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UNBIASED F-TESTS FOR FACTORIAL EXPERIMENTS
WITH CORRELATED DATA

by

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CHAPTER I

INTRODUCTION

Smith and Lewis (1980) showed that if a factorial experiment has a certain correlation structure then the standard F test that is used when independence is assumed can lead to grossly inaccurate results. They provided a technique to correct the standard F-test, so that a true F distribution can be used to test the effects of factors and of interaction terms of the k-way factorial experiment. Also they showed that under the assumption of intraclass correlation, unbiased F tests can be constructed for other designs, such as nested-factorial and split-plot designs. Other works have been done in the area of intraclass correlation. Bhat (1962) showed that a necessary and sufficient condition for the sample variance, S^2 , to be distributed as a constant times a chi-square distribution is that the observations be equicorrelated. Basu, Odell, and Lewis (1974) and Basu, Odell, Lewis, and Kinderman (1975) have looked at multivariate normal populations in which intraclass correlation was present.

In Chapter II, under certain assumptions, a general form of the correlation matrix is derived so that the usual F statistic in a k-way factorial experiment can be corrected by a constant. The correlation matrix developed by Smith and Lewis (1980) is shown to be a special case of the general form. Also a

method is provided for computing the correction constants for the F tests from a given correlation matrix.

In Chapter III, the corrected F tests are shown to be the likelihood ratio tests. From this result, it follows from Lehmann (1959), Chapter 7, that the corrected F is an unbiased uniformly most powerful test among the class of invariant tests. Also an extension is made to the multivariate analysis of variance.

In Chapter IV, a Monte Carlo simulation study is made in which the corrected F statistic on the data and the corrected F statistic on the ranks is examined. These two test statistics are compared under certain situations when not all of the usual assumptions of an ANOVA are present.

CHAPTER II

UNBIASED F TESTS FOR FACTORIAL EXPERIMENTS FOR CORRELATED DATA

2.1 Notation.

The Model for a k-way factorial experiment is

$$\begin{aligned}
 Y_{(i_1, i_2, \dots, i_k) i_0} = & u + A_{i_1}^{(1)} + A_{i_2}^{(2)} + A_{i_1 i_2}^{(12)} + \\
 & A_{i_3}^{(3)} + A_{i_1 i_3}^{(13)} + \dots + A_{i_1 i_2 \dots i_k}^{(12 \dots k)} + \epsilon_{(i_1 i_2 \dots i_k) i_0}
 \end{aligned}
 \tag{2.1}$$

where u is the overall mean, $A_{i_j}^{(j)}$ is the effect of the j^{th}

factor at the i_j^{th} level, $A_{i_1 i_2 \dots i_p}^{(j_1 j_2 \dots j_p)}$ is the interaction of

the $j_1, j_2, \dots, j_p$ factors at the $i_1, i_2, \dots, i_p$ levels, respectively and $1 \leq i_j \leq n_j$, $1 \leq j \leq k$, and $1 \leq i_0 \leq n$.

$\epsilon_{(i_1, i_2, \dots, i_k) i_0}$ is the residual or error term and is dis-

tributed normally with zero mean and variance σ^2 . If the

$\epsilon_{(i_1, i_2, \dots, i_k) i_0}$'s are distributed independently with

homogeneous σ^2 , then the standard F test is used to test

the effects of the factors and the effects of the interaction

terms. Let $n_0 = n_1 n_2 \dots n_k n$.

The next theorem is taken from Smith and Lewis (1980)

and shows how the sum of squares in an ANOVA for a factorial design can be written as a quadratic form.

Theorem 1. Let $SSA^{(i)}$ denote the sum of squares due to

factor $A^{(i)}$, $1 \leq i \leq k$. Denote the sum of squares due to the interaction of factors $A^{(i_1)}$ and $A^{(i_2)}$ by $SSA^{(i_1, i_2)}$, and denote the sum of squares due to the interaction of factors $A^{(i_1)}, A^{(i_2)}, \dots, A^{(i_s)}$ by $SSA^{(i_1 i_2 \dots i_s)}$. Let SSE denote the residual sum of squares. Let M_q be the $q \times q$ matrix whose entries are equal to one. Define $C_q = I_q - \frac{1}{q} M_q$. If U^i is defined to be the matrix such that $SSA^{(i)} = Y^T U^i Y$, and if $U^{i_1 i_2 \dots i_s}$ is defined to be the matrix such that $SSA^{(i_1 \dots i_s)} = Y^T U^{i_1 \dots i_s} Y$, and if U^E is defined to be the matrix such that $SSE = Y^T U^E Y$, then

$$U^1 = (n_2 n_3 \dots n_k n)^{-1} C_{n_1} \otimes M_{n_2} \otimes M_{n_3} \otimes \dots \otimes M_{n_k} \otimes M_n$$

$$U^2 = (n_1 n_3 \dots n_k n)^{-1} M_{n_1} \otimes C_{n_2} \otimes M_{n_3} \otimes \dots \otimes M_{n_k} \otimes M_n$$

$$\vdots$$

$$U^k = (n_1 n_2 \dots n_{k-1} n)^{-1} M_{n_1} \otimes M_{n_2} \otimes \dots \otimes M_{n_{k-1}} \otimes C_{n_k} \otimes M_n$$

$$U^{12} = (n_3 n_4 \dots n_k n)^{-1} C_{n_1} \otimes C_{n_2} \otimes M_{n_3} \otimes \dots \otimes M_{n_k} \otimes M_n$$

$$U^{13} = (n_2 n_4 \dots n_k n)^{-1} C_{n_1} \otimes M_{n_2} \otimes C_{n_3} \otimes M_{n_4} \otimes \dots \otimes M_{n_k} \otimes M_n$$

$$\vdots$$

$$U^{123} = (n_4 n_5 \dots n_k n)^{-1} C_{n_1} \otimes C_{n_2} \otimes C_{n_3} \otimes M_{n_4} \otimes \dots \otimes M_{n_k} \otimes M_n$$

$$\vdots$$

$$U^{12 \dots k} = n^{-1} C_{n_1} \otimes C_{n_2} \otimes \dots \otimes C_{n_k} \otimes M_n$$

and

$$U^E = I_{n_1} \otimes I_{n_2} \otimes \dots \otimes I_{n_k} \otimes C_n, \text{ where } Y \text{ is the}$$

$n_0 \times 1$ vector of observations. The notation $A \otimes B$ denotes the Kronecker product of matrices A and B .

2.2 Lemmas.

Lemma 1. Let the $n \times 1$ random vector Y be distributed $N(0, \Sigma)$ where Σ has full rank. Then the quadratic form $U = Y^T A Y$ is distributed as a $\chi^2(p)$ iff $A\Sigma$ is an idempotent matrix of rank p .

Lemma 2. Let the $n \times 1$ random vector Y be distributed $N(0, \Sigma)$, where Σ has full rank. The quadratic forms $Y^T A Y$ and $Y^T B Y$ are independently distributed iff $A\Sigma B = 0$.

Lemma 3. If A is an idempotent matrix the $\text{RANK}(A) = \text{tr}(A)$.

Lemma 4.
$$\sum_{i=1}^k U^i + \sum_{j>i} U^{ij} + \sum_{k>j>i} U^{ijk} + \dots + U^{12\dots k} + U^E = I - \frac{1}{n_o} M_{n_o}$$
 where the matrices are defined in Theorem 1.

Proof. From the standard ANOVA for a factorial experiment in which the observations are independent, we have $SSA^{(1)} + SSA^{(2)} + \dots + SSA^{(k)} + SSA^{(12)} + \dots + SSA^{(12\dots k)} + SSE = SST$, where SST is the total corrected sum of squares and is equal to $Y^T (I - \frac{1}{n_o} M_{n_o}) Y$. Since $Y^T U^{i_1 i_2 \dots i_q} Y = SSA^{(i_1 i_2 \dots i_q)}$, the lemma is established.

Lemma 5. If A is an idempotent matrix and B is an orthogonal matrix, then $B^T A B$ is an idempotent matrix.

Lemma 6. Let A be an $m \times n$ matrix. Let A^C be any c -inverse of A , and suppose a solution exists to the system $Ax = g$. For each $n \times 1$ vector h , the vector x_o is a solution where $x_o = A^C g + (I - A^C A)h$. Also every solution to the

to the system can be written in this form for some $n \times 1$ vector h .

Proofs for lemmas can be found in Graybill (1976).

Lemma 7. Let $H = \begin{pmatrix} h_1 & h_2 & \dots & h_{n_0} \\ h_2 & 0 & & 0 \\ h_3 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ h_{n_0} & 0 & & 0 \end{pmatrix}.$

Let $B_{n_i} = \begin{pmatrix} \frac{1}{\sqrt{n_i}} & \frac{1}{\sqrt{n_i}} & \frac{1}{\sqrt{n_i}} & \dots & \frac{1}{\sqrt{n_i}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & \dots & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{n_i(n_i-1)}} & \frac{1}{\sqrt{n_i(n_i-1)}} & \frac{1}{\sqrt{n_i(n_i-1)}} & \dots & \frac{-(n_i-1)}{\sqrt{n_i(n_i-1)}} \end{pmatrix}$

Let $q = \frac{1}{\sqrt{n_1 n_2 n_3 \dots n_k n}} = \frac{1}{\sqrt{n_0}}$ where $n_0 = n_1 n_2 \dots n_k n$. Let

$B = B_{n_1} \otimes B_{n_2} \otimes \dots \otimes B_{n_k} \otimes B_n$. Let $B = [E_1, E_2, \dots, E_{n_0}]$ where

the E_i 's are the columns of B . Let $h = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_{n_0} \end{pmatrix}$. Then

$$B^T H B = q \begin{pmatrix} E_1^T h \\ \vdots \\ E_{n_0}^T h \end{pmatrix} \otimes [1, 1, \dots, 1]_{1 \times n_0} + q [E_1^T h, \dots, E_{n_0}^T h] \otimes \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

- $q^2 h_1 M_{n_0}$, where M_{n_0} is the $n_0 \times n_0$ matrix whose entries are equal to one.

Proof: The elements in the first row of B are $[q \ q \ \dots \ q]$. Now,

$$\begin{aligned}
 B^T H B &= B^T \begin{pmatrix} 0 & \dots & 0 \\ h_2 & & 0 \\ \vdots & & \\ h_{n_0} & \dots & 0 \end{pmatrix} B + B^T \begin{pmatrix} h_1 & h_2 & \dots & h_{n_0} \\ 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} B \\
 &= q B^T \begin{pmatrix} 0 & 0 & \dots & 0 \\ h_2 & h_2 & \dots & h_2 \\ \vdots & \vdots & & \vdots \\ h_{n_0} & h_{n_0} & \dots & h_{n_0} \end{pmatrix} + B^T \begin{pmatrix} h_1 & h_2 & \dots & h_{n_0} \\ 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} B \\
 &= q B^T [h \ \dots \ h] - q B^T \begin{pmatrix} h_1 & \dots & h_1 \\ 0 & & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix} + q \begin{pmatrix} h_1 & h_2 & \dots & h_{n_0} \\ \vdots & \vdots & & \vdots \\ h_1 & h_2 & \dots & h_{n_0} \end{pmatrix} B \\
 &= q \begin{pmatrix} E_1^T \\ E_2^T \\ \vdots \\ E_{n_0}^T \end{pmatrix} [h \ \dots \ h] - q^2 h_1 \begin{pmatrix} 1 & \dots & 1 \\ 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{pmatrix} + q \begin{pmatrix} h^T \\ \vdots \\ h^T \end{pmatrix} [E_1 E_2 \ \dots \ E_{n_0}]
 \end{aligned}$$

$$\begin{aligned}
&= q \begin{pmatrix} E_1^T h \\ E_2^T h \\ \vdots \\ E_{n_o}^T h \end{pmatrix} \otimes [1 \ 1 \ \dots \ 1]_{1 \times n_o} + q [E_1^T h \ \dots \ E_{n_o}^T h] \otimes \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{n_o \times 1} \\
&\quad - q^2 h_1 \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & & 1 \\ \vdots & & & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix}_{n_o \times n_o}.
\end{aligned}$$

Thus the lemma is proved.

2.3 Development of the Correlation Matrix V_Y .

From theorem 1, we get that the F ratios used to test the effects of the factors and interaction terms would be

$$\begin{aligned}
&\frac{Y^T U^1 Y / \sigma^2 \text{RANK}(U^1)}{Y^T U^E Y / \sigma^2 \text{RANK}(U^E)}, \frac{Y^T U^2 Y / \sigma^2 \text{RANK}(U^2)}{Y^T U^E Y / \sigma^2 \text{RANK}(U^E)}, \dots, \frac{Y^T U^k Y / \sigma^2 \text{RANK}(U^k)}{Y^T U^E Y / \sigma^2 \text{RANK}(U^E)} \\
&\frac{Y^T U^{12} Y / \sigma^2 \text{RANK}(U^{12})}{Y^T U^E Y / \sigma^2 \text{RANK}(U^E)}, \dots, \frac{Y^T U^{12 \dots k} Y / \sigma^2 \text{RANK}(U^{12 \dots k})}{Y^T U^E Y / \sigma^2 \text{RANK}(U^E)}. \quad \text{Now}
\end{aligned}$$

the question to be answered is "what does the covariance matrix $\sigma^2 V_Y$ for the model (2.1) need to be in order to have each of the statistics $Y^T U^1 Y / \sigma^2 \text{RANK}(U^1)$, $Y^T U^2 Y / \sigma^2 \text{RANK}(U^2)$, $\dots$, $Y^T U^k Y / \sigma^2 \text{RANK}(U^k)$, $Y^T U^{12} Y / \sigma^2 \text{RANK}(U^{12})$, $\dots$, $Y^T U^{12 \dots k} Y / \sigma^2 \text{RANK}(U^{12 \dots k})$ and $Y^T U^E Y / \sigma^2 \text{RANK}(U^E)$ be distributed as a constant times a chi-square and also be distributed independently." V_Y is assumed to be positive definite. Assuming these conditions we will find the general form that the correlation matrix V_Y must be. By Lemma 1, there exist

constants $c_1, c_2, \dots, c_{12}, \dots, c_{12\dots k}$, and c_E such that

$$\frac{U^1 V_Y}{c_1}, \frac{U^2 V_Y}{c_2}, \dots, \frac{U^{12} V_Y}{c_{12}}, \dots, \frac{U^{12\dots k} V_Y}{c_{12\dots k}} \text{ and } \frac{U^E V_Y}{c_E} \quad (2.2)$$

are idempotent.

By Lemma 2, we must have that

$$U^{\Psi_1} V_Y U^{\Psi_2} = 0 \text{ for } \Psi_1, \Psi_2 \in \{1, 2, \dots, k, 12, 13, \dots, 123\dots k, E\} \text{ and } \Psi_1 \neq \Psi_2. \quad (2.3)$$

In the standard case of independence, $V_Y = I$, and we must have all constants $c_1, c_2, \dots, c_{12}, \dots, c_{12\dots k}$, and c_E equal to one. $U^1, U^2, \dots, U^{12}, \dots, U^{12\dots k}$, and U^E are idempotent and $U^{\Psi_1} U^{\Psi_2} = 0$ for $\Psi_1, \Psi_2 \in \{1, 2, \dots, k, 12, 13, \dots, 123, \dots, k, E\}$ and $\Psi_1 \neq \Psi_2$.

By statements (2.2) and (2.3)

$$\left(\frac{U^1}{c_1} + \frac{U^2}{c_2} + \dots + \frac{U^k}{c_k} + \frac{U^{12}}{c_{12}} + \dots + \frac{U^{12\dots k}}{c_{12\dots k}} + \frac{U^E}{c_E} \right) V_Y$$

is an idempotent matrix and by Lemma 3,

$$\begin{aligned} \text{RANK} \left[\left(\frac{U^1}{c_1} + \frac{U^2}{c_2} + \dots + \frac{U^k}{c_k} + \frac{U^{12}}{c_{12}} + \dots + \frac{U^{12\dots k}}{c_{12\dots k}} + \frac{U^E}{c_E} \right) V_Y \right] \\ = \text{RANK} \left[\frac{U^1 V_Y}{c_1} + \frac{U^2 V_Y}{c_2} + \dots + \frac{U^k V_Y}{c_k} + \frac{U^{12} V_Y}{c_{12}} + \dots + \frac{U^{12\dots k} V_Y}{c_{12\dots k}} \right. \\ \left. + \frac{U^E V_Y}{c_E} \right] \end{aligned}$$

$$\begin{aligned}
&= \text{tr} \left[\frac{U^1_V Y}{c_1} + \frac{U^2_V Y}{c_2} + \dots + \frac{U^k_V Y}{c_k} + \frac{U^{12}_V Y}{c_{12}} + \dots + \frac{U^{12\dots k}_V Y}{c_{12\dots k}} + \frac{U^E}{c_E} V_Y \right] \\
&= \text{tr} \left[\frac{U^1_V Y}{c_1} \right] + \text{tr} \left[\frac{U^2_V Y}{c_2} \right] + \dots + \text{tr} \left[\frac{U^k_V Y}{c_k} \right] + \text{tr} \left[\frac{U^{12}_V Y}{c_{12}} \right] + \dots \\
&+ \text{tr} \left[\frac{U^{12\dots k}_V Y}{c_{12\dots k}} \right] + \text{tr} \left[\frac{U^E_V Y}{c_E} \right] = \text{RANK} \left[\frac{U^1_V Y}{c_1} \right] + \text{RANK} \left[\frac{U^2_V Y}{c_2} \right] + \dots \\
&+ \text{RANK} \left[\frac{U^k_V Y}{c_k} \right] + \text{RANK} \left[\frac{U^{12}_V Y}{c_{12}} \right] + \dots + \text{RANK} \left[\frac{U^{12\dots k}_V Y}{c_{12\dots k}} \right] + \text{RANK} \left[\frac{U^E_V Y}{c_E} \right] \\
&= \text{RANK}[U^1] + \text{RANK}[U^2] + \dots + \text{RANK}[U^k] + \text{RANK}[U^{12}] + \dots \\
&+ \text{RANK}[U^{12\dots k}] + \text{RANK}[U^E] = \text{tr}[U^1] + \text{tr}[U^2] + \dots + \text{tr}[U^k] \\
&+ \text{tr}[U^{12}] + \dots + \text{tr}[U^{12\dots k}] + \text{tr}[U^E] = \text{tr}[U^1 + U^2 + \dots + U^k \\
&+ U^{12} + \dots + U^{12\dots k} + U^E] = \text{tr} \left[I - \frac{1}{n_O} M_{n_O} \right] = n_O - 1 \text{ (by Lemma}
\end{aligned}$$

4 where $n_O = n_1 n_2 \dots n_k n$).

Let

$$B_{n_i} = \begin{pmatrix} \frac{1}{\sqrt{n_i}} & \frac{1}{\sqrt{n_i}} & \frac{1}{\sqrt{n_i}} & \dots & \frac{1}{\sqrt{n_i}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & \dots & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{n_i(n_i-1)}} & \frac{1}{\sqrt{n_i(n_i-1)}} & \frac{1}{\sqrt{n_i(n_i-1)}} & \dots & \frac{-(n_i-1)}{\sqrt{n_i(n_i-1)}} \end{pmatrix}$$

$$\text{If } M_{n_i} = \begin{pmatrix} 1 & 1 \dots 1 \\ 1 & 1 \dots 1 \\ \vdots & \vdots \vdots \vdots \\ 1 & 1 \dots 1 \end{pmatrix}_{n_i \times n_i}, \text{ then}$$

$$B_{n_i} M_{n_i} B_{n_i}^T = \begin{pmatrix} \sqrt{n_i} & \sqrt{n_i} & \dots & \sqrt{n_i} \\ 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} B_{n_i}^T$$

$$= \text{diag}_{n_i \times n_i} [n_i \ 0 \dots 0].$$

$$\text{If } C_p = I_p - \frac{1}{p} M_p, \text{ then}$$

$$B_p C_p B_p^T = B_p (I_p - \frac{1}{p} M_p) B_p^T$$

$$= I_p - \frac{1}{p} B_p M_p B_p^T$$

$$= I_p - \text{diag}_{p \times p} [1 \ 0 \dots 0].$$

$$= \text{diag}_{p \times p} [0 \ 1 \ 1 \dots 1].$$

$$\text{Let } B = B_{n_1} \otimes B_{n_2} \otimes B_{n_3} \otimes \dots \otimes B_{n_k} \otimes B_n.$$

$$\text{Since } U^i = \frac{n_i}{n_1 n_2 \dots n_k n} M_{n_1} \otimes M_{n_2} \otimes \dots \otimes M_{n_{i-1}} \otimes C_{n_i} \otimes M_{n_{i+1}}$$

$$\otimes \dots \otimes M_{n_k} \otimes M_n,$$

$$BU^i B^T = \frac{n_i}{n_1 n_2 \dots n_k n} B_{n_1} M_{n_1} B_{n_1}^T \otimes B_{n_2} M_{n_2} B_{n_2}^T \otimes \dots$$

$$\otimes B_{n_{i-1}} M_{n_{i-1}} B_{n_{i-1}}^T \otimes B_{n_i} C_{n_i} B_{n_i}^T \otimes B_{n_{i+1}} M_{n_{i+1}} B_{n_{i+1}}^T$$

$$\otimes \dots \otimes B_{n_k} M_{n_k} B_{n_k}^T \otimes B_n M_n B_n^T$$

$$\begin{aligned}
&= \text{diag}_{n_1 \times n_1} [1 \ 0 \ \dots \ 0] \otimes \text{diag}_{n_2 \times n_2} [1 \ 0 \ \dots \ 0] \otimes \dots \\
&\otimes \text{diag}_{n_{i-1} \times n_{i-1}} [1 \ 0 \ \dots \ 0] \otimes \text{diag}_{n_i \times n_i} [0 \ 1 \ 1 \ \dots \ 1] \\
&\otimes \text{diag}_{n_{i+1} \times n_{i+1}} [1 \ 0 \ \dots \ 0] \otimes \dots \otimes \text{diag}_{n_k \times n_k} [1 \ 0 \ \dots \ 0] \\
&\otimes \text{diag}_{n \times n} [1 \ 0 \ \dots \ 0].
\end{aligned}$$

