ON CORNISH-FISHER EXPANSIONS WITH UNKNOWN CUMULANTS

by

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CHAPTER I

INTRODUCTION

1.1 The Problem

Let $F(; \lambda)$ and Ψ be probability distribution functions such that

$$\lim_{\lambda \rightarrow \infty} F(x; \lambda) = \Psi(x) \quad (1.1)$$

for all x in the support of $F(; \lambda)$. Let x and u be corresponding quantiles of $F(; \lambda)$ and Ψ respectively such that

$$F(x; \lambda) = \Psi(u). \quad (1.2)$$

Under certain conditions (see [10], [25], [34] and [47]) one can solve equation (1.2), obtaining the Cornish-Fisher expansion of u in terms of x given by

$$u = x + \sum_{\ell=1}^{\infty} \delta_{\ell}(\lambda) W_{\ell}(x) \quad (1.3)$$

and the inverse Cornish-Fisher expansion of x in terms of u given by

$$x = u + \sum_{\ell=1}^{\infty} \delta_{\ell}(\lambda) Y_{\ell}(u) \quad (1.4)$$

where δ_ℓ is a function of the cumulants of $F(\cdot; \lambda)$ and where W_ℓ and Y_ℓ are functions independent of λ .

In this dissertation, expansions similar to (1.3) and (1.4) will be obtained which do not require evaluation of the cumulants of $F(\cdot; \lambda)$ and therefore do not require evaluation of δ_ℓ as in (1.3) or (1.4).

1.2 The $\hat{F}_n$ Approximation

Let k_i and α_i be the cumulants of the probability distribution functions $F(\cdot; \lambda)$ and Ψ , respectively. Also let

$$\beta_i = k_i - \alpha_i \quad (1.5)$$

where it is assumed for convenience that $\beta_1 = \beta_2 = 0$. In addition assume that

$$\beta_i = O(\lambda^{1-\frac{i}{2}}), \quad i = 3, 4, \dots, \quad (1.6)$$

and that (1.1) holds. Now the Edgeworth expansion of $F(x; \lambda)$ in terms of $\Psi(x)$ is defined by

$$F(x; \lambda) \sim \Psi(x) + \sum_{i=1}^{\infty} Q_i(x; \lambda), \quad (1.7)$$

where the Q 's are functions of λ and x determined by the β_i and the derivatives, $\{\Psi^{(m)}\}$, of Ψ , and where

$$Q_i(x; \lambda) = O(\lambda^{-\frac{i}{2}}). \quad (1.8)$$

A table of Q 's for $i = 1, 2, \dots, 8$ is given in Table A in the appendix; however for a formal definition and development of the Q 's see [9] or [34].

Under certain conditions (see [9] or [34]) the correspondence in (1.7) is at least an asymptotic relation and in that case it is well known that

$$F(x; \lambda) - F_n(x; \lambda) = O(\lambda^{-\frac{n+1}{2}}) \quad (1.9)$$

where

$$F_n(x; \lambda) = \Psi(x) + \sum_{i=1}^n Q_i(x; \lambda) \quad (1.10)$$

is the n -th Edgeworth approximation of $F(x; \lambda)$.

Now equation (1.9) can be rewritten as

$$F(x; \lambda) = \Psi(x) + \sum_{i=1}^s h_i(\lambda) \Psi^{(m_i)}(x) + O(\lambda^{-\frac{n+1}{2}}) \quad (1.11)$$

where $s \geq n$ and the h_i 's depend on the β_i 's which are functions of λ . In general the m_i in expansion (1.11) will not be distinct. Thus, (1.11) can be rewritten as

$$F(x; \lambda) = \Psi(x) + \sum_{i=1}^k g_i(\lambda) \Psi^{(m_i)}(x) + O(\lambda^{-\frac{n+1}{2}}) \quad (1.12)$$

where (1.12) is just a rearrangement of (1.11) such that the m_i are distinct, the $g_i(\lambda)$ are the resulting coefficients, and k is the resulting number of distinct m_i .

The following theorem is stated without proof. For the proof, see [9].

THEOREM 1.1 If $F(x; \lambda)$ and ψ are analytic functions in a domain R of the complex plane, if (1.9) holds, and if the functions $\{\psi^{(m)}\}$ are bounded in R , then for each integer m

$$F^{(m)}(x; \lambda) = \psi^{(m)}(x) + \sum_{i=1}^k g_i(\lambda) \psi^{(m_i+m)}(x) + O(\lambda^{-\frac{n+1}{2}}) \quad (1.13)$$

uniformly as $\lambda \rightarrow \infty$.

Expansion (1.11), Theorem 1.1, and Cramer's Rule motivates the following definition.

DEFINITION 1.1 Let $\psi^{(i)}$ and $F^{(i)}$ be defined as in (1.13) and let $V_i = \psi^{(i)} - F^{(i)}$ where $i = 1, \dots, k$, and k and n are defined by equation (1.12). Then the following approximation of $F(x; \lambda)$ is defined by the following:

$$\hat{F}_n(x; \lambda) = \frac{\begin{vmatrix} \psi(x) & v_1(x; \lambda) & \dots & v_k(x; \lambda) \\ \psi^{(m_1)}(x) & \psi^{(m_1+1)}(x) & \dots & \psi^{(m_1+k)}(x) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \psi^{(m_k)}(x) & \psi^{(m_k+1)}(x) & \dots & \psi^{(m_k+k)}(x) \end{vmatrix}}{\begin{vmatrix} \psi^{(m_1+1)}(x) & \dots & \psi^{(m_1+k)}(x) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \psi^{(m_k+1)}(x) & \dots & \psi^{(m_k+k)}(x) \end{vmatrix}} \quad (1.14)$$

for all x such that the denominator is nonzero.

The following definition is made for convenience:

DEFINITION 1.2 $\hat{D}_n(x; \lambda) = \hat{F}_n(x; \lambda) - \psi(x)$, that is

$$\hat{D}_n(x; \lambda) = \frac{\begin{vmatrix} 0 & V_1(x; \lambda) & \dots & V_k(x; \lambda) \\ \psi^{(m_1)}(x) & \psi^{(m_1+1)}(x) & \dots & \psi^{(m_1+k)}(x; \lambda) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \psi^{(m_k)}(x) & \psi^{(m_k+1)}(x) & \dots & \psi^{(m_k+k)}(x) \end{vmatrix}}{\begin{vmatrix} \psi^{(m_1+1)}(x) & \dots & \psi^{(m_1+k)}(x) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \psi^{(m_k+1)}(x) & \dots & \psi^{(m_k+k)}(x) \end{vmatrix}} \quad (1.15)$$

The following theorem is stated without proof. For the proof see [9].

THEOREM 1.2 If Theorem 1.1 holds and $\hat{F}_n(x; \lambda)$ is defined, then

$$\hat{F}_n(x; \lambda) - F(x; \lambda) = O(\lambda^{-\frac{n+1}{2}}) \quad (1.16)$$

as $\lambda \rightarrow \infty$.

It might be noted at this point that if the limiting distribution $\psi(x)$ is the Normal (0,1) distribution $\phi(x)$, then

$$\psi^{(m+1)}(x) = \phi^{(m+1)}(x) = (-1)^m H_m(x) \phi^{(1)}(x) \quad (1.17)$$

where the $H_m(x)$ are Hermite Polynomials. Thus $\hat{D}_n(x; \lambda)$ can be expressed in terms of (1.17).

1.3 A Survey of Cornish-Fisher Expansions

A technique is now presented for obtaining quantile expansions of distribution functions. The argument used is the one given in 1937 by Cornish and Fisher [10] which was later reformulated in Fisher and Cornish [26].

Let F and ϕ be probability distribution functions with cumulants k_i and α_i respectively. Let β_i be defined as in (1.5) and again assume for convenience that

$$\beta_1 = \beta_2 = 0. \quad (1.18)$$

Also assume that (1.6) holds and that ϕ is the unit normal, that is

$$\phi(t) = \int_{-\infty}^t (2\pi)^{-1/2} \exp(-\frac{1}{2}z^2) dz. \quad (1.19)$$

Let F and ϕ have density functions denoted respectively by f and $\phi^{(1)}$. Also let x and u be quantiles defined by

$$F(x) = \int_{-\infty}^x f(p) dp = \int_{-\infty}^u \phi^{(1)}(p) dp = \phi(u). \quad (1.20)$$

Now, f can be expressed in terms of $\phi^{(1)}$ by the formal expression

$$f(x) = \exp\left[\sum_{i=1}^{\infty} \beta_i \{(-D)^i / i!\}\right] \phi^{(1)}(x) \quad (1.21)$$

where

$$D^i \phi^{(1)}(x) = \frac{d^i \phi^{(1)}(x)}{dx^i} = \phi^{(i+1)}(x). \quad (1.22)$$

Therefore, using (1.17), equation (1.21) can be expressed in terms of Hermite Polynomials, H_i . Using Taylor's formula, ϕ can be expressed as

$$\phi(u) = \phi(x + (u-x)) = \phi(x) - \sum_{i=1}^{\infty} \frac{(x-u)^i}{i!} H_{i-1}(x) \phi^{(1)}(x). \quad (1.23)$$

Substituting (1.21) and (1.23) into (1.20) and then simplifying gives the identity

$$\begin{aligned} \left[\sum_{j=1}^{\infty} D^{-1} \left\{ \sum_{i=1}^{\infty} \beta_i (-D)^i / i! \right\}^j / j! \right] \phi^{(1)}(x) = \\ \left[- \sum_{i=1}^{\infty} \frac{(x-u)^i}{i!} H_{i-1}(x) \right] \phi^{(1)}(x) \end{aligned} \quad (1.24)$$

using the order properties of β_i and a considerable amount of algebraic manipulations, (1.24) can be rearranged to form the so called Cornish-Fisher expansion of u as a function of x given by

$$\begin{aligned}
u = & x - \frac{1}{6}(x^2-1)\beta_3 - \frac{1}{24}(x^3-3x)\beta_4 + \frac{1}{36}(4x^3-7x)\beta_3^2 \\
& - \frac{1}{120}(x^4-6x^2+3)\beta_5 + \frac{1}{144}(11x^4-42x^2+15)\beta_3\beta_4 \\
& - \frac{1}{648}(69x^4-187x^2+52)\beta_3^3 - \frac{1}{720}(x^5-10x^3+15x)\beta_6 \quad (1.25) \\
& + \frac{1}{360}(7x^5-48x^3+51x)\beta_3\beta_5 + \frac{1}{384}(5x^5-32x^3+35x)\beta_4^2 \\
& - \frac{1}{864}(111x^5-547x^3+456x)\beta_3^2\beta_4 \\
& + \frac{1}{7776}(948x^5-3628x^3+2473x)\beta_3^4 + \dots
\end{aligned}$$

and the inverse Cornish-Fisher expansion of x as a function of u given by

$$\begin{aligned}
x = & u + \frac{1}{6}(u^2-1)\beta_3 + \frac{1}{24}(u^3-3u)\beta_4 - \frac{1}{36}(2u^3-5u)\beta_3^2 \\
& + \frac{1}{120}(u^4-6u^2+3)\beta_5 - \frac{1}{24}(u^4-5u^2+2)\beta_3\beta_4 \\
& + \frac{1}{324}(12u^4-53u^2+17)\beta_3^3 + \frac{1}{720}(u^5-10u^3+15u)\beta_6 \quad (1.26) \\
& - \frac{1}{180}(2u^5-17u^3+21u)\beta_3\beta_5 - \frac{1}{384}(3u^5-24u^3+29u)\beta_4^2 \\
& + \frac{1}{288}(14u^5-103u^3+107u)\beta_3^2\beta_4 \\
& - \frac{1}{7776}(252u^5-1688u^3+1511u)\beta_3^4 + \dots
\end{aligned}$$

It is important to note that the expansions given in

[10] and [26] were derived without the assumption given in (1.18). Hence the expansions contain terms involving β_1 and β_2 but otherwise are identical to the expansions given in (1.25) and (1.26). However, in most applications of Cornish-Fisher expansions found in the literature, the function $F(\cdot; \lambda)$ is in standardized form, that is with mean zero and variance one, and thus β_1 and β_2 satisfy (1.18). It should also be mentioned that equation (1.25) is sometimes called the normalizing expansion.

