EXTREME PROPERTIES OF PRODUCTS
OF QUADRATIC FORMS

by

GLENN E. JOHNSTON, B.S., M.S.

A DISSERTATION

IN

MATHEMATICS

Submitted to the Graduate Faculty
of Texas Tech University in
Partial Fulfillment of
the Requirements for
the Degree of

DOCTOR OF PHILOSOPHY

Approved

August 1970

AC
801
T3
1970
NO.5
cop.2

ACKNOWLEDGMENT

I am deeply indebted to Professor Ali R. Amir-Moéz for his help in guiding this dissertation.

CONTENTS

Chapter	Page
I. INTRODUCTION.	1
II. SUPREMUM OF PRODUCT OF QUADRATIC FORMS.	2
III. INFIMUM OF PRODUCT OF QUADRATIC FORMS	10
REFERENCES	13

CHAPTER I

INTRODUCTION

Purpose 1.1--In the literature, most inequalities concerning proper values of a linear transformation A on a unitary space involve symmetric functions related to A . We study extrema of a product of quadratic forms for a positive transformation on a unitary space related to an orthonormal set. The minimum of a product of quadratic forms of a positive linear transformation A is given in [2]. The maximum is a special case of a theorem of M. Marcus and J. L. McGregor [7]. Certain generalizations of these ideas will be given.

Definitions and notations 1.2--An n -dimensional unitary space will be denoted by E_n . The inner product of two vectors α and β will be denoted by (α, β) . The determinant of a linear transformation A on E_n will be denoted by $\det A$. The adjoint A^* of A will be defined by $(A\xi, \eta) = (\xi, A^*\eta)$, for all ξ and η in E_n . The matrix of a linear transformation A with respect to a given basis will be denoted by (a_{ij}) . A Hermitian linear transformation is called positive if and only if $(A\xi, \xi) > 0$ for all $\xi \neq 0$. An orthonormal set $\{\alpha_1, \dots, \alpha_k\}$ will be indicated by $\{\alpha_1, \dots, \alpha_k\}$ o.n. . A subspace spanned by the set $\{\gamma_1, \dots, \gamma_p\}$ will be denoted by $[\gamma_1, \dots, \gamma_p]$. The expression $A|M$ denotes the linear transformation A restricted to the subspace M .

CHAPTER II

SUPREMUM OF PRODUCT OF QUADRATIC FORMS

Theorem 2.1--Let A be a positive linear transformation on E_n with proper values $m_1 \geq \dots \geq m_n$. Then

$$\sup_{\{\xi_p\}_{o.n.}} F_r \left((A\xi_1, \xi_1), \dots, (A\xi_k, \xi_k) \right) = \binom{k}{r} \left(\frac{1}{k} \sum_{j=1}^k m_j \right)^r,$$

where F_r denotes the r -th elementary symmetric function, i.e.,

$$F_r(t_1, \dots, t_k) = \sum_{1 \leq i_1 < \dots < i_r \leq k} t_{i_1} \cdot \dots \cdot t_{i_r}.$$

The proof is due to M. Marcus and J. L. McGregor [7].

Theorem 2.2--Let A be a positive linear transformation on E_n with proper values $m_1 \geq \dots \geq m_n$. Then

$$\inf_{\substack{M \\ \dim M = h}} \sup_{\substack{\{\xi_p\}_{o.n.} \\ \xi_p \in M}} F_r \left((A\xi_1, \xi_1), \dots, (A\xi_k, \xi_k) \right) = \binom{k}{r} \left(\frac{m_{n-h+1} + \dots + m_{n-h+k}}{k} \right)^r,$$

where $1 < r < k < h < n$, and F_r is the same as in 2.1.