For the matrix $U^{123} = \frac{n_1 n_2 n_3}{n_1 n_2 \dots n_k n} C_{n_1} \otimes C_{n_2} \otimes C_{n_3} \otimes M_{n_4} \otimes \dots$

$$\otimes M_{n_k} \otimes M_n,$$

$$\begin{aligned}
BU^{123} B^T &= \frac{n_1 n_2 n_3}{n_1 n_2 \dots n_k n} B_{n_1} C_{n_1} B_{n_1}^T \otimes B_{n_2} C_{n_2} B_{n_2}^T \otimes B_{n_3} C_{n_3} B_{n_3}^T \\
&\otimes B_{n_4} M_{n_4} B_{n_4}^T \otimes \dots \otimes B_{n_k} M_{n_k} B_{n_k}^T \otimes B_n M_n B_n^T. \\
&= \text{diag}_{n_1 \times n_1} [0 \ 1 \ 1 \ \dots \ 1] \otimes \text{diag}_{n_2 \times n_2} [0 \ 1 \ \dots \ 1] \\
&\otimes \text{diag}_{n_3 \times n_3} [0 \ 1 \ \dots \ 1] \otimes \text{diag}_{n_4 \times n_4} [1 \ 0 \ \dots \ 0] \\
&\otimes \text{diag}_{n_5 \times n_5} [1 \ 0 \ \dots \ 0] \otimes \dots \otimes \text{diag}_{n_k \times n_k} [1 \ 0 \ \dots \ 0] \\
&\otimes \text{diag}_{n \times n} [1 \ 0 \ \dots \ 0].
\end{aligned}$$

For $U^E = I_{n_1} \otimes I_{n_2} \otimes \dots \otimes I_{n_k} \otimes C_n$, we have $BU^E B^T = I_{n_1} \otimes I_{n_2} \otimes \dots$

$$\otimes I_{n_k} \otimes B_n C_n B_n^T = I_{n_1} \otimes I_{n_2} \otimes \dots \otimes I_{n_k} \otimes \text{diag}_{n \times n} [0 \ 1 \ 1 \ \dots \ 1].$$

Now,

$$B \left(\frac{U^1}{c_1} + \frac{U^2}{c_2} + \dots + \frac{U^k}{c_k} + \frac{U^{12}}{c_{12}} + \dots + \frac{U^{12 \dots k}}{c_{12 \dots k}} + \frac{U^E}{c_E} \right) V_Y B^T$$

$$= \left(\frac{BU^1_B{}^T}{c_1} + \frac{BU^2_B{}^T}{c_2} + \dots + \frac{BU^k_B{}^T}{c_k} + \frac{BU^{12}_B{}^T}{c_{12}} + \dots + \frac{BU^{12\dots k}_B{}^T}{c_{12\dots k}} \right. \\ \left. + \frac{BU^E_B{}^T}{c_E} \right) BV_Y{}^T$$

$$= D_1 BV_Y{}^T \text{ where } D_1 \text{ is an } n_o \times n_o \text{ diagonal matrix of rank } n_o - 1.$$

Note that each of the diagonal matrices $BU^1_B{}^T, BU^2_B{}^T, \dots, BU^k_B{}^T, BU^{12}_B{}^T, \dots, BU^{12\dots k}_B{}^T$, and $BU^E_B{}^T$ each have a zero

term for the first element on their diagonals. Hence D_1

has a zero term for the first element of its diagonal. By

Lemma 5, $D_1 BV_Y{}^T$ is an idempotent matrix. Now

$I_{n_o \times n_o} - D_1 BV_Y{}^T$ is an idempotent matrix of rank one. Since

$D_1 BV_Y{}^T$ must have zero for the elements in its first row,

$$I_{n_o} - D_1 BV_Y{}^T = \begin{pmatrix} 1 & 0 & \dots & 0 \\ e_2 & 0 & \dots & 0 \\ e_3 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ e_{n_o} & 0 & \dots & 0 \end{pmatrix}_{n_o \times n_o}, \text{ where } e_2, e_3, \dots, e_{n_o} \text{ are}$$

arbitrary numbers. This can be written as

$$D_1 BV_Y{}^T = I - \begin{pmatrix} 1 & 0 & \dots & 0 \\ e_2 & 0 & \dots & 0 \\ e_3 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ e_{n_o} & 0 & \dots & 0 \end{pmatrix}.$$

By Lemma 6, we have

$$BV_Y B^T = D_1^C \left(I - \begin{pmatrix} 1 & 0 & \dots & 0 \\ e_2 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ e_{n_0} & 0 & \dots & 0 \end{pmatrix} \right) + [I - D_1^C D_1] H, \text{ where}$$

H is an $n_0 \times n_0$ arbitrary matrix. Since $D_1^C D_1 = \text{diag}_{n_0 \times n_0} [0 \ 1 \ 1 \dots 1]$,

$$BV_Y B^T = D_1^C \left(I - \begin{pmatrix} 1 & 0 & \dots & 0 \\ e_2 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ e_{n_0} & 0 & \dots & 0 \end{pmatrix} \right) + \begin{pmatrix} h_1 & h_2 & \dots & h_{n_0} \\ 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

where $h_1, h_2, \dots, h_{n_0}$ are arbitrary constants.

Now,

$$V_Y = B^T D_1^C B - B^T D_1^C \begin{pmatrix} 1 & 0 & \dots & 0 \\ e_2 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ e_{n_0} & 0 & \dots & 0 \end{pmatrix} B + B^T \begin{pmatrix} h_1 & h_2 & \dots & h_{n_0} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} B.$$

$$\text{Since } V_Y \text{ is symmetric, } -D_1^C \begin{pmatrix} 1 & 0 & \dots & 0 \\ e_2 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ e_{n_0} & 0 & \dots & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ h_2 & 0 & \dots & 0 \\ \vdots & & & \vdots \\ h_{n_0} & 0 & \dots & 0 \end{pmatrix}.$$

$$\text{So, } V_Y = B^T D_1^C B + B^T \begin{pmatrix} h_1 & h_2 & \dots & h_{n_0} \\ h_2 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ h_{n_0} & 0 & \dots & 0 \end{pmatrix} B.$$

Since each term in $\left(\frac{BU^1 B^T}{c_1} + \frac{BU^2 B^T}{c_2} + \dots + \frac{BU^k B^T}{c_k} + \frac{BU^{12} B^T}{c_{12}} \right)$

$+ \dots + \frac{BU^{12\dots k}B^T}{c_{12\dots k}} + \frac{BU^E B^T}{c_E}) = D_1$ is a diagonal matrix and

the product of any two terms in the sum is zero, we can write

$$D_1^C = c_1 BU^1 B^T + c_2 BU^2 B^T + \dots + c_k BU^k B^T + c_{12} BU^{12} B^T + \dots \\ + c_{12\dots k} BU^{12\dots k} B^T + c_E BU^E B^T. \text{ This implies}$$

$$V_Y = c_1 U^1 + c_2 U^2 + \dots + c_k U^k + c_{12} U^{12} + \dots + c_{12\dots k} U^{12\dots k} \\ + c_E U^E + B^T \begin{pmatrix} h_1 & h_2 & \dots & h_{n_o} \\ h_2 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ h_{n_o} & 0 & \dots & 0 \end{pmatrix} B.$$

Since V_Y is the correlation matrix, the diagonal elements of V_Y must each be equal to one. Now the diagonal elements of $U^1, U^2, \dots, U^k, \dots, U^{12}, \dots, U^{12\dots k}$, and U^E each have the same value. So the diagonal elements of

$$B^T \begin{pmatrix} h_1 & h_2 & \dots & h_{n_o} \\ h_2 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ h_{n_o} & 0 & \dots & 0 \end{pmatrix} B \text{ must all be the same value. By Lemma 7,}$$

we have

$$B^T H B = q \begin{pmatrix} E_1^T h \\ \vdots \\ E_{n_o}^T h \end{pmatrix} \otimes [1 \ 1 \dots 1]_{1 \times n_o} + q [E_1^T h \dots E_{n_o}^T h] \otimes \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{n_o \times 1} \\ - q^2 h_1 \begin{pmatrix} 1 & 1 \dots 1 \\ 1 & 1 \dots 1 \\ \vdots & \vdots \\ 1 & 1 \dots 1 \end{pmatrix}_{n_o \times n_o},$$

where the E_i 's are the columns of B , $h = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_{n_0} \end{pmatrix}$ and $q = \frac{1}{\sqrt{n_0}}$.

The diagonal elements of $B^T H B$ are $2qE_1^T h - q^2 h_1$, $2qE_2^T h - q^2 h_1$, ..., $2qE_{n_0}^T h - q^2 h_1$. Since the diagonal elements must be equal,

we have for some α , $\begin{pmatrix} E_1^T \\ E_2^T \\ \vdots \\ E_{n_0}^T \end{pmatrix} h = B^T h = \alpha \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$. This implies

$$h = \alpha B \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \alpha \begin{pmatrix} \sqrt{n_0} \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad \text{Hence } h_2 = h_3 = \dots = h_{n_0} = 0.$$

We get

$$V_Y = c_1 U^1 + c_2 U^2 + \dots + c_k U^k + c_{12} U^{12} + \dots + c_{12\dots k} U^{12\dots k} \\ + c_E U^E + q^2 h_1 \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & & 1 \end{pmatrix}.$$

Letting $s = q^2 h_1$,

$$V_Y = c_1 U^1 + c_2 U^2 + \dots + c_k U^k + c_{12} U^{12} + \dots + c_{12\dots k} U^{12\dots k} \\ + c_E U^E + s M_{n_0}. \quad (2.4)$$

Since the product of any two distinct matrices from the set

$\{U^1, U^2, \dots, U^k, U^{12}, U^{13}, \dots, U^{12\dots k}, U^E, \text{ and } M_{n_0}\}$ is equal to

zero and since each of the matrices from the set

$\{B^T U^1 B, B^T U^2 B, \dots, B^T U^k B, B^T U^{12} B, B^T U^{13} B, \dots, B^T U^{12\dots k} B, B^T U^E B,$

and $B^T M_{n_0} B\}$ is a diagonal matrix, we have that V_Y is a positive definite matrix iff each of the constants $c_1, c_2, \dots,$

$c_k, c_{12}, \dots, c_{12\dots k}, c_E,$ and s are positive.

Theorem 2. Let $Z = Y_{(i_1, i_2, \dots, i_k)} i_0$ and

$W = Y_{(j_1, j_2, \dots, j_k)} j_0$. Let $\sigma^2 V_Y$ defined in (2.4) be the

covariance matrix of the factorial experiment defined in (2.1). Then

$\text{cov}(Z, W) = f(\delta(i_1, j_1), \delta(i_2, j_2), \dots, \delta(i_k, j_k), \delta(i_0, j_0))$ where

f is a function of $k+1$ values and $\delta(i_\ell, j_\ell)$

$$= \begin{cases} 1 & \text{if } i_\ell = j_\ell \\ 0 & \text{if } i_\ell \neq j_\ell \end{cases}.$$

Proof. As defined in (2.1), $V_Y = c_1 U^1 + c_2 U^2 + \dots$

$+ c_k U^k + c_{12} U^{12} + \dots + c_{12\dots k} U^{12\dots k} + c_E U^E + s M_{n_0}$. Now,

since $U^1 = \frac{n_1}{n_0} C_{n_1} \otimes M_{n_2} \otimes \dots \otimes M_{n_k} \otimes M_n$ where $n_0 = n_1 n_2 \dots n_k n$,

note that i_1 must be equal to j_1 for the $\text{cov}(Z, W)$ to involve

a diagonal element of U^1 , which is $\sigma^2 \frac{n_1}{n_0} \left(\frac{n_1 - 1}{n_1} \right)$ and that if

$i_1 \neq j_1$ then the $\text{cov}(Z, W)$ involves a nondiagonal element of

U^1 which is $\sigma^2 \frac{n_1}{n_0} (-\frac{1}{n_0})$. Since $U^{12} = \frac{n_1 n_2}{n_0} C_{n_1} \otimes C_{n_2} \otimes M_{n_3} \otimes \dots \otimes M_{n_k} \otimes M_n$, for $i_1 = j_1$, $\frac{n_1-1}{n_1}$ must be a factor of the element from U^{12} that is in $\text{cov}(Z, W)$. If $i_1 \neq j_1$, then $-\frac{1}{n_1}$ must be a factor of the element from U^{12} that is in $\text{cov}(Z, W)$. If $i_2 = j_2$, then likewise $\frac{n_2-1}{n_2}$ must be a factor of the element from U^{12} that is in $\text{cov}(Z, W)$. Otherwise if $i_2 \neq j_2$ then $-\frac{1}{n_2}$ must be a factor of the element from U^{12} .

The term from U^{12} that is involved in $\text{cov}(Z, W)$ can be

$$\text{written } \sigma^2 \frac{n_1 n_2}{n_0} \left[\left(\frac{n_1-1}{n_1} \right)^{\delta(i_1, j_1)} \left(-\frac{1}{n_1} \right)^{1-\delta(i_1, j_1)} \left(\frac{n_2-1}{n_2} \right)^{\delta(i_2, j_2)} \left(-\frac{1}{n_2} \right)^{1-\delta(i_2, j_2)} \right].$$

Since $U^E = I_{n_1} \otimes I_{n_2} \otimes \dots \otimes I_{n_k} \otimes C_n$,

then the term from U^E that is involved in $\text{cov}(Z, W)$ can be

$$\text{written } \sigma^2 \left(\frac{n-1}{n} \right)^{\delta(i_0, j_0)} \left(-\frac{1}{n} \right)^{1-\delta(i_0, j_0)}.$$

Using the above

$$\text{reasoning we can write, } \text{cov}(Z, W) = \sigma^2 \left[\frac{c_1 n_1}{n_0} \left(\frac{n_1-1}{n_1} \right)^{\delta(i_1, j_1)} \left(-\frac{1}{n_1} \right)^{1-\delta(i_1, j_1)} \right.$$

$$\left. + \frac{c_2 n_2}{n_0} \left(\frac{n_2-1}{n_2} \right)^{\delta(i_2, j_2)} \left(-\frac{1}{n_2} \right)^{1-\delta(i_2, j_2)} \right.$$

$$\left. + \dots + c_k \frac{n_k}{n_0} \left(\frac{n_k-1}{n_k} \right)^{\delta(i_k, j_k)} \left(-\frac{1}{n_k} \right)^{1-\delta(i_k, j_k)} + c_{12} \frac{n_1 n_2}{n_0} \right]$$

$$\begin{aligned}
& \left[\left(\frac{n_1-1}{n_1} \right)^{\delta(i_1, j_1)} \left(-\frac{1}{n_1} \right)^{1-\delta(i_1, j_1)} \right] \cdot \left[\left(\frac{n_2-1}{n_2} \right)^{\delta(i_2, j_2)} \left(-\frac{1}{n_2} \right)^{1-\delta(i_2, j_2)} \right] \\
& + \dots + (c_{12\dots k})^n \prod_{\ell=1}^k \left(\frac{n_\ell-1}{n_\ell} \right)^{\delta(i_\ell, j_\ell)} \left(-\frac{1}{n_\ell} \right)^{1-\delta(i_\ell, j_\ell)} \\
& + c_E \left(\frac{n-1}{n} \right)^{\delta(i_0, j_0)} \left(-\frac{1}{n} \right)^{1-\delta(i_0, j_0)} + s]. \text{ Thus the theorem is} \\
& \text{established.}
\end{aligned}$$

Using the above theorem we can write

$$\begin{aligned}
& \text{cov}[Y_{(i_1, i_2, \dots, i_k)t}, Y_{(j_1, j_2, \dots, j_k)t_1}] \quad (2.5) \\
& = \begin{cases} \sigma^2 & \text{if } i_1 = j_1, i_2 = j_2, \dots, i_k = j_k, t = t_1 \\ \rho\sigma^2 & \text{if } i_1 = j_1, i_2 = j_2, \dots, i_k = j_k, t \neq t_1 \\ \alpha_{\{k_1, k_2, \dots, k_q\}} \rho\sigma^2 & \text{if } i_\ell \neq j_\ell \text{ for } \ell \in \{k_1, k_2, \dots, k_q\} \\ & \text{and } i_\ell = j_\ell \text{ for } \ell \notin \{k_1, k_2, \dots, k_q\}. \end{cases}
\end{aligned}$$

The term $\alpha_{\{k_1, k_2, \dots, k_q\}} \rho\sigma^2$ could be just written

$\alpha_{\{k_1, k_2, \dots, k_q\}} \sigma^2$. If ρ is equal to zero then

$\text{cov}[Y_{(i_1, i_2, \dots, i_k)t}, Y_{(j_1, j_2, \dots, j_k)t_1}]$ does not have to be

zero when $i_\ell \neq j_\ell$ for $\ell \in \{k_1, k_2, \dots, k_q\}$ and $i_\ell = j_\ell$ for

$\ell \notin \{k_1, k_2, \dots, k_q\}$. However, the covariance matrix in (2.5)

is written with the factor ρ since this covariance pattern

resembles the one developed by Smith and Lewis (1980).

Now we define the operation $\hat{0}$ as follows.

If $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ and $\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$ are two column vectors,

$$\text{then } \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \hat{0} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} = \begin{pmatrix} (a_1, b_1) \\ (a_1, b_2) \\ \vdots \\ (a_1, b_m) \\ (a_2, b_1) \\ \vdots \\ (a_n, b_m) \end{pmatrix}.$$

This definition can be extended to more than two vectors.

For example, we can write

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \hat{0} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \hat{0} \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \hat{0} \begin{pmatrix} (3, 5) \\ (3, 6) \\ (4, 5) \\ (4, 6) \end{pmatrix}$$

$$= \begin{pmatrix} (1, 3, 5) \\ (1, 3, 6) \\ (1, 4, 5) \\ (1, 4, 6) \\ (2, 3, 5) \\ (2, 3, 6) \\ (2, 4, 5) \\ (2, 4, 6) \end{pmatrix}. \quad \text{Let}$$

$$F = \begin{pmatrix} f_1 \\ \vdots \\ f_{n_1 n_2 \dots n_k} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n_1 \end{pmatrix} \hat{0} \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n_2 \end{pmatrix} \hat{0} \dots \hat{0} \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n_k \end{pmatrix}.$$

Here, the f_i are ordered k -triples. Let $Y_{n \times 1}$ be the vector of observations from the model in (2.1). Then

$$\begin{aligned}
Y_{n \times 1}^T &= [Y_{(11\dots 1)1}, \dots, Y_{(11\dots 1)n}, Y_{(11\dots 2)1}, \dots, Y_{(11\dots 2)n}, \\
&\quad \dots, Y_{(11\dots 1n_k)1}, \dots, Y_{(n_1 n_2 \dots n_k)n}] \\
&= [Y_{f_1 1}, \dots, Y_{f_1 n}, Y_{f_2 1}, \dots, Y_{f_2 n}, \dots, Y_{f_{n_k} 1}, \dots, \\
&\quad Y_{f_{n_1 n_2 \dots n_k} n}].
\end{aligned}$$

Before stating the next theorem, we define some notation.

For all q , $1 \leq q \leq n_1 n_2 \dots n_k$, let $G(f_q) = \{x | x = j, j \text{ is the } j^{\text{th}} \text{ component of } f_q = (i_1, i_2, \dots, i_j, \dots, i_k) \text{ and } i_j \neq 1\}$.

Using the function $G(f_q)$ we can state the first column of $V_Y \sigma^2$ as

$$\text{cov}[Y_{f_1 1}, Y_{f_q t}] = \begin{cases} \sigma^2 & \text{if } q = 1 \text{ and } t = 1 \\ \rho \sigma^2 & \text{if } q = 1 \text{ and } t \neq 1 \\ \rho \sigma^2 \alpha_{G(f_q)} & \text{if } q \neq 1 \end{cases} \quad (2.6)$$

Let $\hat{U}^i$ be the first column of $\frac{n_o}{n_i} U^i$. Let $\hat{U}^{i_1 i_2 \dots i_q}$ be the first column of $\frac{n_o}{n_{i_1} n_{i_2} \dots n_{i_q}} U^{i_1 i_2 \dots i_q}$. Let $\hat{U}^E$ be the first column of U^E . And let $1_{n_o \times 1}$ be an $n_o \times 1$ vector of ones.

Define $\gamma_i = \frac{n_i}{n_i - 1}$, $\gamma_{i_1 i_2 \dots i_p} = \prod_{j=1}^p \frac{n_{i_j}}{n_{i_j} - 1}$, and $\gamma_E = \frac{n}{n-1}$.

This notation is used in the following theorem.

Theorem 3. Given the values of the covariance matrix

$\sigma^2_{V_y}$ in (2.5), the values of $c_1, c_2, \dots, c_k, c_{12}, \dots, c_{12\dots k}, c_E$ and s in (2.4) can be found as follows:

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \\ c_{12} \\ \vdots \\ c_{12\dots k} \\ c_E \\ s \end{pmatrix} = \rho \begin{pmatrix} \gamma_1 (\hat{U}^1)^T \\ \gamma_\alpha (\hat{U}^\alpha)^T \\ \vdots \\ \gamma_{12} (\hat{U}^{12})^T \\ \vdots \\ \gamma_{12\dots k} (\hat{U}^{12\dots k})^T \\ \gamma_E (\hat{U}^E)^T \\ \frac{1}{n_o} \mathbf{1}_{n_o \times 1}^T \end{pmatrix} \left\{ \begin{pmatrix} 1 \\ \alpha_{G(f_2)} \\ \alpha_{G(f_3)} \\ \vdots \\ \alpha_{G(f_q)} \\ \vdots \\ \alpha_{G(f_{n_1 n_2 \dots n_k})} \end{pmatrix} \otimes \mathbf{1}_{n \times 1} \right\} \\ + (1 - \rho) \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ \frac{1}{n_o} \end{pmatrix}_{n_o \times 1}$$

Proof: By statement (2.4) and by the way $\hat{U}^{12\dots q}$ and $\hat{U}^E$ are defined, we can write the first column of V_y as

$$\hat{V}_y = \left[\frac{n_1}{n_o} \hat{U}^1 \frac{n_2}{n_o} \hat{U}^2 \dots \frac{n_k}{n_o} \hat{U}^k \frac{n_1 n_2}{n_o} \hat{U}^{12} \dots \frac{n_1 n_2 \dots n_k}{n_o} \hat{U}^{12\dots k} \right]$$

$$\hat{U}^E \begin{pmatrix} 1 \\ 1 \\ \vdots \\ \vdots \\ 1 \end{pmatrix}_{n_0 \times 1} \cdot \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ \vdots \\ c_k \\ c_{12} \\ \vdots \\ \vdots \\ c_{12 \dots k} \\ c_E \\ s \end{pmatrix}, \quad (2.7)$$

$$= U \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ \vdots \\ c_k \\ c_{12} \\ \vdots \\ \vdots \\ c_{12 \dots k} \\ c_E \\ s \end{pmatrix}, \text{ where } U \text{ is the ap-}$$

propriately defined matrix. Since the multiplication of any two of the matrices $U^1, U^2, \dots, U^k, U^{12}, \dots, U^{12 \dots k}, U^E, M_{n_0}$ is

equal to zero, we have that $\hat{U}^1, \hat{U}^2, \dots, \hat{U}^k, \hat{U}^{12}, \dots, \hat{U}^{12 \dots k},$

$\hat{U}^E$, and $\begin{pmatrix} 1 \\ 1 \\ \vdots \\ \vdots \\ 1 \end{pmatrix}_{n_0 \times 1}$ are all orthogonal. As defined previously, $\gamma_i = \frac{n_i}{n_i - 1}$ and

$\gamma_{i_1 i_2 \dots i_p} = \prod_{j=1}^p \frac{n_{i_j}}{n_{i_j} - 1}$. It's easy to verify that

$\gamma_i \frac{n_i}{n_o} (\hat{U}^i)^T \hat{U}^i = 1$ and that $\gamma_{i_1 i_2 \dots i_p} \frac{n_1 n_2 \dots n_p}{n_o} (\hat{U}^{i_1 \dots i_p})^T \hat{U}^{i_1 \dots i_p} = 1$. Also for $\gamma_E = \frac{n}{n-1}$, $\gamma_E (\hat{U}^E)^T (\hat{U}^E) = 1$.