Let us look at some of the developments involving the Cornish-Fisher type expansions. In the initial paper [10], Cornish and Fisher obtained inverse quantile expansions of the distribution of $z = \frac{1}{2} \log F_{v_1, v_2}$, where the symbol F_{v_1, v_2} represents the F-distribution with v_1, v_2 degrees of freedom. In 1945, Goldberg and Levine [29] obtained both types of Cornish-Fisher expansions for the quantiles of the student-t and chi-square distributions. Comparisons of the accuracy of these expansions were made with other approximations such as the ones derived by Peiser ([45] and [46]) and Fisher [25]. They pointed out the general superiority of the Cornish-Fisher expansion for $\lambda \geq 10$, λ being the degrees of freedom. In 1947, Geary [27] considered the problem of obtaining quantiles for the null distribution of $\sqrt{b_1}$ under the hypothesis of normality where

$$\sqrt{b_1} = \frac{\sum (x - \bar{x})^3 / n}{\{\sum (x - \bar{x})^2 / n\}^{3/2}} \quad (1.27)$$

and where $\bar{x}$ is the sample mean and n the sample size. Geary's approach to the problem involved the technique of integral iteration for $n = 3, \dots, 8$; however the method became so complicated that for $n \geq 8$ he supplied Cornish-Fisher expansions using exact moments up to eight. Several years later, D'agostino and Tietjen [15] compared Geary's Cornish-Fisher expansion with approximations by the normal distribution, the Johnson S_u distributions, the t or Pearson Type VII distributions, and by Monte-Carlo simulations. They concluded that all the approximations, except the normal approximation, appeared satisfactory for $n \geq 8$. In 1949, Chand [8] obtained inverse Cornish-Fisher expansions of the distribution of the arithmetic mean of random samples taken from the normal, rectangular, double-exponential, sech , and sech^2 distributions. The expansions were carried out through terms of $O(\lambda^{-2})$ and gave better than three decimal accuracy for probability levels ranging from .005 to .25. During the same year, Riordan [47] introduced some formulas which he claimed could be used in inverting Cornish-Fisher expansions. Riordan did not give an example of this application, however, he did apply the formulas to the expansions obtained by Hotelling and Frankel

[36] of the t-variate. Considerable interest was born in the 1950's concerning the asymptotic properties of the quantile expansions. In 1956, Wasow [51] proved the existence of asymptotic expansions for the quantiles of a special class of distribution functions, which included interesting distributions such as the Student-t and some members of the gamma family. In 1958, Wallace [50] pointed out that Wasow's results could be used to establish that Cornish-Fisher expansions of quantiles of these special distributions are indeed valid asymptotic expansions. It might also be mentioned that during the same year, Eeden [21] obtained Cornish-Fisher expansions for the quantiles of the non-central t-distribution up to and including terms of order $O(\lambda^{-2})$. In 1959, Bol'shev [2] investigated the connection between asymptotically normal transformations of random variables and the Cornish-Fisher expansions. His conclusions were that the relationship enables asymptotically normal transformations to be constructed by a general method, which he outlines in the paper. In 1960, R.A. Fisher and E.A. Cornish [26] reformulated their technique for deriving the Cornish-Fisher expansion and also extended the number of terms in the previously published inverse expansions of the z, chi-square, and t-distributions. In 1961, Vaderbeck and Cooke [49] compared the Cornish-Fisher expansions of the chi-square quantiles with

the exact values and reported very good results for $\lambda \geq 60$. In 1963, Bol'shev [3] presented conditions for obtaining generalized asymptotic Cornish-Fisher expansions to the case where the limit distributions belong to the K. Pearson family. During the same year, Finney [24] generalized the Cornish-Fisher technique by relating two arbitrary density functions by means of a cumulant adjustment generating function. Generalized quantile expansions are obtained to order $O(\lambda^{-2})$. In 1967, Johnson [40] discussed the asymptotic properties of Edgeworth and Cornish-Fisher type expansions for a special class of density functions. During the same year, Harter and Dubey [33] used the Cornish-Fisher expansions to calculate tables of the distribution of the arithmetic mean of a random sample taken from the Weibull population. In 1968, Hill and Davis [34] provided algorithms for transforming an asymptotic expansion of $F(\cdot; \lambda)$ in terms of the "standard" limiting distribution Ψ into asymptotic relations between the quantiles of these distributions. They illustrated how the algorithms can be used to obtain the Cornish-Fisher expansions. In 1971, Davis [16] used the asymptotic series developed by Box [7] to obtain Cornish-Fisher expansions where the limiting distribution is the chi-square. That is, Davis expressed the percentage points of a given distribution in terms of chi-squared percentiles. During

the same year, D'agostino [14] presented a test of normality based on a statistic D which is a function of the linear unbiased estimator of the population standard deviation to the sample standard deviation given by Downton [17]. D'agostino found that Monte Carlo simulations indicated that Cornish-Fisher expansions adequately approximate the null distribution of D when the sample size is 50 or more. In 1973, Bowman and Shenton ([14] or [15]) approximated the distribution of the skewness statistic $\sqrt{b_1}$ (see (1.27)) in sampling from Pearson Type I distributions. They gave approximate contours of the power of the $\sqrt{b_1}$ statistic for detecting departures of Pearson Type I distributions from normality by using expansions for the first eight moments along with Cornish-Fisher expansions for quantiles. They also pointed out that consideration was being given to using Cornish-Fisher expansions with limiting distributions other than the normal for solving similar type problems. The distribution of the standardized fourth moment b_2 in sampling from three populations was discussed in a paper by Bowman [6] in which contours of the power of the test for b_2 were given for three alternatives. Here again the method involved not only the use of the Cornish-Fisher expansion with the limiting distribution the unit normal but also with the limiting distributions being taken from the class of Johnson S_U dis-

tributions. Norman J. Johnson [37] has considered the fact that the statistic $\sqrt{N}(\bar{x} - \mu)/s$ does not have a t-distribution when sampling from a skewed population. He proposed a correction to this statistic by using the inverse Cornish-Fisher expansion and the jackknife t-statistic. Also during 1973, Draper and Tierney [18] pointed out the need for higher order terms in important expansions which included the two types of Cornish-Fisher expansions. They presented terms to order λ^{-4} in both the Cornish-Fisher expansion and inverse Cornish-Fisher expansion where the limiting distribution was the unit normal. They also discussed techniques used to obtain these expansions some of which involved computer programs in the SNOBOL-4 programming language.

Hence it is obvious that the Cornish-Fisher expansions have been the subject of considerable research for some time. However, this discussion would be somewhat incomplete if it were not pointed out that the Cornish-Fisher expansions have had direct applications to several of the scientific areas other than mathematics and statistics. For example Panse [44] used the expansions in a biological study on quantitative inheritance. The expansions have also been used in areas such as dental research [19], analytical chemistry [41], and etc.

CHAPTER II
GENERALIZED CORNISH-FISHER TYPE EXPANSIONS
AND APPROXIMATIONS

Let $F(\cdot; \lambda)$ and Ψ be probability distribution functions satisfying the limiting condition given in (1.1). Also let x and u be corresponding quantiles of $F(\cdot; \lambda)$ and Ψ respectively such that

$$F(x; \lambda) = \Psi(u). \quad (2.1)$$

Let us now consider the problem of solving for u in terms of x . Let $\Psi^{(1)}$ be the density of Ψ and assume that $\Psi^{(1)}$ is arbitrarily differentiable. From (2.1) and [34] we can write

$$u = x + \sum_{r=1}^{\infty} \frac{C_r(x)}{r!} ((F(x; \lambda) - \Psi(x)) / \Psi^{(1)}(x))^r \quad (2.2)$$

where $C_1(x) = 1$ and

$$C_{r+1}(x) = \left[r \frac{(-\Psi^{(2)}(x))}{\Psi^{(1)}(x)} + D_x \right] C_r(x), \quad r = 1, 2, \dots$$

Under certain conditions, (2.2) can be expressed as

$$u = x + \sum_{r=1}^n \frac{C_r(x)}{r!} ((F(x; \lambda) - \Psi(x)) / \Psi^{(1)}(x))^r + O(\lambda^{-\frac{n+1}{2}}). \quad (2.3)$$

For example, (2.3) will be valid (see [18]) when $F(\cdot; \lambda)$ denotes the distribution function of the standardized sum

$$x = \frac{1}{\sqrt{\lambda}\sigma} \sum_{i=1}^{\lambda} (\xi_i - \mu) \quad (2.4)$$

where $\xi_1, \xi_2, \dots, \xi_\lambda$ are mutually independent random variables each having the same continuous distribution function and the limiting distribution, Ψ , is the unit normal. For other conditions for which (2.3) is valid see ([2], [3], [34], or [35]).