Proof: Let M be any subspace of E_n such that $\dim M = h$ and let P be the orthogonal projection on M . Then, for $\xi \in M$, it follows that $(A|M)\xi = PAP\xi$. Let $s_1 \geq \dots \geq s_h$ be the proper values of $A|M$. Then, by 2.1, we have

$$\sup_{\substack{\{\xi_p\}_{o.n.} \\ \xi_p \in M}} F_r \left((A\xi_1, \xi_1), \dots, (A\xi_k, \xi_k) \right) = \binom{k}{r} \left(\frac{1}{k} \sum_{j=1}^k s_j \right)^r.$$

But $s_i \geq m_{n-h+i}$, see [1, p.464]. Thus,

$$\begin{aligned} & \sup_{\substack{\{\xi_p\}_{o.n.} \\ \xi_p \in M}} F_r \left((A\xi_1, \xi_1), \dots, (A\xi_k, \xi_k) \right) \\ & \geq \binom{k}{r} \left(\frac{m_{n-h+1} + \dots + m_{n-h+k}}{k} \right)^r. \end{aligned}$$

Therefore

$$\begin{aligned} & \inf_{\substack{M \\ \dim M = h}} \sup_{\substack{\{\xi_p\}_{o.n.} \\ \xi_p \in M}} F_r \left((A\xi_1, \xi_1), \dots, (A\xi_k, \xi_k) \right) \\ & \geq \binom{k}{r} \left(\frac{m_{n-h+1} + \dots + m_{n-h+k}}{k} \right)^r. \quad (1) \end{aligned}$$

Now let $\{\alpha_1, \dots, \alpha_n\}$ be an orthonormal set of proper vectors of A corresponding respectively to $m_1, \dots, m_n$ and set $M = [\alpha_{n-h+1}, \dots, \alpha_n]$. Then by 2.1, we have

$$\begin{aligned} & \sup_{\substack{\{\xi_p\}_{o.n.} \\ \xi_p \in M}} F_r \left((A\xi_1, \xi_1), \dots, (A\xi_k, \xi_k) \right) \\ & = \binom{k}{r} \left(\frac{m_{n-h+1} + \dots + m_{n-h+k}}{k} \right)^r. \quad (2) \end{aligned}$$

Comparing (1) and (2) completes the proof.

Theorem 2.3--Let A be a positive linear transformation on E_n with proper values $m_1 \geq \dots \geq m_n$. Then

$$\inf_{\substack{M_1 \subset \dots \subset M_k \\ \dim M_p = i_p}} \sup_{\substack{\{\xi_p\} \text{ o.n.} \\ \xi_p \in M_p}} F_r \left((A\xi_1, \xi_1), \dots, (A\xi_k, \xi_k) \right) \\ = \binom{k}{r} \left(\frac{m_{i_1} + \dots + m_{i_k}}{k} \right)^r,$$

where $(i_1, \dots, i_k)$ is a sequence of integers such that $1 \leq i_1 < \dots < i_k \leq n$.

Proof: Let $M_1 \subset \dots \subset M_k$ be subspaces such that $\dim M_p = i_p$, $p = 1, \dots, k$ and $N_p = [\alpha_{i_p}, \dots, \alpha_n]$, where $\{\alpha_1, \dots, \alpha_n\}$ is an orthonormal set of proper vectors of A corresponding respectively to $m_1, \dots, m_n$. Then we observe that $\dim(M_p \cap N_p) \geq 1$. Thus there exists a subspace M with an orthonormal basis $\{\xi_1, \dots, \xi_k\}$ such that $\xi_p \in M_p$, and another orthonormal basis $\{\delta_1, \dots, \delta_k\}$, where $\delta_p \in N_p$, $p = 1, \dots, k$, see [1, 2.2]. Let $s_1 \geq \dots \geq s_k$ be the proper values of $A|_M$. Then

$$F_r \left((A\xi_1, \xi_1), \dots, (A\xi_k, \xi_k) \right) \leq \binom{k}{r} \left(\frac{s_1 + \dots + s_k}{k} \right)^r.$$