Define

$$\hat{U} = \begin{pmatrix} \gamma_1 (\hat{U}^1)^T \\ \gamma_2 (\hat{U}^2)^T \\ \vdots \\ \gamma_{12} (\hat{U}^{12})^T \\ \vdots \\ \gamma_{12 \dots k} (\hat{U}^{12 \dots k})^T \\ \gamma_E (\hat{U}^E)^T \\ \frac{1}{n_o} [1 \ 1 \dots 1]_{1 \times n_o} \end{pmatrix}.$$

From statements (2.6) we have

$$\sigma^2 \hat{V}_Y = \rho \sigma^2 \begin{pmatrix} 1 \\ \alpha_{G(f_2)} \\ \vdots \\ \alpha_{G(f_3)} \\ \vdots \\ \alpha_{G(f_q)} \\ \vdots \\ \alpha_{G(f_{n_1 n_2 \dots n_k})} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{n \times 1} + \sigma^2 (1 - \rho) \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n_o \times 1}$$

From the above statement and statement (2.7) we can write

$$U \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \\ c_{12} \\ \vdots \\ c_{12\dots k} \\ c_E \\ s \end{pmatrix} = \rho \begin{pmatrix} 1 \\ \alpha_{G(f_2)} \\ \alpha_{G(f_3)} \\ \vdots \\ \alpha_{G(f_q)} \\ \vdots \\ \alpha_{G(f_{n_1 n_2 \dots n_k})} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{n \times 1} + (1 - \rho) \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n_o \times 1}$$

Since $\hat{U}U = I$, we have multiplying the above equation on the left by $\hat{U}$,

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \\ c_{12} \\ \vdots \\ c_{12\dots k} \\ c_E \\ s \end{pmatrix} = \rho \hat{U} \begin{pmatrix} 1 \\ \alpha_{G(f_2)} \\ \alpha_{G(f_3)} \\ \vdots \\ \alpha_{G(f_q)} \\ \vdots \\ \alpha_{G(f_{n_1 n_2 \dots n_k})} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{n \times 1} + (1 - \rho) \hat{U} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n_o \times 1}.$$

This implies

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \\ c_{12} \\ \vdots \\ c_{12\dots k} \\ c_E \\ s \end{pmatrix} = \rho \begin{pmatrix} \gamma_1 (\hat{U}^1)^T \\ \gamma_2 (\hat{U}^2)^T \\ \vdots \\ \gamma_{12} (\hat{U}^{12})^T \\ \vdots \\ \gamma_{12\dots k} (\hat{U}^{12\dots k})^T \\ \gamma_E (\hat{U}^E)^T \\ \frac{1}{n_o} [1 \ 1 \ \dots \ 1]_{1 \times n_o} \end{pmatrix} \left\{ \begin{pmatrix} 1 \\ \alpha_{G(f_2)} \\ \alpha_{G(f_3)} \\ \vdots \\ \alpha_{G(f_q)} \\ \vdots \\ \alpha_{G(f_{n_1 n_2 \dots n_k})} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right\} \\
+ (1 - \rho) \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ \vdots \\ 1 \\ \frac{1}{n_o} \end{pmatrix}_{n_o \times 1} .$$

To get the equality $\hat{U} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}_{n_o \times 1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ \vdots \\ 1 \\ \frac{1}{n_o} \end{pmatrix}_{n_o \times 1}$, note that the

first column of the matrix $(\hat{U}^i)^T$ is the element $\frac{1}{\gamma_i}$ and that the first column of the matrix $(\hat{U}^{i_1 i_2 \dots i_k})^T$ is the element $\frac{1}{\gamma_{i_1 i_2 \dots i_k}}$. Also the first column of $(\hat{U}^E)^T$ is the element

$\frac{1}{\gamma_E}$. Thus the theorem is established.

Note that in the above theorem, the term $\rho\alpha_{G(f_q)}$ could have been written as just $\alpha_{G(f_q)}$. In this way, if $\rho = 0$, the terms $\alpha_{G(f_q)}$ can be nonzero, so that the entire correlation matrix V_Y is not equal to the identity matrix. Also it is easy to show that $c_E = 1 - \rho$.

A computer program was written to compute the correction constants by using the above theorem. The model chosen was a two way layout with interaction and having five observations in each cell. The following values were assigned: $\rho = 0.0$, $\alpha_2 = -.1$, $\alpha_1 = .07$, and $\alpha_{12} = 0.0$. To test the null hypothesis that there is no effect due to the first factor, the correction constant for the F statistic was found to be $\frac{c_E}{c_1} = 6.67$. Note that in this case F would be a conservative test with little power without the correction constant. This follows since $\frac{c_E}{c_1} > 1$. To test the hypothesis that there is no effect due to the second factor, the correction constant for the F statistic is $\frac{c_E}{c_2} = .5405$. To test the hypothesis that there is no effect due to interaction, the correction constant for the F statistic is $\frac{c_E}{c_{12}} = .8696$. Since $\frac{c_E}{c_2} < 1$ and $\frac{c_E}{c_{12}} < 1$, the F statistic in these cases would be liberal test statistics, giving larger type I errors than expected. To find out whether given values for

ρ and the $\alpha_{G(f_q)}$'s make V_Y a positive definite correlation matrix, one merely checks to see that the constants c_1 , c_2 , c_{12} , c_E and s are positive. For example, if $\rho = .8$, $\rho\alpha_2 = .2$, $\rho\alpha_1 = .1$, and $\rho\alpha_{12} = -.15$, then $c_1 = 5.5$, $c_2 = 4.5$, $c_{12} = 1.95$, $c_E = .2$, and $s = .29$. So for these values of ρ , $\rho\alpha_2$, $\rho\alpha_1$, and $\rho\alpha_{12}$, the correlation matrix V_Y is positive definite.

As a special case of the covariance matrix in (2.5), we now consider the covariance matrix defined in Smith and Lewis (1980). Smith and Lewis define their covariance matrix as follows:

$$\text{cov}[Y_{(i_1, i_2, \dots, i_k)t}, Y_{(j_1, j_2, \dots, j_k)t_1}]$$

$$= \begin{cases} \sigma^2 & \text{if } i_1 = j_1, i_2 = j_2, \dots, i_k = j_k, t = t_1 \\ \rho\sigma^2 & \text{if } i_1 = j_1, i_2 = j_2, \dots, i_k = j_k, t \neq t_1 \\ \rho\sigma^2 \alpha_{\{k_1, k_2, \dots, k_q\}} = \rho\sigma^2 \alpha_{k_1} \alpha_{k_2} \dots \alpha_{k_q} & \text{if } i_\ell \neq j_\ell \text{ for } \ell \in \{k_1, k_2, \dots, k_q\} \\ & \text{and } i_\ell = j_\ell \text{ for } \ell \notin \{k_1, k_2, \dots, k_q\} \end{cases}$$

Smith and Lewis (1980) showed that this covariance matrix can be written compactly as $\sigma^2(1-\rho)I_{n_0} + \sigma^2 \rho W_{n_1}(\alpha_1) \otimes \dots \otimes W_{n_k}(\alpha_k) \otimes M_n$, where $W_{n_i}(\alpha_i) = (1-\alpha_i)I_{n_i} + \alpha_i M_{n_i}$

$$= \begin{pmatrix} 1 & \alpha_i & & \alpha_i \\ \alpha_i & 1 & & \alpha_i \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_i & \alpha_i & \cdot & \cdot & : & 1 \end{pmatrix}.$$

The first column of the above covariance matrix can be written,

$$\sigma^2 \hat{V}_Y = \rho \sigma^2 \begin{pmatrix} 1 \\ \alpha_1 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_1 \end{pmatrix}_{n_1 \times 1} \otimes \begin{pmatrix} 1 \\ \alpha_2 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_2 \end{pmatrix}_{n_2 \times 1} \otimes \dots \otimes \begin{pmatrix} 1 \\ \alpha_k \\ \cdot \\ \cdot \\ \cdot \\ \alpha_k \end{pmatrix}_{n_k \times 1} \otimes \begin{pmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix}_{n \times 1} \\ + \sigma^2 (1 - \rho) \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}_{n_0 \times 1}.$$

Before we evaluate the constants $c_1, c_2, \dots, c_k, c_{12}, \dots, c_{12 \dots k}$, and c_E for the above covariance matrix, we state the following lemma.

Lemma 8: Let $P = \begin{pmatrix} 1 \\ \alpha_1 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_1 \end{pmatrix}_{n_1 \times 1} \otimes \begin{pmatrix} 1 \\ \alpha_2 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_2 \end{pmatrix}_{n_2 \times 1} \otimes \begin{pmatrix} 1 \\ \alpha_k \\ \cdot \\ \cdot \\ \cdot \\ \alpha_k \end{pmatrix}_{n_k \times 1} \otimes \begin{pmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix}_{n \times 1}.$

Then $(\hat{U}^i)^T P = \frac{1}{\gamma_i} n(1 - \alpha_i) \prod_{\substack{j=1 \\ j \neq i}}^k (1 + (n_j - 1)\alpha_j),$

$$(\hat{U}^{i_1 i_2 \dots i_q})^T_P = \frac{n}{\gamma_{i_1 \dots i_k}} \left\{ \prod_{j \in \{i_1, i_2, \dots, i_q\}} (1 - \alpha_j) \right\} \left\{ \prod_{j' \notin \{i_1, i_2, \dots, i_q\}} (1 + (n_{j'} - 1) \alpha_{j'}) \right\}$$

$$\text{and } (\hat{U}^E)^T_P = 0.$$

Proof: $(\hat{U}^i)^T_P = ([1 \ 1 \dots 1] \otimes \dots \otimes [1 \ 1 \dots 1]$

$$\otimes [\frac{n_i - 1}{n_i} - \frac{1}{n_i} \dots - \frac{1}{n_i}] \otimes [1 \ 1 \dots 1] \otimes \dots \otimes [1 \ 1 \dots 1]) .$$

$$\left(\begin{pmatrix} 1 \\ \alpha_1 \\ \vdots \\ \alpha_1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \alpha_2 \\ \vdots \\ \alpha_2 \end{pmatrix} \otimes \dots \otimes \begin{pmatrix} 1 \\ \alpha_k \\ \vdots \\ \alpha_k \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right) .$$

This gives,

$$\begin{aligned} (\hat{U}^i)^T_P &= (1 + (n_1 - 1) \alpha_1) \dots (1 + (n_{i-1} - 1) \alpha_{i-1}) \left(\frac{n_i - 1}{n_i} \right) (1 - \alpha_i) \\ &\quad \cdot (1 + (n_{i+1} - 1) \alpha_{i+1}) \dots (1 + (n_k - 1) \alpha_k) n \\ &= \frac{1}{\gamma_i} n (1 - \alpha_i) \prod_{\substack{j=1 \\ j \neq i}}^k (1 + (n_j - 1) \alpha_j) . \end{aligned}$$

Let $\hat{U}^{i_1 i_2 \dots i_q} = v_1 \otimes v_2 \otimes \dots \otimes v_k \otimes v$

$$\text{where } V_j = \begin{cases} \begin{pmatrix} \frac{n_j-1}{n_j} \\ -\frac{1}{n_j} \\ \vdots \\ -\frac{1}{n_j} \end{pmatrix}_{n_j \times 1} & \text{if } j \in \{i_1, i_2, \dots, i_q\} \\ \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{n_j \times 1} & \text{if } j \notin \{i_1, i_2, \dots, i_q\} \end{cases}$$

$$\text{and } V = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{n \times 1}. \quad \text{Since,}$$

$$\begin{pmatrix} \frac{n_j-1}{n_j} \\ -\frac{1}{n_j} \\ \vdots \\ -\frac{1}{n_j} \end{pmatrix}^T \cdot \begin{pmatrix} 1 \\ \alpha_j \\ \vdots \\ \alpha_j \end{pmatrix}_{n_j \times 1} = \frac{n_j-1}{n_j} (1 - \alpha_j) = \frac{1}{\gamma_j} (1 - \alpha_j)$$

$$\text{and } \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}^T_{n_j \times 1} \cdot \begin{pmatrix} 1 \\ \alpha_j \\ \vdots \\ \alpha_j \end{pmatrix}_{n_j \times 1} = 1 + (n_j - 1)\alpha_j,$$

we have,

$$\begin{aligned}
(\hat{U}^{i_1 i_2 \dots i_q})^T P &= V_1^T \begin{pmatrix} 1 \\ \alpha_1 \\ \vdots \\ \alpha_1 \end{pmatrix}_{n_1 \times 1} \otimes V_2^T \begin{pmatrix} 1 \\ \alpha_2 \\ \vdots \\ \alpha_2 \end{pmatrix}_{n_2 \times 1} \otimes \dots \otimes V_k^T \begin{pmatrix} 1 \\ \alpha_k \\ \vdots \\ \alpha_k \end{pmatrix}_{n_k \times 1} \\
&\otimes \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{n \times 1}^T \cdot \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{n \times 1} \\
&= \frac{n}{\gamma_{i_1 i_2 \dots i_q}} \left\{ \begin{matrix} \prod (1 - \alpha_j) \\ j \in \{i_1, i_2, \dots, i_q\} \end{matrix} \right\} \left\{ \begin{matrix} \prod (1 + (n_{j'} - 1) \alpha_{j'}) \\ j' \notin \{i_1, i_2, \dots, i_q\} \end{matrix} \right\}.
\end{aligned}$$

Finally,

$$\begin{aligned}
(\hat{U}^E)^T P &= \begin{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n_1 \times 1}^T & \otimes \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n_2 \times 1}^T & \otimes \dots \otimes \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n_k \times 1}^T & \otimes \begin{pmatrix} \frac{n-1}{n} \\ -\frac{1}{n} \\ \vdots \\ -\frac{1}{n} \end{pmatrix}_{n \times 1}^T \end{pmatrix} \\
&\quad \begin{pmatrix} \begin{pmatrix} 1 \\ \alpha_1 \\ \vdots \\ \alpha_1 \end{pmatrix}_{n_1 \times 1} & \otimes \begin{pmatrix} 1 \\ \alpha_2 \\ \vdots \\ \alpha_2 \end{pmatrix}_{n_2 \times 1} & \otimes \dots \otimes \begin{pmatrix} 1 \\ \alpha_k \\ \vdots \\ \alpha_k \end{pmatrix}_{n_k \times 1} & \otimes \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{n \times 1} \end{pmatrix} \cdot \\
&= 0 \text{ since } \left[\frac{n-1}{n} - \frac{1}{n} - \frac{1}{n} \dots - \frac{1}{n} \right]_{1 \times n} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{n \times 1} = 0.
\end{aligned}$$

Hence the lemma is established.

Now we evaluate the constants $c_1, c_2, \dots, c_k, c_{12}, \dots, c_{12\dots k}$, and c_E for the covariance matrix in Smith (1980). From the previous theorem we have

$$c_i = \rho \gamma_i (\hat{U}^i)^T \begin{pmatrix} 1 \\ \alpha_1 \\ \vdots \\ \alpha_1 \end{pmatrix}_{n_1 \times 1} \otimes \begin{pmatrix} 1 \\ \alpha_2 \\ \vdots \\ \alpha_2 \end{pmatrix}_{n_2 \times 1} \otimes \dots \otimes \begin{pmatrix} 1 \\ \alpha_k \\ \vdots \\ \alpha_k \end{pmatrix}_{n_k \times 1} \otimes \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{n \times 1} + 1 - \rho$$

From the previous lemma we have

$$c_i = 1 - \rho + n\rho(1 - \alpha_i) \prod_{\substack{j=1 \\ j \neq i}}^k (1 + (n_j - 1)\alpha_j).$$

Also from the previous theorem we have

$$c_{i_1 i_2 \dots i_q} = \rho \gamma_{i_1 i_2 \dots i_q} (\hat{U}^{i_1 \dots i_q})^T \begin{pmatrix} 1 \\ \alpha_1 \\ \vdots \\ \alpha_1 \end{pmatrix}_{n_1 \times 1} \otimes \begin{pmatrix} 1 \\ \alpha_2 \\ \vdots \\ \alpha_2 \end{pmatrix}_{n_2 \times 1} \otimes \dots \otimes \begin{pmatrix} 1 \\ \alpha_k \\ \vdots \\ \alpha_k \end{pmatrix}_{n_k \times 1} \otimes \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{n \times 1} + 1 - \rho.$$

From the previous lemma, this becomes

$$c_{i_1 i_2 \dots i_q} = 1 - \rho + n\rho \left\{ \prod_{j \in \{i_1, i_2, \dots, i_q\}} (1 - \alpha_j) \right\} \left\{ \prod_{j' \notin \{i_1, i_2, \dots, i_q\}} (1 + (n_{j'} - 1)\alpha_{j'}) \right\}.$$

Finally from the previous theorem and lemma,

$$c_E = \gamma_E (\hat{U}^E)^T \begin{pmatrix} 1 \\ \alpha_1 \\ \vdots \\ \alpha_1 \end{pmatrix}_{n_1 \times 1} \otimes \begin{pmatrix} 1 \\ \alpha_2 \\ \vdots \\ \alpha_2 \end{pmatrix}_{n_2 \times 1} \otimes \dots \otimes \begin{pmatrix} 1 \\ \alpha_k \\ \vdots \\ \alpha_k \end{pmatrix}_{n_k \times 1} \otimes \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{n \times 1} + 1 - \rho$$

$$= 1 - \rho$$

The value of these constants $c_1, c_2, \dots, c_k, c_{12}, \dots, c_{12\dots k}$ and c_E are in agreement with those found in Smith and Lewis (1980).

As a summary, we state the next theorem.

Theorem 4. Let the factorial experiment be as given

in (2.1). Let each of the statistics $\frac{Y^T U^1 Y}{\sigma^2_{\text{RANK}(U^1)}}$,

$$\frac{Y^T U^2 Y}{\sigma^2_{\text{RANK}(U^2)}} \dots, \frac{Y^T U^k Y}{\sigma^2_{\text{RANK}(U^k)}} \quad , \quad \frac{Y^T U^{12} Y}{\sigma^2_{\text{RANK}(U^{12})}} \dots,$$

$$\frac{Y^T U^{12\dots k} Y}{\sigma^2_{\text{RANK}(U^{12\dots k})}} \quad , \quad \text{and} \quad \frac{Y^T U^E Y}{\sigma^2_{\text{RANK}(U^E)}} \quad \text{be distributed as a chi-}$$

square times a constant and also be distributed independently.

Then the correlation matrix V_Y must be

$$V_Y = c_1 U^1 + c_2 U^2 + \dots + c_k U^k + c_{12} U^{12} + \dots + c_{12\dots k} U^{12\dots k}$$

$$+ c_E U^E + s M_{n_0},$$

where the value of the constants $c_1, c_2, \dots, c_k, c_{12}, \dots,$

$c_{12\dots k}, c_E$, and s can be found as described in Theorem 2.

V_Y is positive definite if and only if the constants $c_1,$

$c_2, \dots, c_k, c_{12}, \dots, c_{12\dots k}, c_E$, and s are all positive.

For testing if there is an effect due to factor A^i , the new test statistic F_1 would be $\frac{c_E}{c_i} F$, where F is the statistic used when there's independence. For testing if there is an effect due to the interaction $A^{i_1 i_2 \dots i_q}$ the new test statistic F_1 would be $\frac{c_E}{c_{i_1 i_2 \dots i_q}} F$ where F is the statistic used when there's independence.

CHAPTER III

DERIVATION OF LIKELIHOOD RATIO TEST

3.1 Introduction and Lemmas.

In this chapter the corrected F statistic given in Chapter 2 for the model (2.1) and for the correlation pattern in (2.5) will be shown to be the likelihood ratio test statistic. An extension will be made to the multivariate factorial model. Also a corrected F test will be given for a factorial design involving covariables.

Lemma 1. If the row vectors of B form an orthogonal set of vectors and the row vectors of A form an orthogonal set of vectors, then the row vectors of $A \otimes B$ form an orthogonal set of vectors.

Proof: Assume that the row vectors of B and that the row vectors of A each form an orthogonal set of vectors. Then $AA^T = D_1$ and $BB^T = D_2$ where D_1 and D_2 are diagonal matrices. This implies that $(A \otimes B)(A \otimes B)^T = (A \otimes B)(A^T \otimes B^T) = AA^T \otimes BB^T = D_1 \otimes D_2$. Since $D_1 \otimes D_2$ is a diagonal matrix we have that the row vectors of $A \otimes B$ form an orthogonal set of vectors.