2.1 Generalized Cornish-Fisher Expansion

Let us assume that (1.9) is valid. Now if (1.9) is substituted into (2.3) and the terms of decreasing orders of λ are collected, the resulting expansion is referred to as the generalized Cornish-Fisher expansion. The expansion up to terms of $O(\lambda^{-9/2})$ is given below:

$$\begin{aligned} u = & x + \{B_1 Q_1\} \\ & + \{B_1 Q_2 + B_2 Q_1^2\} \\ & + \{B_1 Q_2 + B_2 Q_1^2\} \\ & + \{B_1 Q_3 + 2B_2 Q_1 Q_2 + B_3 Q_1^3\} \\ & + \{B_1 Q_4 + B_2 (2Q_1 Q_3 + Q_2^2) + B_3 Q_1^2 Q_2 + B_4 Q_1^4\} \\ & + \{B_1 Q_5 + B_2 (2Q_1 Q_4 + 2Q_2 Q_3) + B_3 (3Q_1^2 Q_3 + 3Q_1 Q_2^2) \end{aligned}$$

$$\begin{aligned}
& + 4B_4Q_1^3Q_2 + B_5Q_1^5\} \\
& + \{B_1Q_6 + B_2(2Q_1Q_5 + 2Q_2Q_4 + Q_3^2) + B_3(3Q_1^2Q_4 + 6Q_1Q_2Q_3 + Q_2^3) \\
& + B_4(4Q_1^3Q_3 + 6Q_1^2Q_2^2) + 5B_5Q_1^4Q_2 + B_6Q_1^6\} \\
& + \{B_1Q_7 + B_2(2Q_1Q_6 + 2Q_2Q_5 + 2Q_3Q_4) \\
& + B_3(3Q_1^2Q_5 + 6Q_1Q_2Q_4 + 3Q_2^2Q_3 + 3Q_1Q_3^2) \\
& + B_4(4Q_1^3Q_4 + 12Q_1^2Q_2Q_3 + 4Q_1Q_2^3) \\
& + B_5(5Q_1^4Q_3 + 10Q_1^3Q_2^2) + 6B_6Q_1^5Q_2 + B_7Q_1^7\} \\
& + \{B_1Q_8 + B_2(2Q_1Q_7 + 2Q_2Q_6 + 2Q_3Q_5 + Q_4^2) \\
& + B_3(3Q_1^2Q_6 + 6Q_1Q_2Q_5 + 3Q_2^2Q_4 + 6Q_1Q_3Q_4 + 3Q_2Q_3^2) \\
& + B_4(4Q_1^3Q_5 + 12Q_1^2Q_2Q_4 + 12Q_1Q_2^2 + 6Q_1^2Q_3^2 + Q_2^4) \\
& + B_5(5Q_1^4Q_4 + 20Q_1^3Q_2Q_3 + 10Q_1^2Q_2^3) \\
& + B_6(6Q_1^5Q_3 + 15Q_1^4Q_2^2) + 7B_7Q_1^6Q_2 + B_8Q_1^8\} + O(\lambda^{-9/2}),
\end{aligned} \tag{2.5}$$

where $B_r = \frac{C_r(x)}{r! (\Psi^{(1)}(x))^r}$ and the Q 's are as in (1.7).

It is possible that the limiting distribution, Ψ , may be other than the Normal; however, if $\Psi(x) =$

$\int_{-\infty}^x (2\pi)^{-1/2} \exp(-\frac{1}{2}x^2) dx$, expansion (2.5) reduces to the

Cornish-Fisher expansion given in (1.25).

2.2 Quantile Expansions With Unknown Cumulants

At this point one might note that to evaluate u in (2.2) the evaluation of F at x is required. For some functions F , this might be a difficult task. Also in order to evaluate u in (2.5), one must be able to evaluate the cumulants of F , which may for some functions be quite difficult. In an attempt to alleviate the above difficulties when the derivatives of F are more easily obtainable than the cumulants, let us consider the following approximation of u , denoted $\hat{u}_{n,L}$ defined as follows:

DEFINITION 2.1 Let $L \in \text{Natural Numbers}$, $L \geq n$. Then

$$\hat{u}_{n,L} = x + \sum_{r=1}^L \frac{C_r(x)}{r!} (\hat{D}_n(x; \lambda) / \psi^{(1)}(x))^r \quad (2.6)$$

where $\hat{D}_n$ is defined in (1.15) and $C_r(x)$ in (2.2).

THEOREM 2.1 Assume that (1.9) and (1.16) are valid, then

$$u - \hat{u}_{n,L} = T_L + \sum_{S=L+1}^{\infty} T_S \quad (2.7)$$

where $T_L = O(\lambda^{-\frac{n+1}{2}})$ and $T_S = O(\lambda^{-\frac{S}{2}})$ as $\lambda \rightarrow \infty$.

Proof: Using equation (2.2),

$$\begin{aligned}
u = x + \sum_{r=1}^L \frac{C_r(x)}{r!} ((F(x;\lambda) - \Psi(x))/\Psi^{(1)}(x))^r \\
+ \sum_{r=L+1}^{\infty} \frac{C_r(x)}{r!} ((F(x;\lambda) - \Psi(x))/\Psi^{(1)}(x))^r
\end{aligned} \tag{2.8}$$

Now (1.9) implies that

$$F(x;\lambda) - \Psi(x) = O(\lambda^{-\frac{1}{2}}) \tag{2.9}$$

and from the properties of the order function it follows that

$$(F(x;\lambda) - \Psi(x))^r = O(\lambda^{-\frac{r}{2}}). \tag{2.10}$$

Hence the terms of (2.8) such that $r = L+2, L+2, \dots$ are of order $O(\lambda^{-\frac{r}{2}})$. Now let us consider

$$\begin{aligned}
& \sum_{r=1}^L \frac{C_r(x)}{r!} ((F(x;\lambda) - \Psi(x))/\Psi^{(1)}(x))^r = \\
& \sum_{r=1}^L \frac{C_r(x)}{r! (\Psi^{(1)}(x))^r} (F(x;\lambda) - \hat{F}_n(x;\lambda) + \hat{F}_n(x;\lambda) - \Psi(x))^r = \\
& \sum_{r=1}^L \frac{C_r(x)}{r!} ((\hat{F}_n(x;\lambda) - \Psi(x))/\Psi^{(1)}(x))^r + T_L \tag{2.11}
\end{aligned}$$

where

$$T_L = \sum_{r=1}^L \sum_{k=1}^r \frac{C_r(x)}{r! (\Psi^{(1)}(x))^r} \binom{r}{k} (F(x; \lambda) - \hat{F}_n(x; \lambda))^k (\hat{F}_n(x; \lambda) - \Psi(x))^{r-k} \quad (2.12)$$

and where $r \geq k$.

It follows from (1.16) that

$$(F(x, \lambda) - \hat{F}_n(x; \lambda))^k = O(\lambda^{-\frac{1}{2}(n+1)k}). \quad (2.13)$$

Using (1.16) and (2.9),

$$\hat{F}_n(x; \lambda) - \Psi(x) = (\hat{F}_n(x; \lambda) - F(x; \lambda)) + (F(x; \lambda) - \Psi(x)) = O(\lambda^{-1/2}) \quad (2.14)$$

and therefore

$$(\hat{F}_n(x; \lambda) - \Psi(x))^{r-k} = O(\lambda^{-\frac{1}{2}(r-k)}). \quad (2.15)$$

From (2.13) and (2.15), it follows that

$$(F(x; \lambda) - \hat{F}_n(x; \lambda))^k (\hat{F}_n(x; \lambda) - \Psi(x))^{r-k} = O(\lambda^{-\frac{1}{2}(nk+r)}). \quad (2.16)$$

Now observing that the maximum of $-\frac{1}{2}(nk+r)$ occurs when $r=k=1$, it follows from (2.12) that

$$T_L = O(\lambda^{-\frac{n+1}{2}}). \quad (2.17)$$

Hence from (2.8) and (2.11) we obtain

$$u = x + \sum_{r=1}^L \frac{C_r(x)}{r!} ((\hat{F}_n(x; \lambda) - \psi(x)) / \psi^{(1)}(x))^r + T_L + \sum_{S=L+1}^{\infty} T_S \quad (2.18)$$

where

$$T_S = \frac{C_S(x)}{S!} ((F(x; \lambda) - \psi(x)) / \psi^{(1)}(x))^S. \quad (2.19)$$

Now since $\hat{D}_n = \hat{F}_n - \psi$, (2.19) reduces to $u - u_{n,L} = T_L + \sum_{S=L+1}^{\infty} T_S$

and the Theorem is established.

THEOREM 2.2 Assume Theorem 2.1 holds and that (2.3) is valid. Then

$$u - \hat{u}_{n,L} = O(\lambda^{-\frac{n+1}{2}}) \quad (2.20)$$

as $\lambda \rightarrow \infty$.

Proof: Using equation (2.3), we have

$$u - \hat{u}_{n,L} = T_L + O(\lambda^{-\frac{L+1}{2}}) \quad (2.21)$$

where T_L is defined in equation (2.12). It now follows from (2.17), (2.21), and the fact that $L \geq n$, that

$$u - \hat{u}_{n,L} = O(\lambda^{-\frac{n+1}{2}}) \text{ which establishes the theorem. Q.E.D.}$$

At this point a few comments are in order concerning equation (2.20). From (2.20) we see that the order of

$u - \hat{u}_{n,L}$ depends only explicitly on n . However, for finite λ , the value of $\hat{u}_{n,L}$ will depend on L and some values of L may be better than others. Ideally one would like to choose L to minimize the error $|u - \hat{u}_{n,L}|$. Unfortunately however no results have been established, other than examples on specific functions. No attempt to establish an optimal value of L will be made here and only those values of L defined by the following definition are considered. The choice of these values of L , i.e. $n+1$, and n , was based on experience rather than theory.

DEFINITION 2.2 Let $\hat{u}_n$ be defined by

$$\hat{u}_n = x + \sum_{r=1}^{2[\frac{n+1}{2}]} \frac{C_r(x)}{r!} (\hat{D}_n(x; \lambda) / \psi^{(1)}(x))^r \quad (2.22)$$

where $[w] = \text{greatest integer } \leq w$.

We have immediately from Definition 2.1, Theorem 2.2, and Definition 2.2 that

$$u - \hat{u}_n = O(\lambda^{-\frac{n+1}{2}}) \quad (2.23)$$

as $\lambda \rightarrow \infty$.

It should be noted at this point that if ψ is the unit normal distribution then (2.22) is the normalizing expansion similar to the Cornish-Fisher expansion given in (1.25). However, to calculate $\hat{u}_n$ we do not need to

evaluate the cumulants of $F(\cdot; \lambda)$ as in the Cornish-Fisher expansion, but instead, we need to be able to evaluate the k derivatives of F (see 1.14 and 1.15).

An example is now in order.