But

$$s_j = \inf_{\substack{N \subset M \\ \dim N = j-1}} \sup_{\substack{\xi \perp N \\ \|\xi\|=1 \\ \xi \in M}} (A\xi, \xi) \leq \sup_{\substack{\xi \perp [\delta_1, \dots, \delta_{j-1}] \\ \|\xi\|=1}} (A\xi, \xi)$$

$$= \sup_{\substack{\xi \in [\delta_j, \dots, \delta_k] \\ \|\xi\|=1}} (A\xi, \xi) \leq \sup_{\substack{\xi \in N_j \\ \|\xi\|=1}} (A\xi, \xi) = m_{i_j}, \quad j = 1, \dots, k.$$

Therefore,

$$\inf_{\substack{M_1 \subset \dots \subset M_k \\ \dim M_p = i_p}} \sup_{\substack{\{\xi_p\} \text{ o.n.} \\ \xi_p \in M_p}} F_r \left((A\xi_1, \xi_1), \dots, (A\xi_k, \xi_k) \right) \leq \binom{k}{r} \left(\frac{m_{i_1} + \dots + m_{i_k}}{k} \right)^r. \quad (1)$$

On the other hand, let $M_p = [\alpha_1, \dots, \alpha_{i_p}]$, $p = 1, \dots, k$.

It is clear that $M_1 \subset \dots \subset M_k$. Let $\{\eta_1, \dots, \eta_k\}$ be orthonormal such that $\eta_p \in M_p$, see [1, p.467], and suppose $M = [\eta_1, \dots, \eta_k]$. Let $t_1 \geq \dots \geq t_k$ be proper values of $A|M$. Then, by 2.1,

$$\sup_{\substack{\{\xi_p\} \text{ o.n.} \\ \xi_p \in M}} F_r \left((A\xi_1, \xi_1), \dots, (A\xi_k, \xi_k) \right) = \binom{k}{r} \left(\frac{t_1 + \dots + t_k}{k} \right)^r.$$

But

$$\begin{aligned} t_j &= \sup_{\substack{N \subset M \\ \dim N = j}} \inf_{\substack{\xi \in N \\ \|\xi\|=1}} (A\xi, \xi) \geq \inf_{\substack{\xi \in [\eta_1, \dots, \eta_j] \\ \|\xi\|=1}} (A\xi, \xi) \\ &\geq \inf_{\substack{\xi \in M_j \\ \|\xi\|=1}} (A\xi, \xi) = m_{i_j}. \end{aligned}$$

Therefore,

$$\inf_{\substack{M_1 \subset \dots \subset M_k \\ \dim M_p = i_p}} \sup_{\substack{\{\xi_p\} \text{ o.n.} \\ \xi_p \in M_p}} F_r \left((A\xi_1, \xi_1), \dots, (A\xi_k, \xi_k) \right) \geq \binom{k}{r} \left(\frac{m_{i_1} + \dots + m_{i_k}}{k} \right)^r. \quad (2)$$

Comparing (1) and (2), the proof is complete.

Definition 2.4--If $j_p \leq i_p$, $p = 1, \dots, k$, we write $(j_1, \dots, j_k) \leq (i_1, \dots, i_k)$. Further, given any sequence $(i_1, \dots, i_k)$ of integers such that $i_1 \leq \dots \leq i_k$ and $i_p \geq p$, $p = 1, \dots, k$, we define $(i'_1, \dots, i'_k)$ to be the strictly increasing sequence of positive integers such that $(i'_1, \dots, i'_k) \leq (i_1, \dots, i_k)$ and $(j_1, \dots, j_k) \leq (i'_1, \dots, i'_k)$ if $(j_1, \dots, j_k)$ is a strictly increasing sequence of positive integers which is less than or equal to $(i_1, \dots, i_k)$.