Define $\otimes$ as in Rao (1973) (pg. 30). Let $A = [A_1 \vdots \dots \vdots A_k]$ and $B = [B_1 \vdots \dots \vdots B_k]$. Then $A \otimes B = [A_1 \otimes B_1 \vdots \dots \vdots A_k \otimes B_k]$. Two properties of $\otimes$ are given in Rao (pg. 30). It is easy to verify that $(A \otimes B) \otimes C = A \otimes (B \otimes C)$

and that $(T_1 \otimes T_2)(A \otimes B) = T_1 A \otimes T_2 B$.

Lemma 2. For $i = 1, 2, \dots, k$, let $A_i A_i^T = D_i$, where D_i is a diagonal matrix and also let $B_i B_i^T = \hat{D}_i$, where $\hat{D}_i$ is a diagonal matrix. If $A = [A_1 : A_2 : \dots : A_k]$ and $B = [B_1 : B_2 : \dots : B_k]$, then the row vectors of $A \otimes B$ form an orthogonal set of vectors, that is, $(A \otimes B)(A \otimes B)^T$ is a diagonal matrix.

Proof: Assume that $A_i A_i^T = D_i$ and $B_i B_i^T = \hat{D}_i$ for $i = 1, 2, \dots, k$. Now

$$\begin{aligned}
 (A \otimes B)(A \otimes B)^T &= [A_1 \otimes B_1 : A_2 \otimes B_2 : \dots : A_k \otimes B_k] \cdot \\
 &\quad [A_1 \otimes B_1 : A_2 \otimes B_2 : \dots : A_k \otimes B_k]^T \\
 &= [A_1 \otimes B_1 : A_2 \otimes B_2 : \dots : A_k \otimes B_k] \begin{bmatrix} A_1^T \otimes B_1^T \\ A_2^T \otimes B_2^T \\ \vdots \\ A_k^T \otimes B_k^T \end{bmatrix} \\
 &= [(A_1 \otimes B_1)(A_1^T \otimes B_1^T) + (A_2 \otimes B_2)(A_2^T \otimes B_2^T) + \dots \\
 &\quad + (A_k \otimes B_k)(A_k^T \otimes B_k^T)] \\
 &= [A_1 A_1^T \otimes B_1 B_1^T + A_2 A_2^T \otimes B_2 B_2^T + \dots + A_k A_k^T \otimes B_k B_k^T] \\
 &= [D_1 \otimes \hat{D}_1 + D_2 \otimes \hat{D}_2 + \dots + D_k \otimes \hat{D}_k].
 \end{aligned}$$

Since $D_1 \otimes \hat{D}_1 + D_2 \otimes \hat{D}_2 + \dots + D_k \otimes \hat{D}_k$ is a diagonal matrix, we have that $(A \otimes B)(A \otimes B)^T$ is a diagonal matrix.

Lemma 3. $(A^T A)^+ = A^+ (A^T)^+$ (A^+ is defined to be the

Moore Penrose generalized inverse of A).

Proof: By Graybill (pg. 29).

Lemma 4. $(A \otimes B)^+ = A^+ \otimes B^+$.

Proof: $(A \otimes B)(A^+ \otimes B^+)(A \otimes B) = AA^+A \otimes BB^+B = A \otimes B$. Also,
 $(A^+ \otimes B^+)(A \otimes B)(A^+ \otimes B^+) = A^+AA^+ \otimes B^+BB^+ = A^+ \otimes B^+$, and
 $[(A^+ \otimes B^+)(A \otimes B)]^T = [A^+A \otimes B^+B]^T$
 $= (A^+A)^T \otimes (B^+B)^T$
 $= A^+A \otimes B^+B$
 $= (A^+ \otimes B^+)(A \otimes B)$.

Finally,

$$\begin{aligned} [(A \otimes B)(A^+ \otimes B^+)]^T &= [AA^+ \otimes BB^+]^T \\ &= (AA^+)^T \otimes (BB^+)^T \\ &= AA^+ \otimes BB^+ \\ &= (A \otimes B)(A^+ \otimes B^+). \end{aligned}$$

Lemma 5. Let a_i for $i = 1, 2, \dots, n$ be $n \times 1$ matrices, then $[a_1 \otimes a_2 \otimes \dots \otimes a_{i-1} \otimes B \otimes a_i \otimes \dots \otimes a_m]C = a_1 \otimes a_2 \otimes \dots \otimes a_{i-1} \otimes BC \otimes a_{i+1} \otimes \dots \otimes a_m$, where B is a $p \times q$ matrix and C is a $q \times r$ matrix.

Proof: $C = 1 \otimes 1 \otimes \dots \otimes 1 \otimes C \otimes 1 \otimes 1 \otimes \dots \otimes 1$ where 1 can be taken to be a 1×1 matrix with the element 1 in it. So,

$$\begin{aligned} &(a_1 \otimes a_2 \otimes \dots \otimes a_{i-1} \otimes B \otimes a_i \otimes \dots \otimes a_m)C \\ &= (a_1 \otimes a_2 \otimes \dots \otimes a_{i-1} \otimes B \otimes a_i \otimes \dots \otimes a_m)(1 \otimes 1 \otimes \dots \\ &\quad \otimes 1 \otimes C \otimes 1 \otimes \dots \otimes 1) \\ &= a_1 \otimes a_2 \otimes \dots \otimes a_{i-1} \otimes BC \otimes a_i \otimes \dots \otimes a_m. \end{aligned}$$

3.2 Derivation of Likelihood Ratio Test.

We will first give a general form for the design matrix X , where $Y = X\beta + \varepsilon$ in the general factorial model given in (2.1). But first we will need to define the matrices $E_1^1, E_2^1, \dots, E_k^1, E_1^2, E_2^2, \dots, E_k^2, \dots, E_1^k, E_2^k, \dots, E_k^k$ and then express X in terms of the E_i^j 's.

Let E_i^1 for $i = 1, 2, \dots, k$ have k partitions such that

$$\begin{aligned} E_1^1 &= [I_{n_1} : 1_{n_1} : 1_{n_1} : \dots : 1_{n_1}] \\ E_2^1 &= [1_{n_2} : I_{n_2} : 1_{n_2} : 1_{n_2} : \dots : 1_{n_1}] \\ &\vdots \\ E_k^1 &= [1_{n_k} : 1_{n_k} : \dots : 1_{n_k} : I_{n_k}] \end{aligned}$$

Let E_i^2 for $i = 1, 2, \dots, k$ have $\binom{k}{2}$ partitions such that

$$\begin{aligned} E_1^2 &= [I_{n_1} : I_{n_1} : I_{n_1} : \dots : 1_{n_1}] \\ E_2^2 &= [I_{n_2} : 1_{n_2} : 1_{n_2} : \dots : 1_{n_2}] \\ &\vdots \\ E_k^2 &= [1_{n_k} : 1_{n_k} : \dots : I_{n_k} : I_{n_k}] \end{aligned}$$

In general, the E_i^j 's have $\binom{k}{j}$ partitions and have the matrix I_{n_i} in $\binom{k-1}{j-1}$ of the partitions such that for any particular partition there are exactly j identity matrices in total for $E_1^j, E_2^j, \dots, E_k^j$.

For example,

$$\begin{aligned} E_1^{k-1} &= [I_{n_1} : I_{n_1} : I_{n_1} : \dots : I_{n_1} : 1_{n_1}] \\ E_2^{k-1} &= [I_{n_2} : I_{n_2} : \dots : I_{n_2} : 1_{n_2} : I_{n_2}] \end{aligned}$$

$$\begin{aligned} & \vdots \\ & E_k^{k-1} = [1_{n_k} \vdots I_{n_k} \vdots I_{n_k} \vdots \dots \vdots I_{n_k} \vdots I_{n_k} \vdots I_{n_k}] \end{aligned}$$

and E_i^{k-1} has $\binom{k}{k-1} = k$ partitions. Now define

$$\begin{aligned} X = [& 1_{n_1 n_2 \dots n_k} \vdots E_1^1 \otimes E_2^1 \otimes \dots \otimes E_k^1 \vdots E_1^2 \otimes E_2^2 \otimes \dots \otimes E_k^2 \vdots \\ & \dots \vdots E_1^{k-1} \otimes E_2^{k-1} \otimes \dots \otimes E_k^{k-1} \vdots I_{n_1 n_2 n_3 \dots n_k}] \otimes 1_n \end{aligned} \quad (3.1)$$

For the k -way factorial model in (2.1), if we let

$$\begin{aligned} T = [& A_1^{(1)} A_2^{(1)} \dots A_{n_1}^{(1)} \dots A_1^{(k)} A_2^{(k)} \dots A_{n_k}^{(k)} A_{11}^{(12)} A_{12}^{(12)} \dots \\ & A_{n_1 n_2}^{(12)} A_{11}^{(13)} \dots A_{n_1 n_2 \dots n_k}^{(12 \dots k)}] \end{aligned}$$

then it can be shown that model (2.1) can be written as $Y = X\beta + \varepsilon$, where X is the matrix defined above and Y and ε are $n_1 n_2 \dots n_k n \times 1$ vectors. The correlation matrix for the vector ε is the same as defined in Chapter 2,

$$\begin{aligned} V_Y = & c_1 U^1 + c_2 U^2 + c_k U^k + c_{12} U^{12} + \dots + c_{12 \dots k} U^{12 \dots k} \\ & + c_E U^E + sM_{n_0}. \end{aligned}$$

From Chapter 2, we have $BV_Y B^T = D_1$ where D_1 is a diagonal matrix and $B = B_{n_1} \otimes B_{n_2} \otimes \dots \otimes B_{n_k}$. Let $G^T = B^T D_1^{\frac{1}{2}}$

where $D_1^{\frac{1}{2}} D_1^{\frac{1}{2}} = D_1$. Define $Z = (G^T)^{-1} Y$, $A = (G^T)^{-1} X$ and

$\eta = (G^T)^{-1} \varepsilon$. Then the model $Y = X\beta + \varepsilon$, where $\varepsilon \sim N(0, \sigma^2 V_Y)$ and $V_Y = G^T G$ is transformed into the model $Z = A\beta + \eta$, where η is distributed as a $N(0, \sigma^2 I)$.

Let $H\beta = 0$ be the null hypothesis for testing that one

of the factors or that the interaction of certain factors has no effect. H is the same matrix that is used in testing the above hypothesis for the case when we have independence, that is when $V_Y = I$. Since $H\beta$ is an estimable hypothesis when $V_Y = I$, there exist a matrix C such that $H = CX$. Now $H = CX = CG^T(G^T)^{-1}X = (CG^T)A$. This implies by Graybill (1976) (pg. 484-485) that $H\beta$ is a set of linearly independent estimable functions of β for the transformed model $Z = A\beta + \eta$, $\eta \sim N(0, \sigma^2 I)$. Graybill (pg. 502) gives the likelihood ratio test statistic for testing $H\beta = 0$ vs $H\beta \neq 0$. The test statistic is

$$W = \frac{(H\hat{\beta})^T [H(A^T A)^C H^T]^{-1} H\hat{\beta}}{Z^T Z - \hat{\beta}^T A^T Z} \left(\frac{n_o - r}{q} \right)$$

where $\hat{\beta} = (A^T A)^C A^T Z$. $(A^T A)^C$ is a conditional inverse of $A^T A$, n_o is equal to the number of rows of X , $r = \text{RANK } X$, and q is equal to the number of rows of H . In terms of X , Y , and V_Y , W can be written as

$$W = \frac{Q_1}{Q_2} \left(\frac{n_o - r}{q} \right) \quad (3.2)$$

where $Q_1 = [H(X^T V_Y^{-1} X)^C X^T V_Y^{-1} Y]^T [H(X^T V_Y^{-1} X)^C H^T]^{-1} H(X^T V_Y^{-1} X)^C X^T V_Y^{-1} Y$ and $Q_2 = Y^T [V_Y^{-1} - V_Y^{-1} X (X^T V_Y^{-1} X)^C X^T V_Y^{-1}] Y$ or we can write Q_1 and Q_2 as

$$Q_1 = [H(X^T B^T D_1^{-1} B X)^C X^T B^T D_1^{-1} B Y]^T [H(X^T B^T D_1^{-1} B X)^C H^T] \\ \cdot H(X^T B^T D_1^{-1} B X)^C X^T B^T D_1^{-1} B Y$$

and

$$Q_2 = Y^T [B^T D_1^{-1} B - B^T D_1^{-1} B X (X^T B^T D_1^{-1} B X)^C X^T B^T D_1^{-1} B] Y.$$

If we multiply B and X, as defined in (3.1), we get

$$BX = (B_{n_1} \otimes B_{n_2} \otimes \dots \otimes B_{n_k} \otimes B_n) ([1_{n_1 n_2 \dots n_k} : E_1^1 \otimes E_2^1 \otimes \dots \otimes$$

$$E_k^1 : E_1^2 \otimes E_2^2 \otimes \dots \otimes E_k^2 : \dots : E_1^{k-1} \otimes E_2^{k-1} \otimes \dots \otimes E_k^{k-1} :$$

$$I_{n_1 n_2 \dots n_k}] \otimes I_n)$$

$$= \begin{bmatrix} \left(\begin{array}{c} \sqrt{n_1 n_2 \dots n_k} \\ 0 \\ \vdots \\ 0 \end{array} \right) \\ \vdots \\ B_{n_1} E_1^1 \otimes B_{n_2} E_2^1 \otimes \dots \otimes B_{n_k} E_k^1 : B_{n_1} E_1^2 \otimes$$

$$B_{n_2} E_2^2 \otimes \dots \otimes B_{n_k} E_k^2 : \dots : B_{n_1} E_1^{k-1} \otimes B_{n_2} E_2^{k-1} \otimes \dots$$

$$\otimes B_{n_k} E_k^{k-1} : B_{n_1} \otimes B_{n_2} \otimes \dots \otimes B_{n_k}] \otimes \begin{bmatrix} \sqrt{n} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

$$\text{Since } B_{n_i} I_{n_i} = \begin{bmatrix} \sqrt{n_i} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ and } B_{n_i} I_{n_i} = B_{n_i}, \text{ and since the par-}$$

tioned matrices of E_i^j are either I_{n_i} or 0 , we have that

the row vectors of $B_{n_i} E_i^j$ form an orthogonal set of vectors.

Now, by Lemmas 1 and 2, the row vectors of each partition

of BX form an orthogonal set of vectors. Hence $(BX)(X^T B^T)$

$= D_2$ where D_2 is a diagonal matrix.

Now we claim that a conditional inverse of $X^T V^{-1} X$ is $X^T B^T D_2^+ D_1 D_2^+ B X$, where for any matrix C , C^+ is defined to be the Moore Penrose generalized inverse as defined in Rao (1973). To prove this claim we have the following:

$$\begin{aligned}
& (X^T V^{-1} X) [X^T B^T D_2^+ D_1 D_2^+ B X] (X^T V^{-1} X) \\
&= X^T B^T D_1^{-1} (B X X^T B^T) D_2^+ D_1 D_2^+ (B X X^T B^T) D_1^{-1} B X \\
&= X^T B^T D_1^{-1} D_2 D_2^+ D_1 D_2^+ D_2 D_1^{-1} B X \\
&= X^T B^T D_2 D_2^+ D_1^{-1} D_1 D_2^+ D_2 D_1^{-1} B X \\
&= X^T B^T D_2 D_2^+ D_2 D_2^+ D_1^{-1} B X \\
&= X^T B^T D_2 D_2^+ D_1^{-1} B X \\
&= X^T B^T D_2 D_2^+ D_1^{-1} B X \\
&= X^T B^T B X X^T B^T (B X X^T B^T)^+ D_1^{-1} B X \\
&= X^T B^T B X X^T B^T (X^T B^T)^+ (B X)^+ D_1^{-1} B X \quad (\text{by Lemma 3}) \\
&= X^T B^T B X (B X)^+ (B X) (B X)^+ D_1^{-1} B X \\
&= X^T B^T B X (B X)^+ D_1^{-1} B X \\
&= X^T B^T (X^T B^T)^+ X^T B^T D_1^{-1} B X \\
&= X^T B^T D_1^{-1} B X \\
&= X^T V_Y^{-1} X.
\end{aligned}$$

Note that if $V_Y = I$, then we get $(X^T X)^C = X^T B^T D_2^+ D_2^+ B X$. Now the matrix used in the quadratic form in the denominator of the W statistic in (3.2) will be simplified.

$$\begin{aligned}
& V_Y^{-1} - V_Y^{-1} X (X^T V_Y^{-1} X)^C X^T V_Y^{-1} \\
&= B^T D_1^{-1} B - B^T D_1^{-1} B X [X^T B^T D_2^+ D_1 D_2^+ B X] X^T B^T D_1^{-1} B
\end{aligned}$$

$$\begin{aligned}
&= B^T D_1^{-1} B - B^T D_1^{-1} D_2 D_2^+ D_1 D_2^+ D_2 D_1^{-1} B \\
&= B^T D_1^{-1} B - B^T D_2 D_2^+ D_1^{-1} D_2^+ D_2 D_1^{-1} B \\
&= B^T D_1^{-1} B - B^T D_2 D_2^+ D_2^+ D_2 D_1^{-1} B \\
&= B^T D_1^{-1} B - B^T D_2 D_2^+ D_1^{-1} B \\
&= B^T [I - D_2 D_2^+] D_1^{-1} B.
\end{aligned}$$

Let

$$E = \begin{pmatrix} \begin{pmatrix} \sqrt{n_1 n_2 \dots n_k} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} & \begin{matrix} \vdots \\ B_{n_1} E_1^1 \otimes B_{n_2} E_2^1 \otimes \dots \otimes B_{n_k} E_k^1 \\ \vdots \\ B_{n_1} E_1^2 \otimes B_{n_2} E_2^2 \otimes \dots \otimes B_{n_k} E_k^2 \\ \vdots \\ B_{n_1} E_1^{k-1} \otimes B_{n_2} E_2^{k-1} \otimes \dots \otimes B_{n_k} E_k^{k-1} \\ \vdots \\ B_{n_1} \otimes B_{n_2} \otimes \dots \otimes B_{n_k} \end{matrix} \end{pmatrix} \text{ and } F = \begin{pmatrix} \sqrt{n} \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n \times 1}.$$

Then $BX = E \otimes F$.

Also $BXX^T B^T = EE^T \otimes FF^T$ and $(BXX^T B^T)^+ = (EE^T)^+ \otimes (FF^T)^+$ by Lemma 4. Now $D_2 D_2^+ = (EE^T)(EE^T)^+ \otimes (FF^T)(FF^T)^+$. Since EE^T is nonsingular and since $FF^T = \text{diag}_{n \times n}[\sqrt{n} \ 0 \ 0 \ \dots \ 0]$, we have $D_2 D_2^+ = I_{n_1 n_2 \dots n_k \times n_1 n_2 \dots n_k} \otimes \text{diag}_{n \times n}[1 \ 0 \ 0 \ \dots \ 0]$.

This implies

$$\begin{aligned}
&B^T [I - D_2 D_2^+] D_1^{-1} B \\
&= B^T [I - (I_{n_1 n_2 \dots n_k \times n_1 n_2 \dots n_k} \otimes \text{diag}_{n \times n}[1 \ 0 \ \dots \ 0])] D_1^{-1} B \\
&= B^T [I_{n_1 n_2 \dots n_k \times n_1 n_2 \dots n_k} \otimes I_{n \times n} - (I_{n_1 n_2 \dots n_k \times n_1 n_2 \dots n_k} \otimes \text{diag}_{n \times n}[1 \ 0 \ \dots \ 0])] D_1^{-1} B
\end{aligned}$$

$$\begin{aligned}
& \otimes \text{diag}_{n \times n} [1 \ 0 \ \dots \ 0]) \cdot D_1^{-1} B \\
&= B^T [I_{n_1 n_2 \dots n_k \times n_1 n_2 \dots n_k} \otimes [I_{n \times n} - \text{diag}_{n \times n} [1 \ 0 \ \dots \ 0]]] D_1^{-1} B \\
&= B^T [I_{n_1 n_2 \dots n_k \times n_1 n_2 \dots n_k} \otimes \text{diag}_{n \times n} [0 \ 1 \ 1 \ \dots \ 1]] D_1^{-1} B \\
&= B^T [(I_{n_1 n_2 \dots n_k \times n_1 n_2 \dots n_k} \otimes \text{diag}_{n \times n} [0 \ 1 \ 1 \ \dots \ 1]) D_1]^\dagger B \\
&= B^T [B U^E B^T D_1]^\dagger B \quad (\text{since } B U^E B^T = I_{n_1 n_2 \dots n_k \times n_1 n_2 \dots n_k}
\end{aligned}$$

$\otimes \text{diag}_{n \times n} [0 \ 1 \ 1 \ \dots \ 1]$ in Chapter 2)

$$\begin{aligned}
&= B^T [B U^E B^T B (c_1 U^1 + c_2 U^2 + \dots + c_k U^k + c_{12} U^{12} + \dots \\
&\quad + c_{12 \dots k} U^{12 \dots k} + c_E U^E + s M_{n_o}) B^T]^\dagger B \\
&= B^T [c_E B U^E B^T]^\dagger B \\
&= \frac{1}{c_E} B^T B [U^E]^\dagger B^T B \\
&= \frac{1}{c_E} U^E.
\end{aligned}$$

When $V_Y = I$, Q_2 in equation (3.2) is equal to $Y^T U^E Y$

$= Y^T [I - X(X^T X)^C X^T] Y$. Therefore when the correlation matrix

V_Y is assumed for the factorial model, we have

$$\begin{aligned}
Q_2 &= Y^T [V_Y^{-1} - V_Y^{-1} X (X^T V_Y^{-1} X)^C X^T V_Y^{-1}] Y \\
&= Y^T [\frac{1}{c_E} U^E] Y \tag{3.3} \\
&= \frac{1}{c_E} Y^T [I - X(X^T X)^C X^T] Y.
\end{aligned}$$

To test the effect of the i th factor, the null hypothesis would be $H\beta = 0$ where

$$\otimes \begin{bmatrix} \sqrt{n_{i+1}} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \otimes \dots \otimes \begin{bmatrix} \sqrt{n_k} \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n_k \times 1} \otimes \begin{bmatrix} \sqrt{n} \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n \times 1} \cdot \begin{bmatrix} I_{n_i-1} \\ -1^T_{n_i-1} \end{bmatrix}$$