EXAMPLE 2.1 Let $\xi_1, \xi_2, \dots, \xi_\lambda$ be mutually independent random variables each distributed as exponential with $E(\xi_i) = 1$. Then

$$Z = \sum_{i=1}^{\lambda} \xi_i \quad (2.24)$$

is distributed as gamma $(\lambda, 1)$ with density

$$g(z) = \begin{cases} \frac{1}{\Gamma(\lambda)} z^{\lambda-1} \exp(-z), & \text{for } z > 0 \\ 0 & , \text{ otherwise} \end{cases} \quad (2.25)$$

Now let

$$X = \frac{Z - \lambda}{(\lambda)^{1/2}} = \frac{1}{(\lambda)^{1/2}} \sum_{i=1}^{\lambda} (\xi_i - 1) \quad , \quad (2.26)$$

which is a standardized sum and the distribution function of X is

$$F(x; \lambda) = \int_{-\infty}^x \lambda^{\frac{1}{2}} g(t\lambda^{\frac{1}{2}} + \lambda) dt \quad (2.27)$$

where g is defined in (2.25). Thus F is the gamma distribution in standard measure whose cumulants are

$$k_i = \begin{cases} 0 & , i = 1 \\ 1 & , i = 2 \\ \lambda^{1-\frac{i}{2}}(i-1)! & , i = 3, 4, \dots \end{cases} \quad (2.28)$$

and $F(x; \lambda) \rightarrow \Psi(x)$ as $\lambda \rightarrow \infty$ for all real x where

$$\Psi(x) = \Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^x \exp(-t^2/2) dt. \quad (2.29)$$

The cumulants of Φ are

$$\alpha_i = \begin{cases} 1 & , i = 2 \\ 0 & , i = 1, 3, 4, \dots \end{cases} \quad (2.30)$$

and therefore

$$\beta_i = \begin{cases} 0 & , i = 1, 2 \\ \lambda^{1-\frac{i}{2}}(i-1) & , i = 3, 4, \dots \end{cases} \quad (2.31)$$

Now it is clear that the $\{\beta_r\}$ satisfy the order relations of (1.6) and in Wasow [51], the expansion (1.9) with Ψ and F defined in (2.27) and (2.29) is shown to be an asymptotic expansion valid uniformly in x . It is pointed out by Coberly [9] that $F - \hat{F}_n = O(\lambda^{-\frac{n+1}{2}})$ for this example and from Draper and Tierney [18], (2.3) is valid. Therefore, conditions are satisfied such that $\hat{u}_n$ exists and such that (2.23) is valid.

Let us now compare the $\hat{u}_n$ expansion with the Cornish-Fisher expansion for estimating F . The symbol u_{cn} will refer to the sum of the terms of (1.25) up to including terms of order $O(\lambda^{-\frac{n}{2}})$. Consider (2.1) which states, in this example

$$F(x; \lambda) = \Phi(u). \quad (2.32)$$

Now $F(x; \lambda)$ can be estimated by

$$\hat{F}(x; \lambda) = \Phi(\hat{u}_n) \quad (2.33)$$

or

$$\sim F(x; \lambda) = \Phi(u_{cn}) \quad (2.34)$$

Several comparisons could be made but the one of interest here is $|F(x; \lambda) - \Phi(\hat{u}_n)|$ with $|F(x; \lambda) - \Phi(u_{cn})|$. Cal-

culations will now be carried out for $n = 1, 2, 3, 4$ in equations (2.33) and (2.34) respectively. In the following $\hat{D}_n(x; \lambda)$ is defined in (1.14).

For $n = 1$,

$$\hat{u}_1 = x + \sum_{r=1}^2 \frac{C_r(x)}{r!} (D_1(x; \lambda) / \phi^{(1)}(x))^r \quad (2.35)$$

where $C_1(x) = 1$, $C_2(x) = x$, and $u = \hat{u}_1 + O(\lambda^{-1})$. The corresponding Cornish-Fisher expansion is

$$u_{c1} = x - \frac{1}{6}\beta_3(x^2 - 1) \quad (2.36)$$

where $u = u_{c1} + O(\lambda^{-1})$ and β_3 is defined in (2.31). Table 2.1 compares $\hat{u}_1$ with u_{c1} .

For $n = 2$,

$$\hat{u}_2 = x + \sum_{r=1}^2 \frac{C_r(x)}{r!} (\hat{D}_2(x; \lambda) / \phi^{(1)}(x))^r \quad (2.37)$$

where $C_1(x) = 1$, $C_2(x) = x$ and $u = \hat{u}_2 + O(\lambda^{-3/2})$. The corresponding Cornish-Fisher expansion is

$$u_{c2} = x - \frac{1}{6}\beta_3(x^2 - 1) - \frac{1}{24}\beta_4(x^3 - 3x) + \frac{1}{36}\beta_3^2(4x^3 - 7x) \quad (2.38)$$

where $u = u_{c2} + O(\lambda^{-3/2})$. The results $\hat{u}_2$ compared with u_{c2} are given in Table 2.1.

For $n = 3$,

$$\hat{u}_3 = x + \sum_{r=1}^4 \frac{C_r(x)}{r!} (\hat{D}_3(x; \lambda) / \phi^{(1)}(x))^r \quad (2.39)$$

where $C_1(x) = 1$, $C_2(x) = x$, $C_3(x) = 2x^2+1$, $C_4(x) = 6x^3+7x$ and $u = \hat{u}_3 + O(\lambda^{-2})$. The corresponding Cornish-Fisher expansion is

$$\begin{aligned} u_{c3} = & x - \frac{1}{6}\beta_3(x^2-1) - \frac{1}{24}\beta_4(x^3-3x) \\ & + \frac{1}{36}\beta_3^2(4x^3-7x) - \frac{1}{120}\beta_5(x^4-6x^2+3) \\ & + \frac{1}{144}\beta_3\beta_4(11x^4-42x^2+52) \end{aligned} \quad (2.40)$$

where $u = u_{c3} + O(\lambda^{-2})$. Table 2.1 compares $\hat{u}_3$ with u_{c3} .

For $n = 4$,

$$\hat{u}_4 = x + \sum_{r=1}^4 \frac{C_r(x)}{r!} (\hat{D}_4(x; \lambda) / \phi^{(1)}(x))^r \quad (2.41)$$

where $C_1(x) = 1$, $C_2(x) = x$, $C_3(x) = 2x^2+1$, $C_4(x) = 6x^3+7x$ and $u = \hat{u}_4 + O(\lambda^{-5/2})$. The Cornish-Fisher expansion which corresponds to $\hat{u}_4$ is

$$u_{c4} = x - \frac{1}{6}\beta_3(x^2-1) - \frac{1}{24}\beta_4(x^3-3x)$$

$$\begin{aligned}
& + \frac{1}{36} \beta_3^2 (4x^3 - 7x) - \frac{1}{120} \beta_5 (x^4 - 6x^2 + 3) \\
& + \frac{1}{144} \beta_3 \beta_4 (11x^4 - 42x^2 + 15) \\
& - \frac{1}{648} \beta_3^3 (69x^4 - 187x^2 + 52) \\
& - \frac{1}{720} \beta_6 (x^5 - 10x^3 + 15x) \\
& + \frac{1}{384} \beta_4^2 (5x^5 - 32x^3 + 35x) \\
& + \frac{1}{360} \beta_3 \beta_5 (7x^5 - 48x^3 + 51x) \\
& - \frac{1}{864} \beta_3^2 \beta_4 (111x^5 - 547x^3 + 456x) \\
& + \frac{1}{7776} \beta_3^4 (948x^5 - 3628x^3 + 2473x)
\end{aligned} \tag{2.42}$$

where $u = u_{c4} + O(\lambda^{-5/2})$.

Table 2.1 gives a comparison of $\hat{u}_4$ and u_{c4} .

TABLE 2.1

λ	x	F	$\phi(\hat{u}_1)$	$\phi(u_{c1})$	$ F - \phi(\hat{u}_1) $	$ F - \phi(u_{c1}) $
5	1	.84748	.84134	.84134	.61308(-2)	.61308(-2)
	2	.95902	.97174	.93976	.12716(-1)	.19261(-1)
	3	.99069	.99977	.96465	.90768(-2)	.26038(-1)
	4	.99812	1.00000	.96113	.18809(-2)	.36991(-1)
15	1	.84362	.84134	.84134	.22738(-2)	.22738(-2)
	2	.96528	.97089	.95923	.56120(-2)	.60537(-2)
	3	.99442	.99860	.98960	.41810(-2)	.48199(-2)
	4	.99930	1.00000	.99663	.69867(-3)	.26756(-2)
25	1	.84276	.84134	.84134	.14132(-2)	.14132(-2)
	2	.96763	.97132	.96407	.36940(-2)	.35562(-2)
	3	.99552	.99805	.99318	.25361(-2)	.23362(-2)
	4	.99955	1.00000	.99865	.44925(-3)	.90065(-3)
100	1	.84172	.84134	.84134	.38344(-3)	.38344(-3)
	2	.97214	.97322	.97128	.10816(-2)	.85277(-3)
	3	.99725	.99774	.99687	.49156(-3)	.38441(-3)
	4	.99984	1.00000	.99977	.15653(-3)	.71556(-4)

TABLE 2.1--continued.

λ	x	F	$\hat{\phi}(u_2)$	$\phi(u_{c2})$	$ F - \hat{\phi}(u_2) $	$ F - \phi(u_{c2}) $
5	1	.84748	.84599	.84928	.14874(-2)	.18005(-2)
	2	.95902	.96553	.96804	.65096(-2)	.90198(-2)
	3	.99069	1.00000	.99775	.93090(-2)	.70594(-2)
	4	.99812	1.00000	.99999	.18809(-2)	.18693(-2)
15	1	.84362	.84325	.84402	.36536(-3)	.39985(-3)
	2	.96528	.96750	.96725	.22168(-2)	.19658(-2)
	3	.99442	.99948	.99605	.50596(-2)	.16289(-2)
	4	.99930	1.00000	.99979	.69867(-3)	.49185(-3)
25	1	.84276	.84257	.84295	.18425(-3)	.19453(-3)
	2	.96763	.96886	.96856	.12387(-2)	.93135(-3)
	3	.99552	.99865	.99624	.31369(-2)	.72758(-3)
	4	.99955	1.00000	.99976	.44925(-3)	.21073(-3)
100	1	.84172	.84169	.84175	.33594(-4)	.19503(-4)
	2	.97214	.97235	.97225	.21194(-3)	.11754(-3)
	3	.99725	.99770	.99732	.45278(-3)	.74042(-4)
	4	.99984	1.00000	.99985	.16044(-3)	.15658(-4)

TABLE 2.1--continued.

λ	x	F	$\phi(\hat{u}_3)$	$\phi(u_{c3})$	$ F - \phi(\hat{u}_3) $	$ F - \phi(u_{c3}) $
5	1	.84748	.84620	.84678	.12793(-2)	.69265(-3)
	2	.95902	.96090	.95186	.18682(-2)	.71665(-2)
	3	.99069	1.00000	.96919	.93095(-2)	.21499(-1)
	4	.99812	1.00000	.87458	.18809(-2)	.12354(0)
15	1	.84362	.84341	.84353	.21133(-3)	.88814(-4)
	2	.96528	.96564	.96449	.36105(-3)	.79572(-3)
	3	.99442	1.00000	.99322	.55836(-2)	.11927(-2)
	4	.99930	1.00000	.99835	.69867(-3)	.95280(-3)
25	1	.84276	.84267	.84272	.86663(-4)	.33470(-4)
	2	.96763	.96778	.96734	.15276(-3)	.28769(-3)
	3	.99552	.99992	.99515	.44021(-2)	.36628(-3)
	4	.99955	1.00000	.99935	.44925(-3)	.19700(-3)
100	1	.84172	.84171	.84172	.13703(-4)	.91285(-5)
	2	.97214	.97215	.97211	.12776(-4)	.17940(-4)
	3	.99725	.99752	.99723	.27269(-3)	.17000(-4)
	4	.99984	1.00000	.99983	.16107(-3)	.49828(-5)

TABLE 2.1--continued.

