[3]{\frac{3(I-\hat{y})}{2}} + \sqrt{\frac{9}{4}(I-\hat{y})^2 - 1}$ and $K_2 = \sqrt[3]{\frac{3(I-\hat{y})}{2}} - \sqrt{\frac{9}{4}(I-\hat{y})^2 - 1}$.

From (17), we know when I increases, $\hat{y}$ increases with I . From (15), when I increases, the term $\hat{y} + \frac{3b\epsilon}{a_0}$ increases, which means the graph of (a_1, b_1) moves upwards with increasing I . Since the point (a_1, b_1) represents the maximum displacement of the trajectory we come to the conclusion that the amplitude increases with the increasing input I , for $I \in (I_1, I_0)$ (see Figure 3).

Figure 3: The graph of $b_1 = \frac{3b\epsilon}{a_1} + \hat{y} - \frac{3b\epsilon}{a_0}$.

Conclusion and Future Work

In this thesis, FitzHugh-Nagumo model is used to model the dynamics of tangential cell in fly's visual cortex, in which the membrane current produced by the visual stimuli is modeled as the external input. An asymptotic method is used to show that the amplitude of the oscillations depend critically on the magnitude of the external input. Furthermore, in a neighborhood of a bifurcation, the amplitude increases monotonically as a function of the external input, this result helps to explain how the visual cortex of fly encodes visual stimulus and detects motions.

The work is limited to only one tangential cell. There are almost 60 tangential cells in fly's visual cortex. The future work is to use a network of oscillators to model tangential cells and the interactions between them, to determine from the oscillator network the direction and velocity vector of an object moving through the field of vision of the eye.

REFERENCES

- [1] Mazokhin-Porshnyakov G.A., *Insect Vision*. Plenum Press, New York, (1969).
- [2] Srinivasan M.V. and Venkatesh S., *From Living Eyes to Seeing Machines*. Oxford University Press, Oxford, (1997).
- [3] Bogoliubov N.N. and Mitropolsky Y.A., *Asymptotic Methods in the Theory of Nonlinear Oscillations*. Hindustan Scientific Press, New Delhi, (1961).
- [4] Minorsky N., *Nonlinear Oscillations*. D.Van Nostrand Company, Inc. Princeton, Englewood Cliffs, New Jersey, (1962).
- [5] Willam C. T., Bifurcation Phenomena in FitzHugh-Nagumo's Nerve Conduction Equations. *Journal of Mathematical Analysis and Applications* 54, 678-690, (1976).
- [6] Mettin R., Parlitz U. and Lauterborn W., Bifurcation Structure of the Driven Van Der Pol Oscillator. *Inter. J. of Bifur. and Chaos*, vol 3, no 6, 1529-1550, (1993).
- [7] Jordan D.W. and Smith P., *Nonlinear Ordinary Differential Equations*. Oxford University Press, Oxford, (1999).
- [8] E.F.Mishchenko and N.Kh.Rozov, *Differential Equations with Small Parameters and Relaxation Oscillations*. Plenum Press, New York, (1980).

PERMISSION TO COPY

In presenting this thesis in partial fulfillment of the requirements for a master's degree at Texas Tech University or Texas Tech University Health Sciences Center, I agree that the Library and my major department shall make it freely available for research purposes. Permission to copy this thesis for scholarly purposes may be granted by the Director of the Library or my major professor. It is understood that any copying or publication of this thesis for financial gain shall not be allowed without my further written permission and that any user may be liable for copyright infringement.

Agree (Permission is granted.)

Student Signature

Date

Disagree (Permission is not granted.)

Student Signature

Date