Theorem 2.5--Let A be a positive transformation on E_n with proper values $m_1 \geq \dots \geq m_n$. Let $(i_1, \dots, i_k)$ be a sequence of positive integers such that $i_1 \leq \dots \leq i_k \leq n$ and $i_p \geq p$, $p = 1, \dots, k$. Then

$$\inf_{\substack{M_1 \subset \dots \subset M_k \\ \dim M_p = i_p}} \sup_{\substack{\{\xi_p\} \text{ o.n.} \\ \xi_p \in M_p}} F_r \left((A\xi_1, \xi_1), \dots, (A\xi_k, \xi_k) \right) \\ = \binom{k}{r} \left(\frac{m_{i'_1} + \dots + m_{i'_k}}{k} \right)^r$$

where $(i'_1, \dots, i'_k)$ is the sequence defined in 2.4.

Proof: Let $M_1 \subset \dots \subset M_k$ be any subspaces with $\dim M_p = i_p$, $p = 1, \dots, k$. Then there exist subspaces $N_1 \subset \dots \subset N_k$ such that $N_p \subset M_p$ and $\dim N_p = i'_p$. Thus

$$\sup_{\substack{\{\xi_p\} \text{ o.n.} \\ \xi_p \in M_p}} F_r \left((A\xi_1, \xi_1), \dots, (A\xi_k, \xi_k) \right)$$

$$\geq \sup_{\substack{\{\xi_p\} \text{ o.n.} \\ \xi_p \in N_p}} F_r \left((A\xi_1, \xi_1), \dots, (A\xi_k, \xi_k) \right).$$

But by 2.3 we have

$$\begin{aligned} & \sup_{\substack{\{\xi_p\} \text{ o.n.} \\ \xi_p \in N_p}} F_r \left((A\xi_1, \xi_1), \dots, (A\xi_k, \xi_k) \right) \\ & \geq \binom{k}{r} \left(\frac{m_{i_1'} + \dots + m_{i_k'}}{k} \right)^r. \end{aligned}$$

Thus,

$$\begin{aligned} & \inf_{\substack{M_1 \subset \dots \subset M_k \\ \dim M_p = i_p}} \sup_{\substack{\{\xi_p\} \text{ o.n.} \\ \xi_p \in M_p}} F_r \left((A\xi_1, \xi_1), \dots, (A\xi_k, \xi_k) \right) \\ & \geq \binom{k}{r} \left(\frac{m_{i_1'} + \dots + m_{i_k'}}{k} \right)^r. \end{aligned} \quad (1)$$

Now let $M_p = [\alpha_1, \dots, \alpha_p]$, $p = 1, \dots, n$, where $\{\alpha_1, \dots, \alpha_n\}$ is an orthonormal set of proper vectors of A corresponding to $m_1, \dots, m_n$. Choose an orthonormal set $\{\delta_1, \dots, \delta_k\}$ such that $\delta_p \in M_{i_p}$, $p = 1, \dots, k$, and

$$\begin{aligned} & \sup_{\substack{\{\xi_p\} \text{ o.n.} \\ \xi_p \in M_{i_p}}} F_r \left((A\xi_1, \xi_1), \dots, (A\xi_k, \xi_k) \right) \\ & = F_r \left((A\delta_1, \delta_1), \dots, (A\delta_k, \delta_k) \right) \\ & = \binom{k}{r} \left(\frac{s_1 + \dots + s_k}{k} \right)^r, \end{aligned}$$

where $s_1 \geq \dots \geq s_k$ are proper values of $A|[\delta_1, \dots, \delta_k]$.

There exists an orthonormal set $\{\eta_1, \dots, \eta_k\}$ such that $\eta_p \in M_{i_p}$, and $[\eta_1, \dots, \eta_k] = [\delta_1, \dots, \delta_k]$, see [1, p.467]. Hence,

$$s_1 + \dots + s_k = \sum_{i=1}^k (A\delta_i, \delta_i) = \sum_{i=1}^k (A\eta_i, \eta_i).$$

But,

$$\sum_{i=1}^k (A\eta_i, \eta_i) = m_{i_1'} + \dots + m_{i_k'}.$$

Therefore,

$$\inf_{\substack{M_{i_p} = [\alpha_1, \dots, \alpha_{i_p}] \\ p = 1, \dots, k}} \sup_{\substack{\{\xi_p\} \text{ o.n.} \\ \xi_p \in M_{i_p}}} F_r \left((A\xi_1, \xi_1), \dots, (A\xi_k, \xi_k) \right) \\ \leq \binom{k}{r} \left(\frac{m_{i_1'} + \dots + m_{i_k'}}{k} \right)^r. \quad (2)$$

Comparing (1) and (2), the proof is complete.