λ	x	F	$\hat{\phi}(u_4)$	$\phi(u_{C4})$	$ F - \hat{\phi}(u_4) $	$ F - \phi(u_{C4}) $
5	1	.84748	.84885	.84774	.13732(-2)	.26875(-3)
	2	.95902	.96054	.95186	.15153(-2)	.45366(-2)
	3	.99069	1.00000	.99790	.93096(-2)	.72143(-2)
	4	.99812	1.00000	1.00000	.18809(-2)	.18809(-2)
15	1	.84362	.84381	.84364	.19152(-3)	.85482(-4)
	2	.96528	.96551	.96561	.22590(-3)	.32300(-3)
	3	.99442	1.00000	.99505	.55836(-2)	.63491(-3)
	4	.99930	1.00000	.99967	.69867(-3)	.36710(-3)
25	1	.84276	.84283	.84276	.68747(-4)	.57344(-5)
	2	.96763	.96771	.96772	.81848(-4)	.91387(-4)
	3	.99552	.99993	.99568	.44144(-4)	.16540(-4)
	4	.99955	1.00000	.99964	.44925(-3)	.94041(-4)
100	1	.84172	.84172	.84172	.34894(-5)	.66678(-5)
	2	.97214	.97214	.97214	.41355(-5)	.29336(-5)
	3	.99725	.99752	.99725	.27254(-3)	.40761(-5)
	4	.99984	1.00000	.99984	.16107(-3)	.15754(-5)

Now, from observing Table 2.1, it is obvious that both approximations, $\hat{u}_n$ and u_{cn} , compare favorably and can be used to obtain reasonable approximations of F . For small λ , $\hat{u}_n$ approximations seem to be in most cases more favorable than the Cornish-Fisher u_{cn} approximations. However for larger λ , $\lambda = 25, 100$; the u_{cn} approximations may be slightly more accurate, especially for $x = 3, 4$.

It might be pointed out that other empirical results [50] have indicated that some of the best applications of the u_{cn} approximations have been to members of the Gamma family. Hence, let us consider an example of a distribution which is not a member of the Gamma family.

Example 2.2

Let

$$\Psi(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^x e^{-\frac{v^2}{2}} dv \quad (2.43)$$

and

$$F(x; \lambda) = G_{\lambda}(x\sigma) \quad (2.44)$$

where

$$\sigma = \sqrt{\frac{\lambda}{\lambda-2}} \quad (2.45)$$

and

$$G_{\lambda}(t) = \int_{-\infty}^t \frac{1}{\sqrt{\lambda\pi}} \frac{\Gamma(\frac{1}{2}(\lambda+1))}{\Gamma(\frac{1}{2}\lambda)} \left(1 + \frac{w^2}{\lambda}\right)^{-\frac{1}{2}(\lambda+1)} dw. \quad (2.46)$$

That is, F is the t -distribution in standard measure. The moments for G_{λ} are

$$\mu_m = \begin{cases} 0 & (m \text{ odd}) \\ \frac{1 \cdot 3 \cdots (m-1) \lambda^{m/2}}{(\lambda-2)(\lambda-4) \cdots (\lambda-m)} & (m \text{ even}) \end{cases} \quad (2.47)$$

which yields cumulants for F to be $k_1=0, k_2=1, k_3=0, k_4=\frac{6}{\lambda-4}, k_5=0, k_6=\frac{240}{(\lambda-4)(\lambda-6)}$, and etc.

The cumulants of Ψ are

$$\alpha_i = \begin{cases} 1 & i = 2 \\ 0 & i=1,3,4,\dots \end{cases} \quad (2.48)$$

and letting $\beta_i = k_i - \alpha_i$, the properties of (1.6) are satisfied. We have that $F(x;\lambda) \rightarrow \Psi(x)$ and, due to Wasow [51], the conditions are satisfied such that (1.9) is valid and such that $F - \hat{F}_n = O(\lambda^{-\frac{n+1}{2}})$. Also, from Draper and Tierney [18], (2.3) is valid, hence (2.23) is valid.

Let us now compare the $\hat{u}_n$ expansion with the Cornish-Fisher expansion u_{cn} for estimating F . Since $k_r = 0$ for r odd, a comparison will only be made for $n = 2$ and $n = 4$. It might be noted that in this example,

$$\beta_1=0, \beta_2=0, \beta_3=0, \beta_4 = \frac{6}{\lambda-4}, \beta_5=0, \text{ and } \beta_6 = \frac{240}{(\lambda-4)(\lambda-6)}. \quad (2.49)$$

For $n = 2$,

$$\hat{u}_2 = x + \sum_{r=1}^2 \frac{C_r(x)}{r!} (\hat{D}_2(x; \lambda) / \psi^{(1)}(x))^r \quad (2.50)$$

and

$$u_{c2} = x - \frac{1}{24} \beta_4 (x^3 - 3x). \quad (2.51)$$

We compare $\hat{u}_2$ with $\hat{u}_{c2}$ in Table 2.2.

For $n = 4$,

$$\hat{u}_4 = x + \sum_{r=1}^4 \frac{C_r(x)}{r!} (\hat{D}_4(x; \lambda) / \psi^{(1)}(x))^r \quad (2.52)$$

and

$$\begin{aligned} u_{c4} = x - \frac{1}{24} \beta_4 (x^3 - 3x) \\ - \frac{1}{720} \beta_6 (x^5 - 10x^3 + 15x) \end{aligned} \quad (2.53)$$

$$+ \frac{1}{384} \beta_4^2 (5x^5 - 32x^3 + 35x) .$$

We compare $\hat{u}_4$ with $\hat{u}_{c4}$ in Table 2.2,

TABLE 2.2

λ	x	F	$\phi(\hat{u}_2)$	$\phi(u_{c2})$	$ F - \phi(\hat{u}_2) $	$ F - \phi(u_{c2}) $
8	1	.85923	.86012	.86971	.88435(-3)	.10474(-1)
	2	.97513	.97361	.96960	.15229(-2)	.55286(-2)
	3	.99574	.99772	.96960	.22587(-1)	.26138(-1)
	4	.99914	1.00000	.77337	.85731(-3)	.22577(0)
15	1	.85014	.85035	.85209	.21427(-3)	.19545(-2)
	2	.97579	.97537	.97468	.41616(-3)	.11051(-2)
	3	.99719	.99777	.99521	.62148(-3)	.19386(-2)
	4	.99968	1.00000	.99759	.31776(-3)	.20971(-2)
25	1	.84644	.84651	.84704	.71744(-4)	.59943(-3)
	2	.97629	.97614	.97593	.14734(-3)	.35461(-3)
	3	.99779	.99800	.99733	.20748(-3)	.45913(-3)
	4	.99984	.99999	.99964	.15087(-3)	.20138(-3)
100	1	.84257	.84257	.84260	.41168(-5)	.31827(-4)
	2	.97699	.97698	.97697	.90464(-5)	.19873(-4)
	3	.99845	.99846	.99843	.11062(-4)	.18600(-4)
	4	.99995	.99995	.99994	.38694(-5)	.32512(-5)

TABLE 2.2--continued.

λ	x	F	$\phi(\hat{u}_4)$	$\phi(u_{c4})$	$ F - \phi(\hat{u}_4) $	$ F - \phi(u_{c4}) $
8	1	.85923	.85815	.82170	.10775(-2)	.27054(-2)
	2	.97513	.97477	.99329	.35956(-3)	.18162(-1)
	3	.99574	.99885	.99993	.23719(-2)	.41849(-2)
	4	.99914	1.00000	.85776	.85731(-3)	.14138(0)
15	1	.85014	.85000	.84884	.13003(-3)	.13038(-3)
	2	.97579	.97573	.97693	.56756(-4)	.11922(-2)
	3	.99719	.99752	.99803	.37046(-3)	.88182(-3)
	4	.99968	1.00000	.99993	.31776(-3)	.24880(-3)
25	1	.84644	.84641	.84625	.25405(-4)	.18419(-3)
	2	.97629	.97628	.97647	.12537(-3)	.17872(-3)
	3	.99779	.99783	.99793	.43588(-4)	.14348(-3)
	4	.99984	1.00000	.99989	.15980(-3)	.48919(-4)
100	1	.84257	.84257	.84257	.30888(-6)	.19349(-5)
	2	.97699	.97699	.97699	.20117(-6)	.19630(-5)
	3	.99845	.99845	.99845	.11585(-4)	.15303(-5)
	4	.99995	.99995	.99995	.30888(-5)	.34374(-6)

Now, from observing Table 2.2, again we find that $\hat{u}_n$ and u_{cn} compare quite favorably. In fact for this example, the $\hat{u}_n$ approximations seem to be more accurate than the Cornish-Fisher u_{cn} approximations for almost all cases, the exceptions being in the extreme tails when $\lambda = 100$.

Before leaving this example, it should be pointed out that for the m -th moment of the Student- t to exist we must have $\lambda > m$. Hence this affects the cumulants, in fact β_4 and β_6 do not exist when $\lambda = 4$ and $\lambda = 6$ respectively. Therefore, equations (2.51) and (2.53) cannot be evaluated for these values of λ . We can however formally evaluate the expansions (2.50) and (2.52) for $\lambda = 4$ and $\lambda = 6$ and obtain the results listed in Table 2.3.

TABLE 2.3

λ	x	F	$\phi(\hat{u}_2)$	$ F - \phi(\hat{u}_2) $
4	1	.88490	.88982	.49247(-2)
	2	.97629	.96953	.67590(-2)
	3	.99338	.99762	.43807(-2)
	4	.99759	1.00000	.24063(-2)
6	1	.86672	.86848	.17629(-2)
	2	.97509	.97229	.27946(-2)
	3	.99480	.99779	.29877(-2)
	4	.99864	.99692	.17278(-2)
λ	x	F	$\phi(\hat{u}_4)$	$ F - \phi(\hat{u}_4) $
4	1	.88490	.86983	.15066(-1)
	2	.97629	.97353	.27551(-2)
	3	.99338	.99995	.65715(-2)
	4	.99759	1.00000	.24063(-2)
6	1	.86672	.86367	.30417(-2)
	2	.97509	.97425	.83476(-3)
	3	.99480	.99975	.79346(-2)
	4	.99864	1.00000	.13568(-2)

From observing Table 2.3, it is aparent that we again obtain favorable results. In fact, in most cases the results are accurate to at least two decimal places.