This gives by Lemma 5,

$$BXH^T = \begin{bmatrix} \sqrt{n_1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n_1 \times 1} \otimes \begin{bmatrix} \sqrt{n_2} \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n_2 \times 1} \otimes \dots \otimes \begin{bmatrix} \sqrt{n_{i-1}} \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n_{i-1} \times 1} \otimes B_{n_i} \begin{bmatrix} I_{n_i-1} \\ -1^T_{n_i-1} \end{bmatrix} \\ \otimes \begin{bmatrix} \sqrt{n_{i+1}} \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n_{i+1} \times 1} \otimes \dots \otimes \begin{bmatrix} \sqrt{n_k} \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n_k \times 1} \otimes \begin{bmatrix} \sqrt{n} \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n \times 1} \cdot$$

Since the first row of the matrix $B_{n_i} \begin{bmatrix} I_{n_i-1} \\ -1^T_{n_i-1} \end{bmatrix}$ is zero, let

$$B_{n_i} \begin{bmatrix} I_{n_i-1} \\ -1^T_{n_i-1} \end{bmatrix} = \begin{bmatrix} 0 \dots 0 \\ \tilde{B}_{n_i} \end{bmatrix} \text{ where } \tilde{B}_{n_i} \text{ is the appropriately de-}$$

fined matrix. Now,

$$D_1 BXH^T = [c_1 BU^1 B^T + c_2 BU^2 B^T + \dots + c_{12\dots k} BU^{12\dots k} B^T + c_E BU^E B^T \\ + sBM_{n_o} B^T] \cdot \left(\begin{bmatrix} \sqrt{n_1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n_1 \times 1} \otimes \begin{bmatrix} \sqrt{n_2} \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n_2 \times 1} \otimes \dots \otimes \begin{bmatrix} \sqrt{n_{i-1}} \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n_{i-1} \times 1} \otimes \right.$$

$$\begin{aligned}
& \begin{pmatrix} 0 & \dots & 0 \\ \tilde{B}_{n_i} \end{pmatrix} \otimes \begin{pmatrix} \sqrt{n_{i+1}} \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n_{i+1} \times 1} \otimes \dots \otimes \begin{pmatrix} \sqrt{n_k} \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n_k \times 1} \otimes \begin{pmatrix} \sqrt{n} \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n \times 1} \\
&= c_i B U^i B^T \cdot \begin{pmatrix} \sqrt{n_1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n_1 \times 1} \otimes \begin{pmatrix} \sqrt{n_2} \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n_2 \times 1} \otimes \dots \otimes \begin{pmatrix} \sqrt{n_{i-1}} \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n_{i-1} \times 1} \\
&\otimes \begin{pmatrix} 0 & \dots & 0 \\ \tilde{B}_{n_i} \end{pmatrix} \otimes \begin{pmatrix} \sqrt{n_{i+1}} \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n_{i+1} \times 1} \otimes \dots \otimes \begin{pmatrix} \sqrt{n_k} \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n_k \times 1} \otimes \begin{pmatrix} \sqrt{n} \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n \times 1}.
\end{aligned}$$

This last equality follows since each of the diagonal matrices $BU^1B^T, BU^2B^T, \dots, BU^{i-1}B^T, BU^{i+1}B^T, \dots, BU^{12 \dots k}B^T, BU^EB^T$, and $BM_{n_0}B^T$ multiplied times BXH^T is equal to the zero matrix. Since $BU^iB^T = \text{diag}_{n_1 \times n_1} [1 \ 0 \ 0 \dots 0]$

$$\otimes \text{diag}_{n_2 \times n_2} [1 \ 0 \dots 0] \otimes \dots \otimes \text{diag}_{n_{i-1} \times n_{i-1}} [1 \ 0 \dots 0]$$

$$\otimes \text{diag}_{n_i \times n_i} [0 \ 1 \ 1 \dots 1] \otimes \text{diag}_{n_{i+1} \times n_{i+1}} [1 \ 0 \dots 0] \otimes \dots$$

$$\otimes \text{diag}_{n_k \times n_k} [1 \ 0 \dots 0] \otimes \text{diag}_{n \times n} [1 \ 0 \dots 0], \text{ which is shown}$$

in Chapter 2, we have that $D_1 B X H^T = c_i B X H^T$. This implies

$$\begin{aligned}
H(X^T V^{-1} X) C_H^T &= H X^T B^T D_2^+ D_1 D_2^+ B X H^T \\
&= H X^T B^T D_2^+ D_2^+ D_1 B X H^T \\
&= c_i H X^T B^T D_2^+ D_2^+ B X H^T \\
&= c_i H(X^T X) C_H^T.
\end{aligned}$$

(3.4)

This last inequality follows since $(X^T X)^C$ was shown to be equal to $X^T B^T D_2^+ D_1^+ B X$. Also we have,

$$\begin{aligned}
 H(X^T V^{-1} X)^C X^T V^{-1} &= H X^T B^T D_2^+ D_1^+ D_2^+ B X X^T B^T D_1^{-1} B \\
 &= H X^T B^T D_2^+ D_1^+ D_2^+ D_2 D_1^{-1} B \\
 &= H X^T B^T D_2^+ D_2^+ D_2 D_1 D_1^{-1} B \\
 &= H X^T B^T D_2^+ D_2^+ B X X^T B^T B \\
 &= H(X^T X)^C X^T
 \end{aligned} \tag{3.5}$$

Therefore, from statements (3.3), (3.4), and (3.5), the W statistic in statement (3.2) for testing the effect of the i th factor is

$$\begin{aligned}
 W &= \frac{c_E}{c_i} \frac{[H(X^T X)^C X^T Y]^T [H(X^T X)^C H^T]^{-1} H(X^T X)^C X^T Y}{Y^T [I - X(X^T X)^C X^T] Y} \left(\frac{n_o - r}{q} \right) \\
 &= \frac{c_E}{c_i} F, \text{ where } F \text{ is the usual test statistic that is used}
 \end{aligned}$$

when the observations in the factorial model are assumed to be independent.

To test the effect of the interaction between the $i_1, i_2, \dots, i_m$ factors, the null hypothesis would be $H\beta = 0$ where

$$\begin{aligned}
 H &= [O[I_{n_{i_1}-1} - 1_{n_{i_1}-1}] \otimes [I_{n_{i_2}-1} - 1_{n_{i_2}-1}] \otimes \dots \\
 &\quad \otimes [I_{n_{i_m}-1} - 1_{n_{i_m}-1}] O], \text{ where } H \text{ has } (n_{i_1}-1)(n_{i_2}-1)\dots
 \end{aligned}$$

$(n_{i_m}-1)$ rows and the same number of columns as the number

of rows in the vector β . The matrix

$$[I_{n_{i_1}-1} \quad -1 \quad -1 \quad n_{i_1}-1] \otimes [I_{n_{i_2}-1} \quad -1 \quad -1 \quad n_{i_2}-1] \otimes \dots \otimes [I_{n_{i_m}-1} \quad -1 \quad -1 \quad n_{i_m}-1]$$

is in the same numbered columns of H as the terms for the interaction between the $i_1, i_2, \dots, i_m$ factors are in the columns of β^T . The columns of X corresponding to the columns of β^T which contain the terms for the interaction between the $i_1, i_2, \dots, i_m$ factors are in a partition of $(E_1^m \otimes E_2^m \otimes \dots \otimes E_k^m) \otimes I_n$ in the expression (3.1). The form of these corresponding columns of X is

$$1_{n_1} \otimes 1_{n_2} \otimes \dots \otimes 1_{n_{i_1}-1} \otimes I_{n_{i_1}} \otimes 1_{n_{i_1}+1} \otimes \dots \otimes 1_{n_{i_2}-1} \otimes I_{n_{i_2}} \\ \otimes 1_{n_{i_2}+1} \otimes \dots \otimes 1_{n_{i_m}-1} \otimes I_{n_{i_m}} \otimes I_{n_{i_m}+1} \otimes \dots \otimes 1_{n_k} \otimes 1_n.$$

So we have

$$BXH^T = (B_{n_1} \otimes B_{n_2} \otimes \dots \otimes B_{n_k} \otimes B_n) [1_{n_1} \otimes 1_{n_2} \otimes \dots \otimes 1_{n_{i_1}-1} \otimes I_{n_{i_1}} \\ \otimes 1_{n_{i_1}+1} \otimes \dots \otimes 1_{n_{i_2}-1} \otimes I_{n_{i_2}} \otimes 1_{n_{i_2}+1} \otimes \dots \otimes 1_{n_{i_m}-1} \otimes I_{n_{i_m}} \\ \otimes 1_{n_{i_m}+1} \otimes \dots \otimes 1_{n_k} \otimes 1_n] \cdot ([I_{n_{i_1}-1} \quad -1 \quad -1 \quad n_{i_1}-1]^T \\ \otimes [I_{n_{i_2}-1} \quad -1 \quad -1 \quad n_{i_2}-1]^T \otimes \dots \otimes [I_{n_{i_m}-1} \quad -1 \quad -1 \quad n_{i_m}-1]^T).$$

This implies

$$BXH^T = \begin{bmatrix} \sqrt{n_1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \otimes \begin{bmatrix} \sqrt{n_2} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \otimes \dots \otimes \begin{bmatrix} \sqrt{n_{i_1}-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \otimes B_{n_{i_1}} \begin{bmatrix} I_{n_{i_1}} & -1 \\ -1^T & -1 \end{bmatrix}$$

$$\begin{aligned}
& \otimes \begin{bmatrix} \sqrt{n_{i_1+1}} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \otimes \dots \otimes \begin{bmatrix} \sqrt{n_{i_2-1}} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \otimes B_{n_{i_2}} \begin{bmatrix} I_{n_{i_2}-1} \\ -1_{n_{i_1}-1} \end{bmatrix} \\
& \otimes \begin{bmatrix} \sqrt{n_{i_2+1}} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \otimes \dots \otimes \begin{bmatrix} \sqrt{n_{i_m-1}} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \otimes B_{n_{i_m}} \begin{bmatrix} I_{n_{i_m}-1} \\ -1_{n_{i_m}}^T \end{bmatrix} \otimes \begin{bmatrix} \sqrt{n_{i_m}} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
& \otimes \dots \otimes \begin{bmatrix} \sqrt{n_k} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \otimes \begin{bmatrix} \sqrt{n} \\ 0 \\ \vdots \\ 0 \end{bmatrix} .
\end{aligned}$$

This last inequality can be shown in a way similar to the proof of Lemma 5. Now the first row of

$$B_{n_{i_j}} \begin{bmatrix} I_{n_{i_j}-1} \\ -1_{n_{i_j}}^T \end{bmatrix} \text{ is zero. Let } B_{n_{i_j}} \begin{bmatrix} I_{n_{i_j}-1} \\ -1_{n_{i_j}}^T \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ & \tilde{B}_{n_{i_j}} & & \end{bmatrix},$$

where $\tilde{B}_{n_{i_j}}$ is the appropriately defined matrix.

This implies

$$\begin{aligned}
D_1 B X H^T &= [c_1 B U^1 B^T + c_2 B U^2 B^T + \dots + c_{12\dots k} B U^{12\dots k} B^T \\
&+ c_E B U^E B^T + s B M_{n_0} B^T] \cdot \left(\begin{bmatrix} \sqrt{n_1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \otimes \begin{bmatrix} \sqrt{n_2} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \otimes \dots \otimes \begin{bmatrix} \sqrt{n_{i_1-1}} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right)
\end{aligned}$$

$$\begin{aligned}
& \otimes \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \\ \tilde{B}_{n_{i_1}} & & \end{pmatrix} \otimes \begin{pmatrix} \sqrt{n_{i_1}+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \otimes \dots \otimes \begin{pmatrix} \sqrt{n_{i_2}-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \\ \tilde{B}_{n_{i_2}} & & \end{pmatrix} \\
& \otimes \begin{pmatrix} \sqrt{n_{i_2}+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \otimes \dots \otimes \begin{pmatrix} \sqrt{n_{i_m}-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & & & \\ \tilde{B}_{n_{i_m}} & & & \end{pmatrix} \otimes \begin{pmatrix} \sqrt{n_{i_m}} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\
& \otimes \dots \otimes \begin{pmatrix} \sqrt{n_k} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \otimes \begin{pmatrix} \sqrt{n} \\ 0 \\ \vdots \\ 0 \end{pmatrix} .
\end{aligned}$$

From the above equality and from the way the diagonal matrices $BU^1_B{}^T$, $BU^2_B{}^T, \dots, BU^{12\dots k}_B{}^T$, $BU^E_B{}^T$, and $BM_{n_o} B^T$ are defined, it can be seen that

$$\begin{aligned}
D_1 BXH^T &= c_{i_1 i_2 \dots i_m} BU^{i_1 i_2 \dots i_m} B^T (BXH^T) \\
&= c_{i_1 i_2 \dots i_m} BXH^T.
\end{aligned} \tag{3.6}$$

From (3.6) it follows that

$$H(X^T V^{-1} X) C_H^T = c_{i_1 i_2 \dots i_m} H(X^T X) C_H^T \tag{3.7}$$

Therefore from statements (3.7), (3.5), and (3.3), the W statistic in statement (3.2) for testing the effect of the interaction between the $i_1, i_2, \dots, i_m$ factors is

$$W = \frac{c_E}{c_{i_1 i_2 \dots i_m}} \frac{[H(X^T X) C_X^T Y]^T [H(X^T X) C_H^T]^{-1} H(X^T X) C_X^T Y}{Y^T [I - X(X^T X) C_X^T] Y} \left(\frac{n_o - r}{q} \right)$$

$$= \frac{c_E}{c_{i_1 i_2 \dots i_m}} F, \text{ where } F \text{ is the usual test statistic that is}$$

used when the observations in the factorial model are assumed to be independent. Hence the corrected F statistic given in Chapter 2 for the model (2.1) and for the correlation pattern in statement (2.5) is the likelihood ratio test statistic.

3.3 Extension to the Multivariate ANOVA.

We wish to extend the likelihood ratio test for testing the different hypotheses of the factorial experiment to the multivariate case. First, a lemma from Marcus and Minc (1964).

Lemma 6. Let A, X, B, and W be matrices such that

$$W = \begin{matrix} & A & X & B \\ \begin{matrix} p \times q \\ p \times m \\ m \times n \\ n \times q \end{matrix} & & & \end{matrix}. \text{ If we denote the } q \text{ columns of } W \text{ by}$$

$W^1, \dots, W^q$ and n columns of X by $X^1, \dots, X^n$ and define

$$\tilde{W} = \begin{pmatrix} W^1 \\ W^2 \\ \vdots \\ W^q \end{pmatrix}_{qp \times 1} \text{ and } \tilde{X} = \begin{pmatrix} X^1 \\ X^2 \\ \vdots \\ X^n \end{pmatrix}_{nm \times 1}$$

then $\tilde{W} = (B^T \otimes A) \tilde{X}$.

$$\text{Let } Y = \begin{pmatrix} y_1^1 & y_1^2 & \dots & y_1^p \\ y_2^1 & y_2^2 & \dots & y_2^p \\ \vdots & \vdots & & \vdots \\ y_{n_0}^1 & y_{n_0}^2 & & y_{n_0}^p \end{pmatrix}_{n_0 \times p}, \text{ where the rows of } Y$$

have covariance Σ , that is

$$\text{cov} \left(\begin{pmatrix} y_i^1 \\ y_i^2 \\ \vdots \\ y_i^p \end{pmatrix} \right) = \Sigma_{p \times p}, \text{ for } i = 1, 2, \dots, n_o. \text{ Let}$$

$$\tilde{Y}^T = [y_1^1 y_1^2 \dots y_1^p y_2^1 y_2^2 \dots y_2^p \dots y_{n_o}^1 y_{n_o}^2 \dots y_{n_o}^p].$$

We will assume that $\text{cov}(\tilde{Y}) = V \otimes \Sigma$, where V is the $n_o \times n_o$ correlation matrix defined in Chapter 2. That is

$$V = c_1 U^1 + c_2 U^2 + \dots + c_k U^k + c_{12} U^{12} + \dots + c_{12 \dots k} U^{12 \dots k} + c_E U^E + s M_{n_o}.$$

The model for the multivariate factorial model can be written $Y = X\beta + \epsilon$ where Y is as defined above, β is a $q \times p$ matrix, ϵ is a $n_o \times p$ matrix equal to

$$\begin{pmatrix} \epsilon_1^1 & \epsilon_1^2 & \dots & \epsilon_1^p \\ \epsilon_2^1 & \epsilon_2^2 & \dots & \epsilon_2^p \\ \vdots & \vdots & & \vdots \\ \epsilon_{n_o}^1 & \epsilon_{n_o}^2 & \dots & \epsilon_{n_o}^p \end{pmatrix}, \text{ and } \epsilon^T = [\epsilon_1^1 \epsilon_1^2 \dots \epsilon_1^p \epsilon_2^1 \epsilon_2^2 \dots \epsilon_2^p \dots$$

$\epsilon_{n_o}^1 \epsilon_{n_o}^2 \dots \epsilon_{n_o}^p]$ is distributed as a $N(0, V \otimes \Sigma)$. Let Y^i be

the i th column of Y , let β^i be the i th column of β , and let ϵ^i be the i th column of ϵ . Then the model $Y^i = X\beta^i + \epsilon^i$ is the factorial model defined by statement (2.1).

Let $V = G^T G$. Let $Z = (G^T)^{-1} Y$, $A = (G^T)^{-1} X$, and $\eta = (G^T)^{-1} \epsilon$. Let $Z = (z_{ij})$. Define $\tilde{Z}^T = (z_{11} z_{12} \dots z_{1p} \dots z_{n_o p})$.

Since $Z^T = I_{p \times p} Y^T G^{-1}$, then by Lemma 6, $\tilde{Z} = ((G^T)^{-1} \otimes I_{p \times p}) \tilde{Y}$.

$$\begin{aligned} \text{This implies } \text{cov}(\tilde{Z}) &= ((G^T)^{-1} \otimes I_{p \times p}) (\text{cov } \tilde{Y}) (G^{-1} \otimes I_{p \times p}) \\ &= ((G^T)^{-1} \otimes I_{p \times p}) (V \otimes \Sigma) (G^{-1} \otimes I_{p \times p}) \\ &= (G^T)^{-1} V G^{-1} \otimes \Sigma \\ &= I_{n_o \times n_o} \otimes \Sigma. \end{aligned}$$

So the rows of Z are independent and

$$Z = A\beta + \eta \text{ where } \begin{pmatrix} \eta_{11} \\ \eta_{12} \\ \vdots \\ \eta_{1q} \\ \vdots \\ \eta_{n_o p} \end{pmatrix} \sim N(0, I_{n_o \times n_o} \otimes \Sigma).$$

If $H\beta = 0$ is an estimable hypothesis, then we can let $L^T \beta$ be a basis set of estimable functions where $L = [H^T, H_1^T]$. By Graybill (pg. 494), $A(L^T)^C L^T = A$. So the model $Z = A\beta + \eta$ can be written $Z = A(L^T)^C L^T \beta + \eta$, or as $Z = U\theta + \eta$, where $U = A(L^T)^C$ and $\theta = L^T \beta$. Note that U is full column rank. Now partition $U = [B_1, B_2]$ and $\theta^T = [\gamma_1^T, \gamma_2^T]$ where $\gamma_1 = H\beta$ and $\gamma_2 = H_1\beta$. So we can write the model as $Z = U\theta + \eta = B_1\gamma_1 + B_2\gamma_2 + \eta$.

We wish to test the hypothesis $H\beta = \gamma_1 = 0$ using the likelihood ratio criterion in Anderson (1958) (pg. 188). The test statistic is

$$\lambda = \frac{|\hat{\Sigma}_{\omega}|^{\frac{1}{2}n_0}}{|\hat{\Sigma}_{\Omega}|^{\frac{1}{2}n_0}}, \quad (3.8)$$

$$\begin{aligned} \text{where } n_0 \hat{\Sigma}_{\Omega} &= [Z^T - Z^T (U (U^T U)^{-1} U^T)] [Z - U (U^T U)^{-1} U^T Z] \\ &= Z^T [I - U (U^T U)^{-1} U^T] [I - U (U^T U)^{-1} U^T] Z \\ &= Z^T [I - U (U^T U)^{-1} U^T] Z \end{aligned}$$

$$\begin{aligned} \text{and } n_0 \hat{\Sigma}_{\omega} &= [Z^T - Z^T B_2 (B_2^T B_2)^{-1} B_2^T] [Z - B_2 (B_2^T B_2)^{-1} B_2^T Z] \\ &= Z^T [I - B_2 (B_2^T B_2)^{-1} B_2^T] [I - B_2 (B_2^T B_2)^{-1} B_2^T] Z \\ &= Z^T [I - B_2 (B_2^T B_2)^{-1} B_2^T] Z \\ &= n_0 \hat{\Sigma}_{\Omega} + Z^T [U (U^T U)^{-1} U^T - B_2 (B_2^T B_2)^{-1} B_2^T] Z \\ &= n_0 \hat{\Sigma}_{\Omega} + [H \hat{\beta}]^T [H (A^T A)^C H^T]^{-1} [H \hat{\beta}]. \end{aligned}$$

From Graybill (pg. 223), we see that $Z^T [U (U^T U)^{-1} U^T - B_2 (B_2^T B_2)^{-1} B_2^T] Z$ can be written $[H \hat{\beta}]^T [H (A^T A)^C H^T]^{-1} [H \hat{\beta}]$ where $\hat{\beta} = (A^T A)^C A^T Z$. To see that $(U^T U)^{-1} = L^T (A^T A)^C L$, we have $U^T U (L^T (A^T A)^C L) = L^C A^T A (L^T)^C L^T (A^T A)^C L$
 $= L^C A^T A (A^T A)^C L = L^C A^T A (A^T A)^C A^T E$, using the fact that there exist a matrix E such that $L = A^T E$. Now $A (A^T A)^C A^T = A A^+$. So $L^C A^T A (A^T A)^C A^T E = L^C A^T A A^+ E = L^C A^T (A^T)^+ A^T E = L^C A^T E = L^C L = I$. Hence, $I - U (U^T U)^{-1} U^T = I - A (A^T A)^C A^T$.