CHAPTER III

INFIMUM OF PRODUCTS OF QUADRATIC FORMS

Lemma 3.1--Let A be a positive transformation on E_n . Then there exists a linear transformation B on E_n such that $A = BB^*$. See [5].

Lemma 3.2--Let (a_{ij}) be the matrix of a positive linear transformation A on E_n with respect to an orthonormal basis χ . Then

$$\det A \leq \prod_{i=1}^n a_{ii}.$$

Proof: By lemma 3.1, there exists a linear transformation B on E_n such that $BB^* = A$. Let the matrix of B with respect to χ be (b_{ij}) . Clearly the matrix of B^* with respect to χ is $(\overline{b_{ji}})$. Suppose $\xi_i = (b_{i1}, \dots, b_{in})$. Then $a_{ij} = (\xi_i, \xi_j)$. Let $\beta_1 = \xi_1$ and

$$\beta_k = \xi_k - \sum_{i=1}^{k-1} s_{ki} \beta_i, \quad (\beta_p, \beta_q) = 0, \quad p \neq q, \quad k = 2, \dots, n.$$

It is easy to show that $s_{ki} = \frac{(\xi_k, \beta_i)}{(\beta_i, \beta_i)}$.

Consider the matrix (c_{ij}) such that $\beta_i = (c_{i1}, \dots, c_{in})$. Then we observe that $\det (b_{ij}) = \det (c_{ij})$. Suppose (c_{ij}) is the matrix of the linear transformation C with respect to χ . Then

$$\det A = \det (BB^*) = \det (CC^*).$$

Since $\{\beta_1, \dots, \beta_n\}$ is non-zero, orthogonal

$$\det (CC^*) = (\beta_1, \beta_1) \cdot \dots \cdot (\beta_n, \beta_n).$$

We can show that $(\beta_k, \beta_k) \leq (\xi_k, \xi_k)$ and consequently

$$\det A = \prod_{i=1}^n (\beta_i, \beta_i) \leq \prod_{i=1}^n (\xi_i, \xi_i) = \prod_{i=1}^n a_{ii}.$$

Theorem 3.3--Let A be a positive linear transformation on E_n with proper values $m_1 \geq \dots \geq m_n$. Then

$$\inf_{\{\xi_1, \dots, \xi_k\} \text{ o.n.}} \begin{vmatrix} (A\xi_1, \xi_1) & \dots & (A\xi_1, \xi_k) \\ \dots & & \dots \\ (A\xi_k, \xi_1) & \dots & (A\xi_k, \xi_k) \end{vmatrix} = m_{n-k+1} \cdot \dots \cdot m_n.$$

This theorem is due to Ky Fan [4]. We give the following

Proof: Let $\xi_i = \alpha_{n-k+i}$, $i = 1, \dots, k$. Then

$$\det \begin{pmatrix} (A\xi_1, \xi_1) & \dots & (A\xi_1, \xi_k) \\ \dots & & \dots \\ (A\xi_k, \xi_1) & \dots & (A\xi_k, \xi_k) \end{pmatrix} = m_{n-k+1} \cdot \dots \cdot m_n.$$

Now let $\{\xi_1, \dots, \xi_k\}$ be any orthonormal set in E_n . Consider $N = [\xi_1, \dots, \xi_k]$ and let P be the orthogonal projection on N . If the proper values of PAP are $t_1 \geq \dots \geq t_k$, then

$$\det \begin{pmatrix} (A\xi_1, \xi_1) & \dots & (A\xi_1, \xi_k) \\ \dots & & \dots \\ (A\xi_k, \xi_1) & \dots & (A\xi_k, \xi_k) \end{pmatrix} = t_1 \cdot \dots \cdot t_k.$$