Hence, Examples 2.1 and 2.2 indicate that $\hat{u}_n$, like u_{cn} , can be used with confidence in its accuracy. It is certainly a welcome alternative when one cannot evaluate the cumulants, and in some instances it may be more desirable when one does know the cumulants.

Example 2.1 does motivate the need for a modification of $\hat{u}_n$ that might improve the approximation result in the extreme tails of the distribution. This problem will be considered in the following section.

2.3 Approximations For the Tails of F.

DEFINITION 2.3

For any arbitrary functions g_{1j} , define $\hat{F}_{g_2}$ as

$$\begin{aligned} \hat{F}_{g_2}(x; \lambda) = & \psi(x) - g_{12}(x) \frac{M_{12}}{M_{11}} (\psi^{(1)}(x) - F^{(1)}(x; \lambda)) \\ & + g_{13}(x) \frac{M_{13}}{M_{11}} (\psi^{(2)}(x) - F^{(2)}(x; \lambda)) \\ & - g_{14}(x) \frac{M_{14}}{M_{11}} (\psi^{(3)}(x) - F^{(3)}(x; \lambda)) \end{aligned}$$

where M_{ij} is the Minor of the element in the i -th row and j -th column of the determinant

$\Psi(x)$	$\Psi^{(1)}(x) - F^{(1)}(x; \lambda)$	$\Psi^{(2)}(x) - F^{(2)}(x; \lambda)$	$\Psi^{(3)}(x) - F^{(3)}(x; \lambda)$
$\Psi^{(3)}(x)$	$\Psi^{(4)}(x)$	$\Psi^{(5)}(x)$	$\Psi^{(6)}(x)$
$\Psi^{(4)}(x)$	$\Psi^{(5)}(x)$	$\Psi^{(6)}(x)$	$\Psi^{(7)}(x)$
$\Psi^{(6)}(x)$	$\Psi^{(7)}(x)$	$\Psi^{(8)}(x)$	$\Psi^{(9)}(x)$

THEOREM 2.3

If

$$\lim_{x \rightarrow \infty} \frac{\Psi^{(i)}(x) - F^{(i)}(x; \lambda)}{\Psi(x) - F(x; \lambda)} = K_i \neq 0 \quad i = 1, 2, 3 \quad (2.54)$$

and

$$g_{1,i+1}(x) = \frac{1}{K_i} \frac{M_{11}}{M_{1,i+1}} \quad (2.55)$$

then

$$\hat{F}_{g_2}(x; \lambda) - F(x; \lambda) = O(\Psi(x) - F(x; \lambda)) \quad (2.56)$$

as $x \rightarrow \infty$ for fixed λ .**Proof:**

$$\lim_{x \rightarrow \infty} \frac{\hat{F}_{g_2} - F}{\Psi - F} = 1 - \frac{1}{K_1} \lim_{x \rightarrow \infty} \frac{\Psi^{(1)} - F^{(1)}}{\Psi - F}$$

$$\begin{aligned}
& + \frac{1}{K_2} \lim_{x \rightarrow \infty} \frac{\Psi^{(2)} - F^{(2)}}{\Psi - F} \\
& - \frac{1}{K_3} \lim_{x \rightarrow \infty} \frac{\Psi^{(3)} - F^{(3)}}{\Psi - F} \\
& = 1 - 1 + 1 - 1 \\
& = 0 .
\end{aligned}$$

Now consider the general procedure for constructing approximations of $F(x; \lambda)$ for large values of x . Let $\Psi(x)$ be chosen such that (2.54) holds and let $g_{1,i+1}$, $i=1,2,3$, be given by (2.55). Then

$$\begin{aligned}
\hat{F}_{g_2}(x; \lambda) &= \Psi(x) - \frac{1}{K_1}(\Psi^{(1)}(x) - F^{(1)}(x; \lambda)) \\
&+ \frac{1}{K_2}(\Psi^{(2)}(x) - F^{(2)}(x; \lambda)) \\
&- \frac{1}{K_3}(\Psi^{(3)}(x) - F^{(3)}(x; \lambda)) .
\end{aligned} \tag{2.57}$$

Example 2.3 Now let us evaluate K_1, K_2 , and K_3 when F is defined as in Example 2.1. Here we find that $K_1 = -\lambda^{1/2}$, $K_2 = \lambda$, and $K_3 = -\lambda^{3/2}$, giving

$$\begin{aligned}\hat{F}_{g_2} = & \psi + \lambda^{-1/2} (\psi^{(1)} - F^{(1)}) + \lambda^{-1} (\psi^{(2)} - F^{(2)}) \\ & + \lambda^{-3/2} (\psi^{(3)} - F^{(3)}). \end{aligned} \quad (2.58)$$

Now substituting $\hat{F}_{g_2}$ into expansion (2.3) and grouping terms of appropriate order, the quantile expansion through terms of order $O(\lambda^{-3/2})$ is obtained:

$$\hat{u}_g = x + \frac{-1}{\phi^{(1)}(x)} \left[\frac{1}{\lambda^{1/2}} (F^{(1)} - \phi^{(1)}) + \frac{1}{\lambda} (F^{(2)} - \phi^{(2)}) \right]. \quad (2.59)$$

Let us now consider $\phi(\hat{u}_g)$ as an approximation of $F(x; \lambda)$ for the following values.

TABLE 2.4.

λ	x	F	$\Phi(\hat{u}_g)$	$ F - \Phi(\hat{u}_g) $
15	1	.84362	.84822	.45966(-2)
	2	.96528	.97460	.93194(-2)
	3	.99442	.99675	.23425(-2)
	4	.99930	.99584	.34567(-2)
25	1	.84276	.84594	.31807(-2)
	2	.96763	.97573	.81046(-2)
	3	.99552	.99759	.20745(-2)
	4	.99955	.99923	.32391(-3)
100	1	.84172	.84271	.98573(-3)
	2	.97214	.97689	.47526(-2)
	3	.99725	.99840	.11500(-2)
	4	.99984	.99993	.88122(-4)

Now, from observing Table 2.4, we find that $\hat{u}_g$ is more accurate than the $\hat{u}_n$ approximations when $x = 4$ and $\lambda = 25, 100$. However for $x = 1, 2, 3$ the approximation is not as good, as would be expected.

CHAPTER III
GENERALIZED INVERSE CORNISH-FISHER TYPE
EXPANSIONS AND APPROXIMATIONS

Again, let $F(\cdot; \lambda)$ and Ψ be probability distribution functions satisfying the limiting condition given in (1.1). Also let x and u be corresponding quantiles of $F(\cdot; \lambda)$ and Ψ respectively such that

$$F(x; \lambda) = \Psi(u). \quad (3.1)$$

In Chapter II, we considered the problem of solving for u in terms of x . In this chapter we investigate the inverse problem, that is solving for x in terms of u . An exact expression for x in terms of u will be obtained by the method given in Hill and Davis [35]. After this, the appropriate approximations will be considered.

Assume that the density function $\Psi^{(1)}$ of the distribution Ψ is arbitrarily differentiable. Let

$$Z_{\lambda}(x) = F(x; \lambda) - \Psi(x) = \int_x^u \Psi^{(1)}(t) dt. \quad (3.2)$$

Lagrange's inversion formula can now be used. The formula states that under certain conditions (see [52], p. 133) that if h and k are analytic functions and

$$w = v + h(w) \quad (3.3)$$

then

$$k(w) = k(v) + \sum_{r=1}^{\infty} D_v^{r-1} [k'(v) (h(v))^r] / r! \quad (3.4)$$

Now consider

$$\psi(x) = \psi(u) - z_{\lambda}(x) \quad (3.5)$$

and define $v = \psi(u)$ and $w = \psi(x)$. Then

$$w = v - z_{\lambda}(\psi^{-1}(w)) \quad (3.6)$$

where ψ^{-1} is the inverse function of ψ . Now (3.6) is in the form of (3.3) and the functional equation (3.6) can be solved for $\psi^{-1}(w) = x$ by setting $k = \psi^{-1}$ in (3.4):

$$\psi^{-1}(w) = \psi^{-1}(v) + \sum_{r=1}^{\infty} (-1)^r (r!)^{-1} D_v^{r-1} \{ [z_{\lambda}(\psi^{-1}(v))]^r / \psi^{(1)}(\psi^{-1}(v)) \}$$

which is equivalent to

$$x = u - \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r! [\psi^{(1)}(u)]^{r-1}} D_u^{r-1} \{ (F(u, \lambda) - \psi(u))^r / \psi^{(1)}(u) \}. \quad (3.7)$$

Under certain conditions (see [34], [2], [3], or [43]), for example if $F(\cdot; \lambda)$ is the distribution of the standardized sum given in (2.4), expansion (3.7) can be written as

$$x=u-\sum_{r=1}^n \frac{(-1)^{r-1}}{r! (\psi^{(1)}(u))^{r-1}} D_u^{r-1} \left\{ \frac{(F(u; \lambda) - \psi(u))^r}{\psi^{(1)}(u)} \right\} + O(\lambda^{-\frac{n+1}{2}}) \quad (3.8)$$

as $\lambda \rightarrow \infty$.

3.1 Generalized Inverse Cornish-Fisher Expansion

Again let us assume that (1.9) and (1.10) are valid. That is, $F(\cdot; \lambda)$ can be written as

$$F(x; \lambda) = \psi(x) + \sum_{i=1}^n Q_i(x; \lambda) + R_n \quad (3.9)$$

where $R_n = O(\lambda^{-\frac{n+1}{2}})$.

Now substituting (3.9) into (3.8) and collecting terms of decreasing orders of λ , we have the Generalized Inverse Cornish-Fisher Expansion up to terms of order $O(\lambda^{-3/2})$, which is

$$x = u - \frac{1}{\psi^{(1)}} [Q_1^{(0)} + Q_2^{(0)}] + \left(\frac{1}{\psi^{(1)}}\right)^2 (Q_1^{(0)} Q_1^{(1)}) + \frac{-\psi^{(2)}}{2(\psi^{(1)})^3} (Q_1^{(0)})^2 + O(\lambda^{-3/2}) \quad (3.10)$$

where

$$Q_i^{(0)} = Q_i(u; \lambda),$$

$$Q_i^{(j)} = D_u^{(j)} Q_i(u; \lambda), \quad j = 1, 2, \dots$$

and the Q 's are defined in (1.7).

Actually, by carefully selecting terms $Q_i^{(j)}$ of appropriate order, (3.10) can be expanded to terms of $O(\lambda^{-\frac{n+1}{2}})$.

Again, one might note that the limiting distribution may be other than the Normal. However, if $\Psi(x) = \Phi(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-1/2x^2} dx$, expansion (3.10) reduces to the inverse Cornish-Fisher expansion up to terms of $O(\lambda^{-3/2})$ given in (1.26).