So we have

$$\begin{aligned} n_0 \hat{\Sigma}_{\Omega} &= Z^T [I - U (U^T U)^{-1} U^T] Z \\ &= Z^T [I - A (A^T A)^C A^T] Z \end{aligned}$$

$$\begin{aligned}
&= Y^T [V^{-1} - X(X^T V^{-1} X) C_X^T V^{-1}] Y \\
&= \frac{1}{c_E} Y^T [I - X(X^T X) C_X^T] Y.
\end{aligned} \tag{3.9}$$

The last equality is by statement (3.3). Now,

$$\begin{aligned}
&[H\hat{\beta}]^T [H(A^T A) C_H^T]^{-1} [H\hat{\beta}] \\
&= [H(X^T V^{-1} X) C_X^T V^{-1} Y]^T [H(X^T V^{-1} X) C_H^T]^{-1} H(X^T V^{-1} X) C_X^T V^{-1} Y \\
&= \begin{cases} \frac{1}{c_i} [H(X^T X) C_X^T Y]^T [H(X^T X) C_H^T]^{-1} H(X^T X) C_X^T Y & \text{if } H\beta = 0 \text{ is the hypothesis for testing the} \\ & \text{effect due to the } i\text{th factor} \\ \frac{1}{c_{i_1 i_2 \dots i_q}} [H(X^T X) C_X^T Y]^T [H(X^T X) C_H^T]^{-1} H(X^T X) C_X^T Y & \text{if } H\beta = 0 \text{ is the hypothesis for testing the effect} \\ & \text{of the interaction due to the } i_1 i_2 \dots i_q \text{ factors} \end{cases}
\end{aligned} \tag{3.10}$$

This last equality follows from statements (3.4), (3.5), and (3.7).

We have that

$$\lambda = \frac{|\hat{\Sigma}_\Omega|^{\frac{1}{2}n_0}}{|\hat{\Sigma}_\Omega + (H\hat{\beta})^T [H(A^T A) C_H^T]^{-1} [H\hat{\beta}]|^{\frac{1}{2}n_0}}.$$

Let $Q_3 = [H(X^T X) C_X^T Y]^T [H(X^T X) C_H^T]^{-1} H(X^T X) C_X^T Y$. Using the correction constants in the F test developed in the previous section, the λ statistic for testing the effect of the i th factor is by statements (3.9) and (3.10),

$$\lambda = \frac{|Y^T(I - X(X^T X)^{-1}C_X^T)Y|^{\frac{1}{2}n_0}}{|Y^T(I - X(X^T X)^{-1}C_X^T)Y + \frac{C_E}{C_i} \cdot Q_3|^{\frac{1}{2}n_0}}$$

For testing the effect of the interaction due to the $i_1, i_2, \dots, i_q$ factors, the λ statistic is by statements (3.9) and (3.10),

$$\lambda = \frac{|Y^T(I - X(X^T X)^{-1}C_X^T)Y|^{\frac{1}{2}n_0}}{|Y^T(I - X(X^T X)^{-1}C_X^T)Y + \frac{C_E}{C_{i_1 i_2 \dots i_q}} \cdot Q_3|^{\frac{1}{2}n_0}}.$$

3.4 Corrected F Test for Analysis of Covariance.

Smith (1980) obtained unbiased F tests for factorial designs involving covariables in which a certain correlation structure was assumed for the observations. This extension to the Analysis of Covariance can be made using the more general correlation structure given by statement (2.5).

Let the experimental model be the same as in statement (2.1), except for the addition of concomittant variables or covariables, $X_{\alpha 1}, X_{\alpha 2}, \dots, X_{\alpha s}$. Therefore the model becomes

$$\begin{aligned} Y_{(i_1, i_2, \dots, i_k)j} = & u + A_{i_1}^{(1)} + A_{i_2}^{(2)} + A_{i_1 i_2}^{(12)} + \\ & A_{i_3}^{(3)} + A_{i_1 i_3}^{(13)} + \dots + A_{i_1 i_2 \dots i_k}^{(12 \dots k)} + \sum_{t=1}^s b_t X_{(i_1, \dots, i_k)jt} \\ & + \varepsilon_{(i_1 i_2 \dots i_k)jo} \end{aligned} \quad (3.11)$$

where $\sum_{t=1}^S b_t X_{(i_1, \dots, i_k)jt}$ represents the regression of $Y_{(i_1, \dots, i_k)}$ on its concomitant variables $X_{(i_1, \dots, i_k)j1}, \dots, X_{(i_1, \dots, i_k)js}$.

Let $\gamma \in \{1, 2, 12, 3, 13, \dots, 12 \dots k\}$. Let H_0 be the null hypothesis to test the effect of the interaction due to the factors in γ , if γ is more than one number. Or if γ is a single number let H_0 be the null hypothesis to test the effect of factor γ .

The next theorem shows how to correct the F test. No proof is given. But the same method used by Smith (1980) to find unbiased F test in the Analysis of Covariance can be used to prove the theorem.

Theorem 5. Assume the model given in statement (3.11). Then to test the hypothesis H_0 against its alternative, the corrected F statistic, to be a true F distribution, would be

$$F = \frac{Y^T (U_2^\gamma - U_2^E) Y / m}{Y^T U_2^E Y / m}$$

where $U_2^E = (c_E)^{-1} [U^E - U^E \bar{X} (\bar{X}^T U^E \bar{X})^{-1} \bar{X}^T U^E]$,

$U_2^\gamma = U_3^\gamma - U_3^\gamma \bar{X} (\bar{X}^T U_3^\gamma \bar{X})^{-1} \bar{X}^T U_3^\gamma$, $U_3^\gamma = (c_\lambda)^{-1} U^\gamma + (c_E)^{-1} U^E$, and

$$\underline{\bar{X}} = \begin{pmatrix} X_{(1 \ 1 \dots 1)11} & \cdot & \cdot & \cdot & X_{(1 \ 1 \dots 1)1s} \\ X_{(1 \ 1 \dots 1)21} & \cdot & \cdot & \cdot & X_{(1 \ 1 \dots 1)2s} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ X_{(n_1 n_2 \dots n_k)n1} & \cdot & \cdot & \cdot & X_{(n_1 n_2 \dots n_k)ns} \end{pmatrix}_{n_o \times s}$$

where $n_o = n_1 n_2 \dots n_k n$. Also

$$m_\gamma = \text{RANK} [U^\gamma + U^E - (U^\gamma + U^E) \underline{\bar{X}} (\underline{\bar{X}}^T (U^\gamma + U^E) \underline{\bar{X}})^{-1} \cdot \underline{\bar{X}}^T (U^\gamma + U^E)]$$

and

$$m = \text{RANK} [U^E - U^E \underline{\bar{X}} (\underline{\bar{X}}^T U^E \underline{\bar{X}})^{-1} \underline{\bar{X}}^T U^E].$$

CHAPTER IV

SIMULATION STUDY ON TESTING INTERACTION IN A TWO-WAY ANOVA WITH CORRELATED OBSERVATIONS

4.1 Introduction.

In the usual analysis of data using an analysis of variance, the assumptions are almost always made that the model is linear and the error terms have independent identically normal distributions. It is well known that in many cases the F test used in the ANOVA is robust and has reasonably good power when the normality assumption is not satisfied. If the data have the properties of the assumptions used in the Central Limit Theorem, then as the sample size gets large, the test statistic used in the ANOVA will approach asymptotically an F distribution. In this sense the test statistic used in the ANOVA can be thought of as a nonparametric test. When the data are normal, the test statistic used in the ANOVA is the likelihood ratio test statistic and it has been proved in the literature that such tests have desirable properties. It follows from Lehmann (1959), Chapter 7, that the corrected F used on normal data with the correlation pattern specified in Chapter 1 is an unbiased uniformly most powerful test among the class of invariant test.

If either the assumption that the model is linear or that the data is independent is not satisfied, then there may be considerable loss of perhaps both robustness and

power. In Chapter 2, the correction constant for correcting the F statistic was arrived at by making a transformation on correlated multivariate observations to get independent normal observations. Unfortunately, if data other than multivariate normal data is used, say for example, multivariate lognormal data, then making the same transformation on the data will give uncorrelated data. But independence does not follow. Also the distribution of the transformed data may no longer resemble a lognormal distribution.

If the marginals of multivariate data are normal and the data are uncorrelated, then independence does not necessarily follow unless the data have the standard multivariate normal distribution. For example, let $f(x,y)$ and $g(x,y)$ be two standard bivariate normal distributions, both with means equal to zero and variance equal to one for the variables X and Y . But let the correlation between X and Y be .5 for $f(x,y)$ and -.5 for $g(x,y)$. Then $p(x,y) = .5f(x,y) + .5g(x,y)$ is not the standard bivariate normal density. However, $p(x,y)$ is a joint density in which the marginal distributions are normal and the variables X and Y are uncorrelated but not independent.

When data are nonnormal, there are alternate procedures to use for analyzing a factorial experiment which are fairly robust and have more power than the standard F test in the ANOVA. Conover and Iman (1976) examined three test statistics on a nonlinear lognormal model in a simulation study. The test statistics were the standard F statistic,

the F statistic on the ranks of the data, and a test statistic by Methra and Sen (1969), which is an aligned rank procedure. For the situations they considered, the F statistic used on the ranks showed to be robust and showed to have considerable power advantages over the other two test statistics.

In the simulation study presented in this chapter, a comparison will be made between the corrected F statistic on the data and the corrected F statistic on the ranks. To use the F statistic on the ranks one simply ranks the data from smallest to largest and then replaces the data by the ranks in the standard F statistic. The corrected F will be denoted by cF and the corrected rank transformed F will be denoted as CFR and the uncorrected rank transformed F statistic will be denoted by FR. This simulation study will compare the robustness and power of these statistics, hoping that the corrected F developed under normal theory will behave well under a variety of situations when not all of the usual assumptions of an ANOVA are present.

4.2 The Linear Model and a Method of Generating Correlated Data.

The model used is

$$Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ijk} \quad (4.1)$$

where $i = 1, 2$, $j = 1, 2$, and $k = 1, 2, 3, 4, 5$. The overall mean is μ . The effect of the i th level of factor A is α_i . The effect of the j th level of factor B is β_j .

$(\alpha\beta)_{ij}$ is the effect due to the interaction of factors A and

B. The mean and variance of the ε_{ijk} 's are equal for all i, j, k . The hypothesis is $H_0: (\alpha\beta)_{ij} = 0$ vs. the alternative $H_A: (\alpha\beta)_{ij} \neq 0$. For each situation considered in this simulation, the experiment is repeated 1000 times, which means 20,000 observations are generated.

It is well known that correlated normal data can be generated by making a linear transformation on independent normal data. To generate correlated lognormal data, consider the following case. Let X be distributed as $N(\mu_1, \sigma_1^2)$ and Y be distributed as $N(\mu_2, \sigma_2^2)$ and let X and Y have a bivariate normal distribution with correlation coefficient $\rho(X, Y)$. Now define $Z = \exp(X)$ and $W = \exp(Y)$. Z and W can be considered to have a joint lognormal distribution. The correlation coefficient for the joint pdf is given by

$$\rho(Z, W) = \frac{\exp[\rho(X, Y)\sigma_1\sigma_2] - 1}{[(\exp(\sigma_1^2) - 1)(\exp(\sigma_2^2) - 1)]} \quad (4.2)$$

Therefore a multivariate joint lognormal distribution could be generated using this technique.

Now to generate a correlated multivariate vector whose components have marginal uniform, double exponential, or Cauchy distributions is not so easy. Through simulation studies such as Iman and Davenport (1980), it has been shown that the Pearson and Spearman correlations are very close, practically identical for many symmetric continuous distributions such as in cases when the marginals of the joint distribution have either the uniform or normal distri-

bution. In these cases, generating many samples so that the data have a certain rank correlation, say V , should give data that have a data correlation essentially equal to V . Generating correlated uniform, double exponential, or any rank correlated distribution can be implemented quite well using Iman and Conover's (1980) distribution free approach of inducing rank correlation among input variables.

Method B of Iman and Conover (1980) gives an $n \times k$ matrix X where each row vector has the same rank correlation structure and each of the k components have the desired marginal distributions. One question that remains is whether or not these n vectors can be treated as an independent sample. Due to the fact that the elements in each column of X have already been ordered using Iman and Conover's technique, it seems that some independence between rows has been lost. In the simulation study in this chapter, correlated multivariate normal and lognormal vectors generated by Iman and Conover's technique and correlated multivariate normal and lognormal vectors generated using methods discussed above which assure independence were used. The results were very close, as will be shown in Tables 41-53. Therefore it is assumed that the n vectors in the above matrix X can be treated as an independent or perhaps pseudo-independent sample.

4.3 Correlated Normal Error Terms in the Linear Model.

The first correlation structure imposed on the model

(4.2) is

$$V_Y = (1 - \rho)I_{20 \times 20} + \rho W_2(\alpha_1) \otimes W_2(\alpha_2) \otimes M_n \quad (4.3)$$

where $\rho = .15$, $\alpha_1 = .1$, and $\alpha_2 = .02$. This correlation structure was developed in Smith and Lewis (1980). We can state this another way. The correlation within cells is .15. The correlation for observations at different levels of factor A but at the same level of factor B is .015. The correlation for observations at different levels of factor B but at the same level of factor A is .003. Finally, the correlation for pairs of observations at different levels of both factors is .0003. Each error term in the model (4.1) is distributed $N(0,1)$. The correction constant derived from this correlation for correcting the F statistic worked out to be .562355.

In each of the figures presented in this simulation study, cumulative plots are given for the observed critical levels vs. the theoretical critical levels. A critical level is the smallest α for which H_0 could be rejected. The straight line drawn as a 45 degree angle is used as a theoretical curve to compare with the observed empirical cumulative plots. The column headed by D gives the observed value of the Kolmogorov goodness of fit statistic, which is the maximum vertical deviation of the cumulative plot from the straight line drawn at a 45 degree angle. The last column gives the exact two sided critical level associated with that observed D value, for $n = 1000$. On

each figure, the proportion of the rejections at the alpha levels .01, .025, .05, .1, .15, .2, .25, .3, .35, .4, .45, .5, .55, .6, .65, .70, .75, .80, .85, .90, and .95 are graphed and straight lines are used to connect these points. Since the factorial experiment was repeated 1000 times for each figure, the cumulative plots are fairly smooth and straight lines were used to show this smoothness rather than a stair-case cumulative plot at the above chosen alpha levels.

In the first four figures the error terms have the standard multivariate normal distribution with the correlation structure given by statement (4.3). The above correlation structure was chosen because it has very small correlations and perhaps the experiment would have been run assuming independence. In Figure 1, the severity of the consequences can be seen if the correlation structure is ignored. The type I error is much too large for the standard F.

The cumulative plots of FR and cFR are not graphed on Figure 1 since FR and F have very close values and also cF and cFR have very close values. These values are shown in Table 41. Therefore both cF and cFR follow their hypothesized distributions quite well when no effects or interaction is present.

In Figure 2 the nuisance parameters of $\alpha_1 = .5$, $\alpha_2 = -.5$, $\beta_1 = .3$, and $\beta_2 = -.7$ are added to the linear

FIG. 1

The Null Distribution of F and cF When No Effects Are Present.
Error Terms in Linear Model are Correlated Normals.

TABLE I

Test Statistic	Proportion of Rejections at			$\underline{D}$	$\underline{\hat{\alpha}}$
	$\alpha = .01$	$.05$	$.10$		
F	.042	.126	.206	.1455	0.0
cF	.01	.049	.096	.0157	.9657
FR	.039	.125	.193	.1250	0.0
cFR	.012	.045	.100	.0310	.2917

FIG. 2

The Null Distribution of cF and cFR in the Presence of Effects due to the Two Factors. Error Terms in the Linear Model Are Correlated Normals.

TABLE II

Test Statistic	Proportion of Rejections at			$\underline{D}$	$\underline{\hat{\alpha}}$
	$\alpha = .01$	$.05$	$.10$		
cF	.01	.049	.096	.0157	.9657
cFR	.006	.011	.071	.0500	.0133

model. Now the cF cumulative plot does not change since the quadratic forms in the cF statistic are not sensitive to effects due to the two factors. However, the effects do change the ranks somewhat. From Figure 2, we see that the cFR cumulative plot becomes a more conservative test, that is, the proportion of rejections are less than the theoretical alpha levels.

In Figure 3, the power of the two cumulative plots are compared when there is no effects and the interaction terms are $(\alpha\beta)_{11} = .8$, $(\alpha\beta)_{12} = .3$, $(\alpha\beta)_{21} = -.8$, and $(\alpha\beta)_{22} = .5$. At many points the power of the cF and cFR plot can be considered to be practically the same when one considers the standard deviation of the observed proportion of rejections.

In Figure 4, both effects and interaction terms are present. That is, $\alpha_1 = .5$, $\alpha_2 = -.5$, $\beta_1 = .3$, $\beta_2 = -.7$, $(\alpha\beta)_{11} = .8$, $(\alpha\beta)_{12} = .3$, $(\alpha\beta)_{21} = -.8$, $(\alpha\beta)_{22} = .5$. Again the cF plot is exactly the same as in Figure 3. However, the cFR plot is slightly lower than it was in Figure 3, showing that the effects due to the factors tend to lower the power slightly.

When the data are multivariate normal, the cF statistic is the likelihood ratio test statistic. It is expected that the cF cumulative plot will follow its hypothesized curve very well when there is no interaction and also that the cF cumulative plot have more power than the cFR statistic when there is interaction. However, from these four figures, the

FIG. 3

The Nonnull Distribution of cF and cFR When No Effects are Present Except Interaction. Error Terms in the Linear Model Are Correlated Normals.

TABLE III

<u>Test Statistic</u>	<u>Proportion of Rejections at</u>						
	<u>$\alpha = .01$</u>	<u>.025</u>	<u>.05</u>	<u>.10</u>	<u>.15</u>	<u>.20</u>	<u>.25</u>
cF	.138	.233	.366	.457	.538	.616	.670
cFR	.139	.228	.304	.428	.519	.579	.635

FIG. 4

The Nonnull Distribution of cF and cFR When Effects due to Factors and Interaction are Present. Error Terms in the Linear Model are Correlated Normals.

TABLE IV

<u>Test Statistic</u>	<u>Proportion of Rejections at</u>						
	<u>$\alpha = .01$</u>	<u>.025</u>	<u>.05</u>	<u>.10</u>	<u>.15</u>	<u>.20</u>	<u>.25</u>
cF	.138	.233	.336	.457	.538	.616	.670
cFR	.099	.175	.262	.384	.469	.544	.608

cFR statistic is shown to perform reasonably well in comparison to the cF statistic. The cFR statistic tends to be a more conservative test.

The simulation shown in the first four figures was repeated for correlated joint normal data generated by method B in Iman and Conover (1980). The results are given in Tables 41-46. In comparing the data in this table, one should take into account the standard deviation of the observed proportion of rejections. If p is the observed proportion of rejections at level α , then the estimate of the standard deviation of p is $((1-p)p/1000)^{1/2}$. For example, in Table 41, p is equal to .045 at $\alpha = .05$ for the cFR cumulative distribution generated using independent vectors of correlated multivariate normal observations. So the estimate of $(\text{Var}(p))^{1/2}$ is $((1 - .045)(.045)/1000)^{1/2} = .0066$. Notice that $p = .05$ at the α level of .05 for the cFR cumulative distribution generated using method B of Iman and Conover (1980). It is easily seen that these two values, .05 and .045, are within a standard deviation of each other. Most of the values in Tables 41-46 are within a standard deviation of each other. The conclusion from comparing the results generated by these two methods is that the data generated by Iman and Conover's technique can be treated as independent vectors of correlated observations.

4.4 Correlated Lognormal Error Terms in the Linear Model.

In this section the same linear model is used as in section 4.3, and the correlation pattern of the previous section is taken to be the rank correlation structure induced upon the joint lognormal error terms. A comparison will be made between the results generated by using independent vectors whose components have a certain rank correlation structure and the results generated by vectors induced with the same rank correlation structure and obtained using Iman and Conover's distribution-free technique.

The lognormal distribution is a skewed distribution and one in which rank correlation and data correlations are somewhat different. Let $Y = (Y_{111}, Y_{112}, \dots, Y_{225})$ be a multivariate normal vector with the correlation pattern V_Y of section 4.3 and let each Y_{ijk} be distributed as a $N(0,1)$. Then the correlation pattern V_Z of $Z = (Z_{111}, Z_{112}, \dots, Z_{225})$ where $Z_{ijk} = \exp(Y_{ijk})$ can be gotten by equation (4.2). So the correlation structure V_Z is

$$\rho(Z_{ijk}, Z_{i_1, j_1, k_1}) = \begin{cases} 1 & \text{if } i = i_1, j = j_1, k = k_1 \\ .09418 & \text{if } i = i_1, j = j_1, k \neq k_1 \\ .00879 & \text{if } i \neq i_1, j = j_1 \\ .00175 & \text{if } i = i_1, j \neq j_1 \\ .00018 & \text{if } i \neq i_1, j \neq j_1 \end{cases} \quad (4.4)$$

It is easy to see that the correlation matrix V_Y and

V_z are certainly different and that V_z is closer to the identity matrix. One might readily assume that the correlation structure can be ignored and that the standard F test may be used. The correction constant to the F test is derived from the correlation matrix V_y , and is the same value as in the previous section, that is, $c = .562355$. For the correlated multivariate normal vectors, the Spearman and the Pearson correlated matrices can be considered to be virtually the same. Now, a monotone transformation of the components of each multivariate normal vector does not change the rank correlations. Note that theoretically, the rank correlation of two populations is not defined. However in practice, the correlations of any population are always estimated. For large sample size, the estimated correlations of the multivariate lognormal vectors $Z = (Z_{111}, Z_{112}, \dots, Z_{225})$ using the ranks turns out to be very close to the values in the matrix V_y . So V_y will be referred to as the rank correlation matrix or the Spearman correlation matrix of the multivariate lognormal vectors. c_s will be used to denote the correction constant derived from the Spearman correlations.

The input data for the results shown in Figures 5, 6, 7, and 8 is obtained by first generating 1000 independent multivariate normal vectors. The components of each vector have mean zero and unit variance. The correlation structure of each vector is given by statement (4.3). For

FIG. 5

The Null Distribution of F, cF, FR, cFR When No Effects are Present. Error Terms in the Linear Model are Correlated Lognormals.