But by the minimax principle,

$$\inf_{\substack{N \subset E_n \\ \dim N = p}} \sup_{\substack{\xi \in M \\ \|\xi\|=1}} (A\xi, \xi) = m_{n-p+1}$$

and

$$\inf_{\substack{M \subset N \\ \dim M = p}} \sup_{\substack{\xi \in M \\ \|\xi\|=1}} (A\xi, \xi) = t_{k-p+1}.$$

This implies that

$$t_{k-p+1} \geq m_{n-p+1}, \quad p = 1, \dots, k.$$

This completes the proof.

Theorem 3.4--Let A be a positive linear transformation on E_n with proper values $m_1 \geq \dots \geq m_n$. Then

$$\inf_{\{\xi_1, \dots, \xi_k\} \text{ o.n.}} \prod_{i=1}^k (A\xi_i, \xi_i) = m_{n-k+1} \cdot \dots \cdot m_n.$$

Proof: Consider $\det \left((A\xi_i, \xi_j) \right)$, $i, j = 1, \dots, k$. By 3.2 and 3.3 we have

$$\prod_{i=1}^k (A\xi_i, \xi_i) \geq \det \left((A\xi_i, \xi_j) \right) \geq m_{n-k+1} \cdot \dots \cdot m_n.$$

Now let $\{\alpha_1, \dots, \alpha_n\}$ be the orthonormal set of proper vectors

of A corresponding respectively to $m_1, \dots, m_n$. Then

$$(A\alpha_{n-k+1}, \alpha_{n-k+1}) \cdot \dots \cdot (A\alpha_n, \alpha_n) = m_{n-k+1} \cdot \dots \cdot m_n.$$

This proves the theorem.

Theorem 3.5--Let A satisfy the hypothesis of 2.5. Then

$$\begin{aligned} & \sup_{\substack{M_1 \subset \dots \subset M_k \\ \dim M_p = i_p}} \inf_{\substack{\{\xi_p\} \text{ o.n.} \\ \xi_p \in M_p}} \prod_{i=1}^k (A\xi_i, \xi_i) \\ &= \sup_{\substack{M_1 \subset \dots \subset M_k \\ \dim M_p = i_p}} \inf_{\substack{\{\xi_p\} \text{ o.n.} \\ \xi_p \in M_p}} \det \left((A\xi_i, \xi_j) \right), \end{aligned}$$

where $(i'_1, \dots, i'_k)$ is the sequence defined in 2.4.

The proof of this theorem and other generalizations of it follow ideas similar to the ones in 2.3 and 2.5; thus, we omit it.

REFERENCES

- [1]. A. R. Amir-Moéz, "Extreme Properties of Eigenvalues of a Hermitian Transformation and Singular Values of the Sum and Product of Linear Transformations", Duke Math. Journal, Vol. 23, 1956, pp. 463-476
- [2]. E. F. Beckenbach, R. Bellman, Inequalities, Springer-Verlag, 1964.
- [3]. Ky Fan, "On a Theorem of Weyl Concerning Eigenvalues of Linear Transformations", Proc. of Nat. Acad. of Science, Vol. 35, 1949, pp. 652-655.
- [4]. _____, "On a Theorem of Weyl Concerning Eigenvalues of Linear Transformations II", Proc. of Nat. Acad. of Science, Vol. 36, 1950, pp. 31-35.
- [5]. I. N. Herstein, Topics in Algebra, Blaisdell Pub. Co., 1964, p. 304.
- [6]. A. Horn, "Eigenvalues of Sums of Hermitian Matrices", Pac. Journal of Math., Vol. 12, 1962, pp. 225-241.
- [7]. M. Marcus, J. L. McGregor, "Extremal Properties of Hermitian Matrices", Canadian Journal of Math., Vol. 8, 1956, pp. 524-531.