3.2 Inverse Quantile Expansions with Unknown Cumulants

Let us consider the following approximation of x , denote $\hat{x}_n$, defined as follows:

DEFINITION 3.1: Assuming $\hat{D}_n$ exists, where $\hat{D}_n$ is as in (1.15), $\hat{x}_n$ is defined by

$$\hat{x}_n = u + \sum_{r=1}^n A_r(u) E_{r,n}(u; \lambda) \quad (3.11)$$

where

$$A_r(u) = \frac{(-1)^r}{r! [\psi^{(1)}(u)]^{r-1}} \quad (3.12)$$

and where $E_{r,n}(u;\lambda)$ is obtained by expanding

$$D_u^{r-1} \left[\frac{(F(u;\lambda) - \Psi(u))^r}{\psi^{(1)}(u)} \right] \text{ and then substituting } \hat{D}_n \text{ for}$$

$$F(u;\lambda) - \Psi(u).$$

To aid in calculating $\hat{x}_n$, a Table of $E_{r,n}$'s, Table 3.1, will be given for $r = 1, 2, 3, 4$. For convenience, let

$$z^{(r)}(u;\lambda) = F^{(r)}(u;\lambda) - \psi^{(r)}(u). \quad (3.13)$$

TABLE 3.1

TABLE OF $E_{r,n}$'s, $r = 1, 2, 3, 4$

$$E_{1,n} = (\hat{D}_n / \psi^{(1)})$$

$$E_{2,n} = ((2\psi^{(1)} \hat{D}_n z^{(1)} - (\hat{D}_n)^2 \psi^{(2)}) / (\psi^{(1)})^2)$$

$$E_{3,n} = ((3(\psi^{(1)})^2 (\hat{D}_n)^2 z^{(2)} + 6(\psi^{(1)})^2 (\hat{D}_n) (z^{(1)})^2 - 6(\hat{D}_n) (z^{(1)}) \psi^{(1)} \psi^{(2)} - (\hat{D}_n)^3 \psi^{(1)} \psi^{(3)} + (\hat{D}_n)^3 (\psi^{(2)})^2) / (\psi^{(1)})^3)$$

$$E_{4,n} = ((4(\psi^{(1)})^3 (\hat{D}_n)^3 z^{(3)} + 36(\psi^{(1)})^3 (\hat{D}_n)^2 z^{(1)} z^{(2)} + 24(\psi^{(1)})^3 \hat{D}_n (z^{(1)})^3 - (\hat{D}_n)^4 (\psi^{(1)})^2 \psi^{(4)} - 12(\hat{D}_n)^3 z^{(1)} (\psi^{(1)})^2 \psi^{(3)} - 12(\psi^{(1)})^2 \psi^{(2)} (\hat{D}_n)^3 z^{(2)} - 36(\psi^{(1)})^2 \psi^{(2)} (\hat{D}_n)^2 (z^{(1)})^2 + 24(\hat{D}_n)^3 (z^{(1)}) \psi^{(1)} (\psi^{(2)})^2 - 6(\hat{D}_n)^4 (\psi^{(2)})^3 + 6(\hat{D}_n)^4 \psi^{(1)} \psi^{(2)} \psi^{(3)}) / (\psi^{(1)})^4)$$

$$\hat{D}_n = \hat{D}_n(u; \lambda) \text{ (defined in (1.15))}, E_{r,n} =$$

$$E_{r,n}(u; \lambda),$$

$$z^{(j)} = z^{(j)}(u; \lambda) \quad j = 1, 2, \dots$$

$$\psi^{(j)} = \psi^{(j)}(u) \quad j = 1, 2, \dots$$

THEOREM 3.1 Let $F(x;\lambda) = \Psi(u)$ and assume that (1.9), (1.16), and (3.8) are valid. Then

$$x - \hat{x}_n = O(\lambda^{-\frac{n+1}{2}}) \quad (3.14)$$

Proof: We have from (3.8) that

$$x = u + \sum_{r=1}^n \frac{(-1)^r}{r! [\Psi^{(1)}(u)]^{r-1}} D_u^{r-1} \left[\frac{(F(u;\lambda) - \Psi(u))^r}{\Psi^{(1)}(u)} \right] + O(\lambda^{-\frac{n+1}{2}}).$$

This expression can be written as

$$x = u + \sum_{r=1}^n A_r(u) B_r(u;\lambda) + O(\lambda^{-\frac{n+1}{2}}) \quad (3.15)$$

where $A_r(u)$ is defined in (3.12) and

$$B_r(u;\lambda) = \sum_{i=1}^{k_r} h_i(u) (F(u;\lambda) - \Psi(u))^{p_{j1}} (z^{(1)})^{p_{j2}} \dots (z^{(r-1)})^{p_{jr}} \quad (3.16)$$

where $h_i(u)$ is a function of u independent of λ and

$$\sum_{t=1}^r p_{jt} = r. \quad \text{Now in (3.16), consider}$$

$$F(u;\lambda) - \Psi(u) = (F(u;\lambda) - \hat{F}_n) + (\hat{F}_n - \Psi(u)) = (F(u;\lambda) - \hat{F}_n) + \hat{D}_n. \quad (3.17)$$

Substituting (3.17) in (3.16) and expanding yeilds

$$B_r(u; \lambda) = E_{r,n}(u; \lambda) + O(\lambda^{-\frac{n+1}{2}}) \quad (3.18)$$

where $E_{r,n}(u; \lambda)$ is as in Definition 3.1.

Therefore, using (3.17) and (3.15)

$$x = u + \sum_{r=1}^n A_r(u) E_{r,n}(u; \lambda) + O(\lambda^{-\frac{n+1}{2}}) = \hat{x}_n + O(\lambda^{-\frac{n+1}{2}}). \quad (3.19)$$

Hence theorem 3.1 is proved.

3.3 Comparison of Inverse Cornish-Fisher Expansion with $\hat{x}_n$.

Evaluation of the inverse Cornish-Fisher expansion requires explicit knowledge of the first $n+2$ cumulants of $F(; \lambda)$, when evaluating up to terms of order $O(\lambda^{-\frac{n+1}{2}})$. On the other hand, evaluation of $\hat{x}_n$ requires knowledge of the first k (see (1.12)) derivatives of $F(; \lambda)$. Hence, $\hat{x}_n$ may be a valuable alternative when the cumulants of $F(; \lambda)$ are difficult to evaluate but are known to possess the order properties of (1.5).

Let us now consider some example in which the inverse Cornish-Fisher expansion will be compared with $\hat{x}_n$.

Example 3.1 Chi-Square Quantiles

Let $\delta_1, \delta_2, \dots, \delta_\lambda$ be mutually independent random variables each distributed as chi-square with 1 d.f.

Then $\sum_{i=1}^{\lambda} \delta_i = W$ is distributed as $\chi^2(\lambda)$, that is the density of W is

$$g(w) = \begin{cases} \frac{1}{\Gamma(\frac{\lambda}{2}) 2^{\lambda/2}} (w)^{\frac{\lambda}{2}-1} e^{-\frac{w}{2}} & \text{for } w > 0 \\ 0 & \text{otherwise} \end{cases} \quad (3.20)$$

Now let

$$x = \frac{w-\lambda}{\sqrt{2\lambda}} = \frac{1}{\sqrt{\lambda}\sqrt{2}} \sum_{i=1}^{\lambda} (\delta_i - 1) \quad (3.21)$$

Then (3.21) is a standardized sum and the distribution function of x is

$$F(x; \lambda) = \int_{-\infty}^x \sqrt{2\lambda} g(\sqrt{2\lambda} t + \lambda) dt \quad (3.22)$$

where g is defined in (3.20). Thus F is the chi-square distribution in standard measure whose cumulants are

$$k_i = \begin{cases} 0 & i = 1 \\ 1 & i = 2 \\ \lambda^{1-\frac{i}{2}} 2^{\frac{i}{2}-1} (i-1)! & i = 3, 4, \dots \end{cases} \quad (3.23)$$

and $F(x;\lambda) \rightarrow \Psi(x)$ as $\lambda \rightarrow \infty$ for all real x where

$$\Psi(x) = \Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^x e^{-\frac{t^2}{2}} dt \quad (3.24)$$

and where the cumulants of Φ are given in (2.30). Hence

$$\beta_i = \begin{cases} 0 & i = 1, 2 \\ \lambda^{1-\frac{i}{2}} \frac{i-1}{2} (i-1)! & i = 3, 4, \dots \end{cases} \quad (3.25)$$

It is clear that $\{\beta_i\}$ satisfy the order relations of (1.6) and in Wasow [51], the expansion (1.7) with F and Φ defined in (3.22) and (3.24) is shown to be an asymptotic expansion valid uniformly in x . The derivatives of Φ are bounded in the region $R = \{z | z = x+iy, -\infty < x < \infty, -1 < y < 1\}$, therefore (1.9) and (1.16) are valid. Also due to Wasow [51], (3.8) is valid for F and Ψ in (3.22) and (3.24), respectively.

Now let

$$F(x;\lambda) = \Phi(u). \quad (3.26)$$

For $n = 1$, Definition 3.1 gives

$$\hat{x}_1 = u + A_1(u)E_{1,1}(u;\lambda) \quad (3.27)$$

where $x = \hat{x}_1 + O(\lambda^{-1})$. Now using the relation

$$\hat{x}_1 = \frac{\chi_1^2 - \lambda}{\sqrt{2\lambda}} \quad (3.28)$$

(3.27) becomes

$$\hat{\chi}_1^2 = \lambda + \sqrt{2\lambda}u + \sqrt{2\lambda}A_1(u)E_{1,1}(u;\lambda) \quad (3.29)$$

where $\chi^2 = \chi_1^2 + O(\lambda^{-1/2})$. The corresponding inverse Cornish-Fisher expansion is

$$\chi_{c1}^2 = \lambda + \sqrt{2\lambda}u + \frac{2}{3}(u^2 - 1) \quad (3.30)$$

where $\chi^2 = \chi_{c1}^2 + O(\lambda^{-1/2})$. Table 3.2 gives the comparison of $\hat{\chi}_1^2$ and χ_{c1}^2 with the exact chi-square quantile, χ^2 .

For $n = 2$, Definition 3.1 gives

$$\hat{x}_2 = u + A_1(u)E_{1,2}(u;\lambda) + A_2(u)E_{2,2}(u;\lambda) \quad (3.31)$$

where $x = \hat{x}_2 + O(\lambda^{-3/2})$. Now using the relation

$$\hat{x}_2 = \frac{\chi_2^2 - \lambda}{\sqrt{2\lambda}} \quad (3.32)$$

(3.31) becomes

$$\hat{\chi}_2^2 = \lambda + \sqrt{2\lambda} u + \sum_{r=1}^2 \sqrt{2\lambda} A_r(u) E_{r,2}(u; \lambda) \quad (3.33)$$

where $\chi^2 = \hat{\chi}_2^2 + O(\lambda^{-1})$. The corresponding inverse Cornish-Fisher expansion is

$$\chi_{C2}^2 = \lambda + \sqrt{2\lambda} u + \frac{2}{3}(u^2 - 1) + \frac{1}{\sqrt{\lambda}} \left(\frac{u^3 - 7u}{9\sqrt{2}} \right) \quad (3.34)$$

where $\chi^2 = \chi_{C2}^2 + O(\lambda^{-1})$. Table 3.2 gives the comparison of $\hat{\chi}_2^2$ and χ_{C2}^2 with the exact chi-square quantile, χ^2 .