TABLE V

Test Statistic	Proportion of Rejections at			<u>D</u>	<u>$\hat{\alpha}$</u>
	<u>$\hat{\alpha} = .01$</u>	<u>.05</u>	<u>.10</u>		
F	.017	.095	.180	.1568	0.0
$c_S F$	.004	.021	.062	.0414	.0648
FR	.039	.125	.193	.1249	0.0
$c_S FR$	.012	.045	.100	.0310	.2917

FIG. 6

The Null Distribution of c_F and c_{FR} in the Presence of Effects due to the Two Factors. Error Terms in the Linear Model are Correlated Lognormals.

TABLE IV

Test Statistic	Proportion of Rejections at			$\underline{D}$	$\underline{\hat{\alpha}}$
	$\alpha = .01$	.05	.10		
c_S^F	.004	.021	.062	.0414	.0648
c_S^{FR}	.007	.024	.061	.0478	.0208

FIG. 7

The Nonnull Distribution of c_F and c_{FR} When No Effects are Present Except Interaction. Error Terms in the Linear Model are Correlated Lognormals. Correction Constant is Derived From the Spearman Correlations.

TABLE VII

Test Statistic	Proportion of Rejections at						
	$\hat{\alpha} = .01$	<u>.025</u>	<u>.05</u>	<u>.10</u>	<u>.15</u>	<u>.20</u>	<u>.25</u>
c_S^F	.058	.114	.196	.294	.370	.432	.477
c_S^{FR}	.16	.238	.318	.427	.520	.581	.626

FIG. 8

The Nonnull Distribution of c_F and c_{FR} When Effects due to Factors and Interaction are Present. Error Terms in the Linear Model are Correlated Lognormals.

TABLE VIII

Test Statistic	Proportion of Rejections at						
	$\alpha = .01$	<u>.025</u>	<u>.05</u>	<u>.10</u>	<u>.15</u>	<u>.20</u>	<u>.25</u>
c_S^F	.058	.114	.196	.294	.370	.432	.477
c_S^{FR}	.082	.150	.234	.330	.403	.465	.510

each normal vector $X = (X_{111}, X_{112}, \dots, X_{225})$, the error terms of the linear model (4.1) are taken to be

$$(\epsilon_{111}, \epsilon_{112}, \dots, \epsilon_{115}, \dots, \epsilon_{225}) = (\exp X_{111}, \exp X_{112}, \dots, \exp X_{225}).$$

In Figure 5, if the correlation structure is ignored, then the type I error for both FR and F is shown to be too high. In Figure 5, c_{sFR} follows the 45 degree line fairly well. However, in Figure 6, the effects $\alpha_1 = .5$, $\alpha_2 = -.5$, $\beta_1 = .3$, and $\beta_2 = -.7$ affect the ranks of the data somewhat and the proportion of rejections for the c_{sFR} statistic, in Table VI are lower than those in Table V where the effects are due only to the interaction terms $(\alpha\beta)_{11} = .8$, $(\alpha\beta)_{12} = .3$, $(\alpha\beta)_{21} = -.8$, and $(\alpha\beta)_{22} = .5$. In Figure 7 the power of the c_{sFR} statistic is much higher than the power of the c_{sF} statistic. However, in Figure 8, with the effects and interaction terms given for Figures 6 and 7, the cumulative plot of the c_{sFR} statistic is still above the cumulative plot of the c_{sF} statistic although the difference between the two is less than that shown in Figure 7. For these situations considered, the c_{sFR} statistic performs rather well.

The above simulation shown in Figures 5, 6, 7, and 8 was repeated using method B of Iman and Conover (1980). Joint lognormal data were generated and induced with the rank correlation given by statement (4.3). The results are shown in Tables 47-52. When the standard deviation of

the observed proportion of rejections is considered, a comparison of the two methods show that the results are very close. Therefore for this situation it is again assumed that the data generated by Iman and Conover's technique can be treated as independent vectors of correlated observations.

4.5 Correlated Lognormal Error Term in the Linear Model. A Comparison for Two Different Correction Constants.

This section is intended to give a comparison between the corrected F and corrected FR statistics when the correction constant c_p is derived from the Pearson correlations and when c_s is derived from the Spearman correlations. The rank correlation structure for this section is

$$\text{RCORR}[\epsilon_{ijk}, \epsilon_{i_1 j_1 k_1}] = \begin{cases} 1 & \text{if } i = i_1, j = j_1, k = k_1 \\ .5 & \text{if } i = i_1, j = j_1, k \neq k_1 \\ .4 & \text{if } i \neq i_1, j = j_1 \\ .35 & \text{if } i = i_1, j \neq j_1 \\ .33 & \text{if } i \neq i_1, j \neq j_1 \end{cases} \quad (4.5)$$

where RCORR stands for rank correlation. The Pearson correlations can be gotten from equation (4.2). The correction constant derived from the Pearson correlations is $c_p = .6241797$. The correction constant derived from the Spearman correlations is $c_s = .5555554$. Since the correction constant from the Spearman correlations is smaller and is used in Figures 13, 14, 15 and 16, the proportions of rejections in these figures will be less than the proportion of rejections in Figures 9, 10, 11 and 12. The

FIG. 9

The Null Distribution of F and cF When No Effects are Present. Error Terms in Linear Model are Correlated Lognormals.

TABLE IX

<u>Test Statistic</u>	<u>Proportion of Rejections at</u>			<u>D</u>	<u>$\hat{\alpha}$</u>
	<u>$\alpha = .01$</u>	<u>.05</u>	<u>.10</u>		
c_P^F	.013	.042	.092	.0510	.0111
c_P^{FR}	.02	.058	.103	.0244	.5928

FIG. 10

The Null Distribution of c_F and c_{FR} in the Presence of Effects due to the Two Factors. Error Terms in the Linear Model are Correlated Log-normals.

TABLE X

Test Statistic	Proportion of Rejections at			$\underline{D}$	$\hat{\alpha}$
	$\alpha = .01$	$.05$	$.10$		
c_P^F	.013	.042	.092	.0510	.0111
c_P^{FR}	.008	.039	.067	.0940	0.0

FIG. 11

The Nonnull Distribution c_F and c_{FR} When No Effects Are Present Except Interaction. Error Terms in the Linear Model are Correlated Lognormals.

TABLE XI

<u>Test Statistic</u>	<u>Proportion of Rejections at</u>						
	<u>$\alpha = .01$</u>	<u>.025</u>	<u>.05</u>	<u>.10</u>	<u>.15</u>	<u>.20</u>	<u>.25</u>
c_p^F	.289	.377	.461	.562	.615	.664	.694
c_p^{FR}	.346	.431	.561	.610	.678	.726	.763

FIG. 12

The Nonnull Distribution of c_p^F and c_p^{FR} When Effects due to Factors and Interaction are Present. Error Terms in the Linear Model are Correlated Lognormals.

TABLE XII

Test Statistic	Proportion of Rejection at						
	$\alpha = .01$	.025	.05	.10	.15	.20	.25
c_p^F	.289	.377	.461	.562	.615	.664	.694
c_p^{FR}	.208	.303	.401	.511	.589	.642	.682

FIG. 13

The Null Distribution of c_S^F and c_S^{FR} When No Effects are Present.
Error Terms in Linear Model are Correlated Lognormals.

TABLE XIII

Test Statistic	Proportion of Rejections at			<u>D</u>	<u>$\hat{\alpha}$</u>
	<u>$\alpha = .01$</u>	<u>.05</u>	<u>.10</u>		
c_S^F	.011	.032	.076	.0318	.2644
c_S^{FR}	.017	.048	.090	.0269	.4666

FIG. 14

The Null Distribution of c_S^F and c_S^{FR} in the Presence of Effects due to the Two Factors. Error Terms in the Linear Model are Correlated Lognormals.

TABLE XIV

Test Statistic	Proportion of Rejections at			<u>D</u>	<u>$\hat{Q}$</u>
	<u>$\alpha = .01$</u>	<u>.05</u>	<u>.10</u>		
c_S^F	.011	.032	.076	.0318	.2644
c_S^{FR}	.006	.029	.059	.1147	0.0

FIG. 15

The Nonnull Distribution of c_S^F and c_S^{FR} When No Effects are Present Except Interaction. Error Terms in the Linear Model are Correlated Lognormals.

TABLE XV

Test Statistic	Proportion of Rejections at						
	$\alpha = .01$	$.025$	$.05$	$.10$	$.15$	$.20$	$.25$
c_S^F	.265	.344	.425	.522	.599	.642	.682
c_S^{FR}	.313	.397	.481	.577	.649	.705	.747

FIG. 16

The Nonnull Distribution of c_S^F and c_S^{FR} When Effects due to Factors and Interaction are Present. Error Terms in the Linear Model are Correlated Lognormal.

TABLE XVI

Test Statistic	Proportion of Rejections at						
	$\alpha = .01$	<u>.025</u>	<u>.05</u>	<u>.10</u>	<u>.15</u>	<u>.20</u>	<u>.25</u>
c_S^F	.265	.344	.425	.522	.599	.642	.682
c_S^{FR}	.185	.273	.362	.471	.554	.620	.663

model in this section is the same as in the previous section except that the correlation pattern is different for the error terms. The same values for effects and interaction are used.

In Figure 13, the level of .2644 from the Kolmogorov goodness of fit test for the c_F statistic indicates that the corrected F statistic follows its theoretical curve better when the correction constant is derived from the Spearman correlations. If one is interested in insuring that the type I error is not larger than its theoretical value, the correction constant derived from the Spearman correlation should be used. In Figures 13 and 14, the c_{sFR} cumulative plot seems to stay below its theoretical curve, indicating that the type I error is smaller than expected. When effects and interaction terms are present in Figures 12 and 16, the corrected FR cumulative plots are overall closer to the corrected F cumulative plots. However, when no effects are present except interaction, it is clear that the corrected FR statistic using the correction constant derived from the Pearson correlations, has the most power. Also Figures 15 and 16 show that actual power of the corrected FR and F statistics does not decrease very much when the correction constant c_s is used.

4.6 Correlated Uniform Error Terms in the Linear Model.

In section 4.4 and 4.3 it was pointed out that cumulative c_F and cFR plots obtained from data generated by

FIG. 17

The Null Distribution of cF and cFR When No Effects are Present. Error Terms in the Linear Model are Correlated Uniform Deviates.

TABLE XVII

Test Statistic	Proportion of Rejections at			$\underline{D}$	$\hat{\alpha}$
	$\alpha = .01$	$.05$	$.10$		
cF	.012	.052	.101	.0135	.9933
cFR	.009	.043	.096	.0394	.0902

FIG. 18

The Null Distribution of cF and cFR in the Presence of Effects due to the Two Factors. Error Terms in the Linear Model are Correlated Uniform Deviates.

TABLE XVIII

Test Statistic	Proportion of Rejections at			$\underline{D}$	$\underline{\hat{\alpha}}$
	$\alpha = .01$	$.05$	$.10$		
cF	.012	.052	.101	.0135	.9933
cFR	.007	.042	.088	.0530	.0073

FIG. 19

The Nonnull Distribution of cF and cFR When No Effects are Present, Except Interaction. Error Terms in the Linear Model are Correlated Uniform Deviates.

TABLE XIX

Test Statistic	Proportion of Rejections at						
	$\alpha = .01$	<u>.025</u>	<u>.05</u>	<u>.10</u>	<u>.15</u>	<u>.20</u>	<u>.25</u>
cF	.073	.136	.205	.314	.407	.477	.540
cFR	.074	.130	.205	.313	.398	.464	.536

FIG. 20

The Nonnull Distribution of cF and cFR When Effects due to Factors and Interaction are Present. Error Terms in the Linear Model are Correlated Uniform Deviates.

TABLE XX

Test Statistic	Proportion of Rejections at						
	$\alpha = .01$	.025	.05	.10	.15	.20	.25
cF	.073	.136	.205	.314	.407	.477	.540
cFR	.058	.111	.175	.273	.344	.428	.494

method B of Iman and Conover (1980) and the cumulative cF and cFR plots obtained by using independent vectors whose components had the desired correlations gave very similar results. Therefore, in this section, the correlated joint uniform error terms are generated using Iman and Conover's technique and these vectors are treated as independent vectors of correlated observations. The rank correlation structure for the joint uniform error terms is the same as in the previous section and is given in statement (4.5). The value of c is .555554. Each component of the error vector is distributed uniformly over the interval (0,6). The values used for effects and interaction are the same as used in the previous section.

It is clear from Figures 17, 18, 19 and 20, that the cF statistic follows its theoretical curve very well when there is no interaction and has at least as much power as the cFR statistic when there is interaction. The cFR statistic would be a more conservative statistic, having lower empirical type I errors at most alpha levels.

4.7 Correlated Double Exponentially Distributed Error Terms in the Linear Model.

Again, the correlated error terms, which are joint double exponentially distributed in this section, are generated using Iman and Conover's technique and these vectors are treated as independent vectors of correlated observations. The rank correlations are the same as is given in

FIG. 21

The Null Distribution of cF and cFR When No Effects are Present.
Error Terms in Linear Model are Correlated Double Exponential Deviates.

TABLE XXI

Test Statistic	Proportion of Rejections at			$\underline{D}$	$\hat{\alpha}$
	$\alpha = .01$	$.05$	$.10$		
cF	.01	.043	.092	.0214	.7487
cFR	.009	.043	.096	.0394	.0902

FIG. 22

The Null Distribution of cF and cFR in the Presence of Effects due to the Two Factors. Error Terms in the Linear Model are Correlated Double Exponential Deviates.

TABLE XXII

Test Statistic	Proportion of Rejections at			<u>D</u>	<u>Q</u>
	<u>$\alpha = .01$</u>	<u>.05</u>	<u>.10</u>		
cF	.01	.043	.092	.0214	.7487
cFR	.009	.047	.098	.0453	.0328

FIG. 23

The Nonnull Distribution of cF and cFR When No Effects are Present Except Interaction. Error Terms in the Linear Model are Correlated Double Exponential Deviates.

TABLE XXIII

Test Statistic	Proportion of Rejections at						
	$\alpha = .01$	<u>.025</u>	<u>.05</u>	<u>.10</u>	<u>.15</u>	<u>.20</u>	<u>.25</u>
cF	.029	.067	.108	.180	.257	.318	.378
cFR	.046	.078	.126	.185	.269	.324	.393

FIG. 24

The Nonnull Distribution of cF and cFR When Effects due to Factors and Interaction are Present. Error Terms in the Linear Model are Correlated Double Exponential Deviates.

TABLE XXIV

<u>Test Statistic</u>	<u>Proportion of Rejections at</u>						
	<u>$\alpha = .01$</u>	<u>.025</u>	<u>.05</u>	<u>.10</u>	<u>.15</u>	<u>.20</u>	<u>.25</u>
cF	.029	.067	.108	.180	.257	.318	.378
cFR	.027	.068	.111	.176	.246	.310	.370

statement (4.5). The values used in the simulation for effects and interaction are the same as have been used in the previous sections. The density function of the double exponent is taken to be $f(x) = (\exp(-|x|/2))/4$. The value of c is .5555554.

It is interesting to see how the cF and cFR statistics compare when used on a heavier tailed distribution. In Figures 21 and 22, while the cF cumulative plot follows its theoretical curve very well, the cFR cumulative plot appears to have slightly lower empirical type I error. In Figures 23 and 24, the empirical powers of the cF and cFR statistic appear to be very close, especially when the standard deviation of the proportion of rejections is taken into consideration.

4.8 Correlated Cauchy Distributed Error Terms in the Linear Model.

It is well known that the Central Limit theorem does not hold for independent Cauchy random variables. Also it is difficult to define a correlation structure on a vector with Cauchy marginal distributions, since mathematically, the moments of a Cauchy random variable do not exist. Using method B of Iman and Conover (1980), a rank correlation structure is imposed on a 1000×20 matrix X , so that the components of each row have the standard Cauchy distribution, $f(x) = (\pi(1+x^2))^{-1}$. From the remarks made at the end of sections 4.3 and 4.4, the rows of the matrix X will

FIG. 25

The Null Distribution of cF and cFR When No Effects are Present.
Error Terms in Linear Model are Correlated Cauchy Deviates.

TABLE XXV

Test Statistic	Proportion of Rejections at			<u>D</u>	<u>$\hat{\alpha}$</u>
	<u>$\alpha = .01$</u>	<u>.05</u>	<u>.10</u>		
cF	.005	.02	.048	.0892	0.0
cFR	.011	.041	.092	.0353	.1657

FIG. 26

The Null Distribution of cF and cFR in the Presence of Effects due to the Two Factors. Error Terms in the Linear Model are Correlated Cauchy Deviates.

TABLES XXVI

Test Statistics	Proportion of Rejections at			$\underline{D}$	$\underline{\hat{Q}}$
	$\alpha = .01$	$.05$	$.10$		
cF	.005	.02	.048	.0892	0.0
cFR	.005	.044	.093	.0516	.0097

FIG. 27

The Nonnull Distribution of cF and cFR When No Effects are Present Except Interaction. Error Terms in the Linear Model are Correlated Cauchy Deviates.

TABLE XXVII

Test Statistic	Proportion of Rejections at						
	$\alpha = .01$	.025	.05	.10	.15	.20	.25
cF	.015	.036	.061	.121	.177	.237	.287
cFR	.055	.101	.150	.229	.295	.356	.422

FIG. 28

The Nonnull Distribution of cF and cFR When Effects due to Factors and Interaction are Present. Error Terms in the Linear Model are Correlated Cauchy Deviates.

TABLE XXVIII

Test Statistic	Proportion of Rejections at						
	$\alpha = .01$	<u>.025</u>	<u>.05</u>	<u>.10</u>	<u>.15</u>	<u>.20</u>	<u>.25</u>
cF	.015	.036	.061	.121	.177	.237	.287
cFR	.035	.071	.127	.200	.262	.320	.398

be treated as independent vectors of correlated data. The Spearman correlations imposed on the matrix X is given by statement (4.5). The correction constant is $c = .5555554$. The values used in the simulation for effects and interaction are the same as used in the previous sections.

As might be expected, from Figures 25, 26, 27 and 28, it is clear that the cF statistic does not follow its theoretical curve very well when no interaction is present and has very little power when interaction is present. However, the cFR statistic seems to stay below its theoretical curve for most values, thereby keeping the type I error small. In addition, the power of the cFR seems to be reasonable, in view of the fact that a very heavy tailed distribution is being used.

4.9 Nonlinear Model Using Correlated Data.

Let X_{ijk} be correlated normal random variables with means

$$\alpha_i + \beta_j + (\alpha\beta)_{ij}$$

and unit variances for $i = 1, 2$, $j = 1, 2$, and $k = 1, 2, 3, 4, 5$.

Let the correlation structure of the X_{ijk} 's be

$$\rho(X_{ijk}, X_{i_1 j_1 k_1}) = \begin{cases} 1 & \text{if } i = i_1, j = j_1, k = k_1 \\ .6201 & \text{if } i = i_1, j = j_1, k \neq k_1 \\ .5231 & \text{if } i \neq i_1, j = j_1 \\ .4709 & \text{if } i = i_1, j \neq j_1 \\ .4492 & \text{if } i \neq i_1, j \neq j_1 \end{cases} \quad (4.6)$$

Let $Y_{ijk} = \exp(X_{ijk})$ be the observable lognormal variates. Now the correlation structure of the Y_{ijk} 's can be derived from equation (4.2) and are

$$\rho(Y_{ijk}, Y_{i_1 j_1 k_1}) = \begin{cases} 1 & \text{if } i = i_1, j = j_1, k = k_1 \\ .5 & \text{if } i = i_1, j = j_1, k \neq k_1 \\ .4 & \text{if } i \neq i_1, j = j_1 \\ .35 & \text{if } i = i_1, j \neq j_1 \\ .33 & \text{if } i \neq i_1, j \neq j_1 \end{cases} \quad (4.7)$$

Notice that the assumptions of having a linear model or of having equal variances is no longer satisfied. Since the Pearson and Spearman correlations for the X_{ijk} 's can be thought of as being essentially equal and a monotone transformation does not change the Spearman correlations, the correlation structure in (4.6) can be thought of as being essentially the Spearman correlations of the Y_{ijk} 's. Let c_p be the correction constant to the F statistic derived from the Pearson correlations. Let c_s be the correction constant to the F statistic derived from the Spearman correlation. Then $c_p = .5555554$ and $c_s = .502291$.

Figures 29, 30, 35, and 36 show that the $c_p F$, $c_p FR$, $c_s F$ and $c_s FR$ statistics are greatly affected by the addition of the effects $\alpha_1 = .5$, $\alpha_2 = -.5$, $\beta_1 = .3$ and $\beta_2 = -.7$. In the previous sections under the assumption of the observations coming from a linear model, the quadratic forms of the F statistic were not sensitive to effects. Figures 30 and 36 show that the $c_p F$ and $c_s F$ statistics become more

FIG. 29

The Null Distribution of c_F and c_{FR} When No Effects are Present. Non-linear Lognormal Model is Used.

TABLE XXIX

Test Statistic	Proportion of Rejections at			$\underline{D}$	$\hat{\alpha}$
	$\alpha = .01$	$.05$	$.10$		
c_P^F	.015	.042	.092	.0452	.0336
c_P^{FR}	.022	.049	.106	.0280	.4131

FIG. 30

The Null Distribution of c_F and c_{FR} in the Presence of Effects due to the Two Factors. Nonlinear Lognormal Model is Used.

TABLE XXX

Test Statistic	Proportion of Rejections at			$\underline{D}$	$\hat{\alpha}$
	$\alpha = .01$	$.05$	$.10$		
c_p^F	.038	.116	.208	.1890	0.0
c_p^{FR}	.004	.015	.047	.0600	.0015

FIG. 31

The Nonnull Distribution of c_F and c_{FR} When No Effects are Present Except Interaction. Nonlinear Lognormal Model is Used.

TABLE XXXI

Test Statistic	Proportion of Rejections at						
	$\alpha = .01$	.025	.05	.10	.15	.20	.25
c_p^F	.137	.234	.361	.520	.616	.690	.748
c_p^{FR}	.324	.450	.567	.680	.753	.808	.850

FIG. 32

The Nonnull Distribution of c_F and c_{FR} When Effects due to Factors and Interaction are Present. Nonlinear Lognormal Model is Used.

TABLE XXXII

Test Statistic	Proportion of Rejections at						
	$\alpha = .01$	<u>.025</u>	<u>.05</u>	<u>.10</u>	<u>.15</u>	<u>.20</u>	<u>.25</u>
c_P^F	.249	.399	.558	.740	.816	.886	.931
c_P^{FR}	.187	.298	.398	.538	.657	.717	.769

FIG. 33

The Null Distribution of c_F and c_{FR} in the Presence of Effects due to the Two Factors. Nonlinear Lognormal Model is Used. Effects are Reduced.

TABLE XXXIII

Test Statistic	Proportion of Rejections at			$\underline{D}$	$\hat{\alpha}$
	$\alpha = .01$	$.05$	$.10$		
c_P^F	.016	.051	.102	.0768	0.0
c_P^{FR}	.013	.043	.086	.0300	.3291

FIG. 34

The Nonnull Distribution of c_F and c_{FR} When Effects due to Factors and Interaction are Present. Nonlinear Lognormal Model is Used. Effects are Reduced.

TABLE XXXIV

Test Statistic	Proportion of Rejections at						
	$\alpha = .01$	.025	.05	.10	.15	.20	.25
c_p^F	.134	.232	.338	.502	.586	.664	.726
c_p^{FR}	.285	.424	.538	.676	.749	.802	.847

liberal test statistics and that the c_{pFR} and c_{sFR} statistics become more conservative. Therefore, in the situation when the same effects are present, one would expect the c_{pF} and c_{sF} statistic to have more empirical power than the c_{pFR} and c_{sFR} statistics respectively. This situation is demonstrated in Figures 32 and 38. Since the effects are amplified by the exponential transformation in this non-linear model, the effects were reduced to $\alpha_1 = .4$, $\alpha_2 = -.5$, $\beta_1 = .3$, and $\beta_2 = 0.0$ in Figures 33, 34, 39 and 40. However, in Figures 33 and 39, the c_{pF} and c_{sF} respectively have somewhat larger empirical type I errors than their theoretical values, except for alpha levels less than .2. In Figures 29, 30, and 33, in which interaction is not present, the c_{pFR} statistic's cumulative plot either follows its theoretical curve or stays below it. Therefore the c_{pFR} statistic would be more desirable to use than the c_{sFR} statistic, since c_p is greater than c_s , thus giving the c_{pFR} statistic slightly more power.

4.10 Conclusion.

When no interaction terms are present, the corrected FR cumulative plots mostly either follow their theoretical curves fairly closely or stay below their theoretical curve. Therefore the corrected FR statistic appears to be more robust than the corrected F statistic. For populations that are very skewed or have many "outliers", even in the case where the observations are correlated, it is hoped that by

FIG. 35

The Null Distribution of c_F and c_{FR} When No Effects are Present.
Nonlinear Lognormal Model is Used.

TABLE XXXV

Test Statistic	Proportion of Rejections at			$\underline{D}$	$\hat{\alpha}$
	$\alpha = .01$	$.05$	$.10$		
c_S^F	.013	.034	.079	.0290	.3707
c_S^{FR}	.018	.039	.092	.0280	.4131

FIG. 36

The Null Distribution of c_F and c_{FR} in the Presence of Effects due to the Two Factors. Nonlinear Lognormal Model is Used.

TABLE XXXVI

Test Statistic	Proportion of Rejections at			$\underline{D}$	$\hat{\alpha}$
	$\alpha = .01$	$.05$	$.10$		
c_S^F	.031	.097	.179	.1654	0.0
c_S^{FR}	.003	.013	.034	.1322	0.0

FIG. 37

The Nonnull Distribution of c_F and c_{FR} When No Effects are Present Except Interaction. Nonlinear Lognormal Model is Used.

TABLE XXXVII

Test Statistic	Proportion of Rejections at						
	$\alpha = .01$	$.025$	$.05$	$.10$	$.15$	$.20$	$.25$
c_S^F	.107	.199	.324	.482	.667	.891	.731
c_S^{FR}	.278	.406	.531	.653	.729	.786	.830

FIG. 38

The Nonnull Distribution of c_F and c_{FR} When Effects due to Factors and Interaction are Present. Nonlinear Lognormal Model is Used.

TABLE XXXVIII

Test Statistic	Proportion of Rejection at						
	$\alpha = .01$	.025	.05	.10	.15	.20	.25
c_S^F	.214	.357	.494	.697	.792	.859	.910
c_S^{FR}	.159	.252	.367	.513	.622	.692	.746

FIG. 39

The Null Distribution of c_F and c_{FR} in the Presence of Effects due to the Two Factors. Nonlinear Lognormal Model is Used. Effects are Reduced.

TABLE XXXIX

<u>Test Statistic</u>	<u>Proportion of Rejections at</u>			<u>$\underline{D}$</u>	<u>$\hat{\alpha}$</u>
	<u>$\alpha = .01$</u>	<u>.05</u>	<u>.10</u>		
c_S^F	.014	.042	.084	.0626	.0008
c_S^{FR}	.009	.032	.072	.0373	.1243

FIG. 40

The Nonnull Distribution of c_F and c_{FR} When Effects due to Factors and Interaction are Present. Nonlinear Lognormal Model is Used. Effects are Reduced.

TABLE XL

Test Statistic	Proportion of Rejections at						
	$\alpha = .01$	$.025$	$.05$	$.10$	$.15$	$.20$	$.25$
c_S^F	.107	.198	.299	.458	.560	.640	.699
c_S^{FR}	.253	.390	.497	.650	.727	.778	.830

using the ranks, the data are transformed into something more resembling normality. In the figures involving the double exponential, the Cauchy, or the lognormal, the corrected FR statistic appeared in most cases to be as robust as the corrected F and outperformed the corrected F on many of the empirical power figures. Conover and Iman (1976) recommend that 'Analysis of the data and of the ranks can be examined side by side to see if they agree, indicating validity of the parametric assumptions or if they disagree, indicating that the data should be reexamined for possible violations of the parametric assumptions.' This advice seems to apply equally as well for the case where the observable data are correlated.

TABLE XLI

Proportion of Rejections When No Interaction Is Present. Error Terms in Linear Model are Correlated Normals. Method B is Iman and Conover's Technique. Values of Effects are $\alpha_1 = .5$, $\alpha_2 = -.5$, $\beta_1 = .3$, and $\beta_2 = -.7$. Alpha Levels Are Between .01 and .25.

ALPHA LEVELS

<u>TEST</u>	<u>.01</u>	<u>.025</u>	<u>.05</u>	<u>.1</u>	<u>.15</u>	<u>.2</u>	<u>.25</u>
<u>NO EFFECTS</u>							
F	.042	.082	.126	.206	.270	.333	.392
cF	.01	.024	.049	.096	.143	.191	.245
FR	.039	.078	.125	.193	.256	.315	.366
cFR	.012	.027	.045	.100	.137	.180	.235
<u>METHOD B AND NO EFFECTS</u>							
F	.044	.078	.121	.199	.266	.333	.394
cF	.010	.023	.054	.097	.144	.194	.239
FR	.042	.079	.120	.198	.259	.328	.386
cFR	.010	.022	.05	.098	.137	.180	.229
<u>EFFECTS ARE PRESENT</u>							
FR	.028	.057	.091	.168	.226	.286	.343
cFR	.006	.011	.035	.071	.103	.159	.203
<u>METHOD B AND EFFECTS ARE PRESENT</u>							
FR	.036	.061	.101	.176	.237	.293	.341
cFR	.005	.017	.041	.074	.120	.168	.210

TABLE XLII

Proportion of Rejections When No Interaction Is Present. Error Terms in Linear Model Are Correlated Normals. Method B is Iman and Conover's Technique. Value of Effects are $\alpha_1 = .5$, $\alpha_2 = -.5$, $\beta_1 = .3$, and $\beta_2 = -.7$. Alpha Levels Are Between .3 and .6.

ALPHA LEVELS

<u>TEST</u>	<u>.3</u>	<u>.35</u>	<u>.4</u>	<u>.45</u>	<u>.5</u>	<u>.55</u>	<u>.6</u>
<u>NO EFFECTS</u>							
F	.429	.479	.529	.585	.614	.660	.695
cF	.305	.375	.401	.443	.496	.557	.603
FR	.424	.462	.506	.556	.619	.652	.688
cFR	.284	.329	.386	.437	.483	.529	.600

METHOD B AND NO EFFECTS

F	.443	.490	.533	.565	.596	.646	.689
cF	.293	.357	.403	.464	.509	.559	.589
FR	.444	.484	.515	.554	.594	.641	.681
cFR	.283	.348	.403	.454	.495	.536	.577

EFFECTS ARE PRESENT

FR	.389	.449	.501	.559	.601	.635	.670
cFR	.252	.308	.357	.419	.466	.529	.593

METHOD B AND EFFECTS ARE PRESENT

FR	.399	.434	.481	.532	.580	.627	.665
cFR	.264	.308	.348	.411	.452	.513	.563

TABLE XLIII

Proportion of Rejections When No Interaction Is Present. Error Terms in Linear Model Are Correlated Normals. Method B Is Iman and Conover's Technique. Values of Effects Are $\alpha_1 = .5$, $\alpha_2 = -.5$, $\beta_1 = .3$, and $\beta_2 = -.7$. Alpha Levels Are Between .65 and .95.

ALPHA LEVELS

<u>TEST</u>	<u>.65</u>	<u>.70</u>	<u>.75</u>	<u>.80</u>	<u>.85</u>	<u>.90</u>	<u>.95</u>
<u>NO EFFECTS</u>							
F	.740	.780	.811	.852	.893	.931	.971
cF	.654	.704	.760	.800	.853	.911	.961
FR	.714	.761	.802	.850	.898	.935	.969
cFR	.651	.692	.731	.797	.850	.902	.939
<u>METHOD B AND NO EFFECTS</u>							
F	.724	.760	.804	.839	.875	.920	.954
cF	.641	.699	.739	.795	.841	.894	.945
FR	.727	.770	.809	.842	.884	.927	.976
cFR	.639	.686	.748	.805	.843	.889	.936
<u>EFFECTS ARE PRESENT</u>							
FR	.710	.753	.781	.813	.859	.911	.967
cFR	.633	.679	.725	.774	.813	.869	.952
<u>METHOD B AND EFFECTS ARE PRESENT</u>							
FR	.699	.747	.787	.830	.887	.925	.976
cFR	.625	.671	.724	.770	.831	.899	.957

TABLE XLIV

Proportion of Rejections When the Interaction Terms $(\alpha\beta)_{11} = .8$, $(\alpha\beta)_{12} = .3$, $(\alpha\beta)_{21} = -.8$, and $(\alpha\beta)_{22} = .5$ Are Present. Error Terms In Linear Model Are Correlated Normal. Method B is Iman and Conover's Technique. Values of Effects are $\alpha_1 = .5$, $\alpha_2 = -.5$, $\beta_1 = .3$, and $\beta_2 = -.7$. Alpha Levels Are Between .01 and .25.

ALPHA LEVELS

<u>TEST</u>	<u>.01</u>	<u>.025</u>	<u>.05</u>	<u>.1</u>	<u>.15</u>	<u>.2</u>	<u>.25</u>
<u>NO EFFECTS</u>							
cF	.138	.233	.336	.457	.538	.616	.670
cFR	.139	.228	.304	.428	.519	.579	.635

METHOD B AND NO EFFECTS

cF	.131	.236	.330	.469	.561	.618	.669
cFR	.130	.227	.315	.431	.526	.597	.654

EFFECTS ARE PRESENT

cFR	.099	.175	.262	.384	.469	.544	.608
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METHOD B AND EFFECTS ARE PRESENT

cFR	.093	.171	.252	.380	.466	.542	.604
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TABLE XLV

Proportion of Rejections When the Interaction Terms $(\alpha\beta)_{11} = .8$, $(\alpha\beta)_{12} = .3$, $(\alpha\beta)_{21} = -.8$, and $(\alpha\beta)_{22} = .5$ Are Present. Error Terms in the Linear Model Are Correlated Normal. Method B is Iman and Conover's Technique. Values of Effects Are $\alpha_1 = .5$, $\alpha_2 = -.5$, $\beta_1 = .3$, and $\beta_2 = -.7$. Alpha Levels Are Between .3 and .6.

ALPHA LEVELS

<u>TEST</u>	<u>.3</u>	<u>.35</u>	<u>.4</u>	<u>.45</u>	<u>.5</u>	<u>.55</u>	<u>.6</u>
<u>NO EFFECTS</u>							
cF	.714	.751	.789	.810	.835	.856	.878
cFR	.689	.724	.758	.797	.825	.843	.858

<u>METHOD B AND NO EFFECTS</u>							
cF	.710	.749	.789	.815	.839	.859	.880
cFR	.697	.735	.766	.793	.821	.852	.867

<u>EFFECTS ARE PRESENT</u>							
cFR	.659	.695	.741	.771	.791	.813	.846

<u>METHOD B AND EFFECTS ARE PRESENT</u>							
cFR	.659	.706	.743	.771	.797	.819	.846

TABLE XLVI

Proportion of Rejections When the Interaction Terms $(\alpha\beta)_{11} = .8$, $(\alpha\beta)_{12} = .3$, $(\alpha\beta)_{21} = -.8$, and $(\alpha\beta)_{22} = .5$ Are Present. Error Terms in the Linear Model Are Correlated Normal. Method B is Iman and Conover's Technique. Values of Effects Are $\alpha_1 = .5$, $\alpha_2 = -.5$, $\beta_1 = .3$, and $\beta_2 = -.7$. Alpha Levels Are Between .65 and .95.

ALPHA LEVELS

<u>TEST</u>	<u>.65</u>	<u>.7</u>	<u>.75</u>	<u>.8</u>	<u>.85</u>	<u>.9</u>	<u>.95</u>
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NO EFFECTS

cF	.902	.918	.934	.953	.965	.973	.985
cFR	.880	.893	.915	.936	.949	.968	.979

METHOD B AND NO EFFECTS

cF	.899	.914	.937	.956	.965	.976	.991
cFR	.886	.904	.918	.935	.955	.974	.986

EFFECTS ARE PRESENT

cFR	.871	.894	.915	.933	.954	.971	.994
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METHOD B AND EFFECTS ARE PRESENT

cFR	.876	.901	.926	.943	.955	.974	.993
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TABLE XLVII

Proportion of Rejections When No Interaction Is Present. Error Terms in Linear Model Are Correlated Lognormal. Method B is Iman and Conover's Technique. Values of the Effects are $\alpha_1 = .5$, $\alpha_2 = -.5$, $\beta_1 = .3$, and $\beta_2 = -.7$. Alpha Levels Are Between .01 and .25.

ALPHA LEVELS							
<u>TEST</u>	<u>.01</u>	<u>.025</u>	<u>.05</u>	<u>.1</u>	<u>.15</u>	<u>.2</u>	<u>.25</u>
<u>NO EFFECTS</u>							
F	.017	.047	.095	.180	.258	.317	.397
cF	.004	.009	.021	.062	.116	.168	.235
FR	.039	.078	.125	.193	.256	.315	.366
cFR	.012	.027	.045	.100	.137	.180	.235
<u>METHOD B AND NO EFFECTS</u>							
F	.016	.047	.082	.171	.252	.314	.383
cF	0.0	.011	.017	.061	.105	.154	.222
FR	.042	.079	.120	.198	.259	.328	.386
cFR	.010	.022	.05	.098	.137	.180	.229
<u>EFFECTS ARE PRESENT</u>							
FR	.019	.050	.086	.174	.238	.299	.355
cFR	.007	.013	.024	.061	.104	.169	.216
<u>METHOD B AND EFFECTS ARE PRESENT</u>							
FR	.023	.055	.096	.164	.218	.295	.351
cFR	.007	.017	.028	.074	.109	.155	.196

TABLE XLVIII

Proportion of Rejection When No Interaction Is Present. Error Terms in Linear Model Are Correlated Lognormals. Method B is Iman and Conover's Technique. Values of the Effects are $\alpha_1 = .5$, $\alpha_2 = -.5$, $\beta_1 = .3$, and $\beta_2 = -.7$. Alpha Levels Are Between .3 and .6.

ALPHA LEVELS							
<u>TEST</u>	<u>.3</u>	<u>.35</u>	<u>.4</u>	<u>.45</u>	<u>.5</u>	<u>.55</u>	<u>.6</u>
<u>NO EFFECTS</u>							
F	.449	.503	.552	.601	.639	.685	.716
cF	.282	.347	.412	.470	.518	.579	.627
FR	.424	.462	.506	.556	.619	.652	.688
cFR	.284	.329	.386	.437	.483	.529	.600
<u>METHOD B AND NO EFFECTS</u>							
F	.433	.481	.533	.586	.631	.674	.716
cF	.281	.333	.401	.455	.504	.559	.619
FR	.444	.484	.515	.554	.594	.641	.681
cFR	.283	.348	.403	.454	.495	.536	.577
<u>EFFECTS ARE PRESENT</u>							
FR	.406	.458	.509	.561	.600	.649	.699
cFR	.260	.320	.370	.421	.477	.536	.589
<u>METHOD B AND EFFECTS ARE PRESENT</u>							
FR	.402	.450	.494	.548	.583	.617	.659
cFR	.255	.315	.368	.421	.465	.524	.570

TABLE XLIX

Proportion of Rejections When No Interaction Is Present. Error Terms in Linear Model Are Correlated Lognormals. Method B is Iman and Conover's Technique. Values of the Effects are $\alpha_1 = .5$, $\alpha_2 = -.5$, $\beta_1 = .3$, and $\beta_2 = -.7$. Alpha Levels Are Between .65 and .95.

ALPHA LEVELS							
<u>TEST</u>	<u>.65</u>	<u>.70</u>	<u>.75</u>	<u>.80</u>	<u>.85</u>	<u>.90</u>	<u>.95</u>
<u>NO EFFECTS</u>							
F	.761	.807	.838	.881	.914	.948	.979
cF	.681	.722	.783	.830	.883	.925	.969
FR	.714	.761	.802	.850	.898	.935	.969
cFR	.651	.692	.731	.797	.850	.902	.939
<u>METHOD B AND NO EFFECTS</u>							
F	.756	.798	.830	.872	.908	.941	.973
cF	.669	.723	.776	.820	.873	.920	.963
FR	.727	.770	.809	.842	.884	.927	.976
cFR	.639	.686	.748	.805	.843	.889	.936
<u>EFFECTS ARE PRESENT</u>							
FR	.737	.780	.817	.863	.899	.934	.971
cFR	.647	.702	.757	.808	.864	.911	.958
<u>METHOD B AND EFFECTS ARE PRESENT</u>							
FR	.689	.739	.776	.826	.871	.927	.966
cFR	.613	.664	.715	.766	.826	.886	.955

TABLE L

Proportion of Rejections When the Interaction Terms $(\alpha\beta)_{11} = .8$, $(\alpha\beta)_{12} = .3$, $(\alpha\beta)_{21} = -.8$, and $(\alpha\beta)_{22} = .5$ Are Present. Error Terms in Linear Model Are Correlated Lognormals. Method B is Iman and Conover's Technique. Values of Effects are $\alpha_1 = .5$, $\alpha_2 = -.5$, $\beta_1 = .3$, and $\beta_2 = -.7$. Alpha Levels Are Between .01 and .25.

ALPHA LEVELS

<u>TEST</u>	<u>.01</u>	<u>.025</u>	<u>.05</u>	<u>.1</u>	<u>.15</u>	<u>.2</u>	<u>.25</u>
<u>NO EFFECTS</u>							
cF	.058	.114	.196	.294	.370	.432	.477
cFR	.160	.238	.318	.427	.520	.581	.626

<u>METHOD B AND NO EFFECTS</u>							
cF	.062	.118	.172	.288	.362	.426	.475
cFR	.158	.234	.315	.446	.523	.581	.624

<u>EFFECTS ARE PRESENT</u>							
cFR	.082	.150	.234	.330	.403	.465	.510

<u>METHOD B AND EFFECTS ARE PRESENT</u>							
cFR	.072	.127	.204	.325	.408	.456	.515

TABLE LI

Proportion of Rejections When the Interaction Terms $(\alpha\beta)_{11} = .8$, $(\alpha\beta)_{12} = .3$, $(\alpha\beta)_{21} = -.8$, and $(\alpha\beta)_{22} = .5$ Are Present. Error Terms in the Linear Model Are Correlated Lognormals. Method B Is Iman and Conover's Technique. Values of the Effects Are $\alpha_1 = .5$, $\alpha_2 = -.5$, $\beta_1 = .3$, and $\beta_2 = -.7$. Alpha Levels Are Between .3 and .6.

ALPHA LEVELS

<u>TEST</u>	<u>.3</u>	<u>.35</u>	<u>.4</u>	<u>.45</u>	<u>.5</u>	<u>.55</u>	<u>.6</u>
<u>NO EFFECTS</u>							
cF	.537	.578	.611	.637	.679	.710	.742
cFR	.676	.707	.739	.775	.798	.816	.840

METHOD B AND NO EFFECTS

cF	.530	.575	.608	.646	.683	.715	.746
cFR	.665	.697	.731	.758	.801	.823	.843

EFFECTS ARE PRESENT

cFR	.573	.615	.645	.679	.723	.756	.782
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METHOD B AND EFFECTS ARE PRESENT

cFR	.559	.604	.643	.682	.719	.749	.777
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TABLE LII

Proportion of Rejection When the Interaction Terms $(\alpha\beta)_{11} = .8$, $(\alpha\beta)_{12} = .3$, $(\alpha\beta)_{21} = -.8$, and $(\alpha\beta)_{22} = .5$ Are Present. Error Terms in the Linear Model Are Correlated Lognormals. Method B is Iman and Conover's Technique. Values of the Effects Are $\alpha_1 = .5$, $\alpha_2 = -.5$, $\beta_1 = .3$, and $\beta_2 = -.7$. Alpha Levels Are Between .65 and .95.

ALPHA LEVELS

<u>TEST</u>	<u>.65</u>	<u>.7</u>	<u>.75</u>	<u>.8</u>	<u>.85</u>	<u>.9</u>	<u>.95</u>
<u>NO EFFECTS</u>							
cF	.776	.817	.851	.886	.917	.936	.973
cFR	.866	.882	.900	.922	.940	.964	.980

METHOD B AND NO EFFECTS

cF	.777	.826	.850	.878	.908	.943	.970
cFR	.867	.891	.915	.931	.950	.967	.983

EFFECTS ARE PRESENT

cFR	.813	.838	.864	.895	.926	.954	.980
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METHOD B AND EFFECTS ARE PRESENT

cFR	.805	.831	.855	.877	.907	.940	.971
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