For $n = 3$, Definition 3.1 gives

$$\hat{x}_3 = u + \sum_{r=1}^3 A_r(u) E_{r,3}(u; \lambda) \quad (3.35)$$

where $x = \hat{x}_3 + O(\lambda^{-2})$. Now using the relation

$$\hat{x}_3 = \frac{\chi_3^2 - \lambda}{\sqrt{2\lambda}}, \quad (3.36)$$

(3.35) becomes

$$\hat{\chi}_3^2 = \lambda + \sqrt{2\lambda} u + \sum_{r=1}^3 \sqrt{2\lambda} A_r(u) E_{r,3}(u, \lambda) \quad (3.37)$$

where $\chi^2 = \hat{\chi}_3^2 + O(\lambda^{-3/2})$. The corresponding inverse

Cornish-Fisher expansion is

$$\chi_{C3}^2 = \lambda + \sqrt{2\lambda}u + \frac{2}{3}(u^2 - 1) + \frac{1}{\sqrt{\lambda}}\left(\frac{u^3 - 7u}{9\sqrt{2}}\right) - \frac{1}{\lambda}\left(\frac{6u^4 + 14u^2 - 32}{405}\right) \quad (3.38)$$

where $\chi^2 = \chi_{C3}^2 + O(\lambda^{-3/2})$. Table 3.2 gives the comparison of $\hat{\chi}_3^2$ and χ_{C3}^2 with the exact chi-square quantile, χ^2 .

TABLE 3.2
QUANTILES OF CHI-SQUARE DISTRIBUTION

	λ	.750	.900	.950
χ^2		12.5489	15.9871	18.3070
χ_{c1}^2	10	12.6530	16.1580	18.4930
$\hat{\chi}_1^2$		12.7120	16.0950	19.3648
χ^2		23.8277	28.4120	31.4104
χ_{c1}^2	20	23.9025	28.5335	31.5400
$\hat{\chi}_1^2$		23.9456	28.4855	32.2743
χ^2		45.6160	51.8050	55.7585
χ_{c1}^2	40	45.6694	51.8908	55.8490
$\hat{\chi}_1^2$		45.7014	51.8556	55.8106

TABLE 3.2--continued
QUANTILES OF CHI-SQUARE DISTRIBUTION

	λ	.750	.900	.950
χ^2		66.9814	74.3970	79.0819
χ^2_{c1}	60	67.0252	74.4669	79.1554
$\hat{\chi}^2_1$		67.0520	74.4377	79.6719
χ^2		88.1303	96.5782	101.879
χ^2_{c1}	80	88.1683	96.6387	101.943
$\hat{\chi}^2_1$		88.1918	96.6132	102.408
χ^2		109.141	118.498	124.342
χ^2_{c1}	100	109.175	118.552	124.399
$\hat{\chi}^2_1$		109.197	118.530	124.826

TABLE 3.2--continued.

QUANTILES OF CHI-SQUARE DISTRIBUTION

	λ	.975	.990	.995
χ^2		20.4831	23.2093	25.1882
χ^2_{c1}	10	20.6595	23.3450	25.2761
$\hat{\chi}_1^2$		19.0729	21.8276	23.7645
χ^2		34.1696	37.5662	39.9968
χ^2_{c1}	20	34.2902	37.6544	40.0476
$\hat{\chi}_1^2$		33.0136	36.4767	38.9266
χ^2		59.3417	63.6907	66.7659
χ^2_{c1}	40	59.4247	63.7488	66.7955
$\hat{\chi}_1^2$		58.4278	62.8582	65.9842

TABLE 3.2--continued.
QUANTILES OF CHI-SQUARE DISTRIBUTION

	λ	.975	.990	.995
χ^2		83.2976	88.3794	91.9517
χ_{c1}^2	60	83.3646	88.4252	91.9734
$\hat{\chi}_1^2$		82.5119	87.6761	91.3071
χ^2		106.629	112.329	116.321
χ_{c1}^2	80	106.686	112.368	116.339
$\hat{\chi}_1^2$		105.926	111.707	115.761
χ^2		129.561	135.807	140.169
χ_{c1}^2	100	129.612	135.841	140.184
$\hat{\chi}_1^2$		128.919	135.243	139.667

TABLE 3.2--continued.
QUANTILES OF CHI-SQUARE DISTRIBUTION

	λ	.750	.900	.950
χ^2		12.5489	15.9871	18.3070
χ_{c2}^2	10	12.5434	15.9889	18.3175
$\hat{\chi}_2^2$		12.5012	16.0304	18.4612
χ^2		23.8277	28.4120	31.4104
χ_{c2}^2	20	23.8249	28.4129	31.4159
$\hat{\chi}_2^2$		23.7994	28.4348	31.5017
χ^2		45.6160	51.8050	55.7585
χ_{c2}^2	40	45.6146	51.8055	55.7613
$\hat{\chi}_2^2$		45.6000	51.8169	55.8106

TABLE 3.2.--continued.
QUANTILES OF CHI-SQUARE DISTRIBUTION

	λ	.750	.900	.950
χ^2		66.9814	74.3970	79.0819
χ^2_{c2}	60	66.9805	74.3973	79.0838
$\hat{\chi}^2_2$		66.9702	74.4050	79.1190
χ^2		88.1303	96.5782	101.879
χ^2_{c2}	80	88.1295	96.5784	101.881
$\hat{\chi}^2_2$		88.1215	96.5843	101.908
χ^2		109.141	118.498	124.342
χ^2_{c2}	100	109.141	118.498	124.343
$\hat{\chi}^2_2$		109.134	118.503	124.366

TABLE 3.2--continued.

QUANTILES OF CHI-SQUARE DISTRIBUTION

	λ	.975	.990	.995
χ^2		20.4831	23.2093	25.1882
χ_{c2}^2	10	20.5057	23.2532	25.2527
$\hat{\chi}_2^2$		19.7811	22.9494	25.2346
χ^2		34.1696	37.5662	39.9968
χ_{c2}^2	20	34.1814	37.5895	40.0311
$\hat{\chi}_2^2$		33.7366	37.4371	40.0646
χ^2		59.3417	63.6907	66.7659
χ_{c2}^2	40	59.3478	63.7029	66.7838
$\hat{\chi}_2^2$		59.0906	63.6284	66.8150

TABLE 3.2--continued.
QUANTILES OF CHI-SQUARE DISTRIBUTION

	λ	.975	.990	.995
χ^2		83.2976	88.3794	91.9517
χ^2_{c2}	60	83.3018	88.3877	91.9639
$\hat{\chi}^2_2$		83.1189	88.3388	91.9875
χ^2		106.629	112.329	116.321
χ^2_{c2}	80	106.632	112.335	116.330
$\hat{\chi}^2_2$		106.489	112.299	116.349
χ^2		129.561	135.807	140.169
χ^2_{c2}	100	129.564	135.812	140.177
$\hat{\chi}^2_2$		129.447	135.783	140.192

TABLE 3.2--continued.

QUANTILES OF CHI-SQUARE DISTRIBUTION

	λ	.750	.900	.950
χ^2		12.5489	15.9871	18.3070
χ^2_{C3}	10	12.5493	15.9872	18.3052
$\hat{\chi}^2_3$		12.5255	16.0353	18.2819
χ^2		23.8277	28.4120	31.4104
χ^2_{C3}	20	23.8279	28.4120	31.4097
$\hat{\chi}^2_3$		23.8158	28.4340	31.3939
χ^2		45.6160	51.8050	55.7585
χ^2_{C3}	40	45.6160	51.8050	55.7582
$\hat{\chi}^2_3$		45.6105	51.8144	55.7486

TABLE 3.2--continued.

QUANTILES OF CHI-SQUARE DISTRIBUTION

	λ	.750	.900	.950
χ^2		66.9814	74.3970	79.0819
χ^2_{C3}	60	66.9815	74.3970	79.0818
$\hat{\chi}_3^2$		66.9781	74.4025	79.0749
χ^2		88.1303	96.5782	101.879
χ^2_{C3}	80	88.1303	96.5782	101.879
$\hat{\chi}_3^2$		88.1279	96.5820	101.874
χ^2		109.141	118.498	124.342
χ^2_{C3}	100	109.141	118.498	124.342
$\hat{\chi}_3^2$		109.139	118.501	124.338

TABLE 3.2--continued.
QUANTILES OF CHI-SQUARE DISTRIBUTION

	λ	.975	.990	.995
χ^2		20.4831	23.2093	25.1882
χ^2_{C3}	10	20.4785	23.1990	25.1724
$\hat{\chi}^2_3$		20.3948	23.1037	25.3181
χ^2		34.1696	37.5662	39.9968
χ^2_{C3}	20	34.1678	37.5624	39.9909
$\hat{\chi}^2_3$		34.1056	37.5394	40.2133
χ^2		59.3417	63.6907	66.7659
χ^2_{C3}	40	59.3410	63.6893	66.7638
$\hat{\chi}^2_3$		59.3094	63.7000	66.9731

TABLE 3.2--continued.
QUANTILES OF CHI-SQUARE DISTRIBUTION

	λ	.975	.990	.995
χ^2		83.2976	88.3794	91.9517
χ^2_{C3}	60	83.2972	88.3787	91.9505
$\hat{\chi}_3^2$		83.2767	88.3964	92.1303
χ^2		106.629	112.329	116.321
χ^2_{C3}	80	106.628	112.328	116.320
$\hat{\chi}_3^2$		106.613	112.348	116.476
χ^2		129.561	135.807	140.169
χ^2_{C3}	100	129.561	135.806	140.169
$\hat{\chi}_3^2$		129.549	135.826	140.306

Observing Table 3.2, we find that $\hat{\chi}_n^2$ and the Cornish-Fisher χ_{cn}^2 are both reasonable approximations of the exact values of the chi-square quantiles. There are some instances where $\hat{\chi}_1^2$ seems to be more accurate than χ_{c1}^2 ; however, in general χ_{cn}^2 seems to be slightly more accurate than $\hat{\chi}_n^2$. Both approximations lose some degree of accuracy as you move into the tails of the distribution, especially for small λ .

Let us now compare the inverse expansions in estimating the quantiles of the Student-t distribution.

Example 3.2. Consider example 2.2 and let

$$F(x; \lambda) = \Psi(u) \quad (3.39)$$

where Ψ and F are defined in (2.43) and (2.44), respectively. Also let σ be defined as in (2.45).

For $n = 2$, Definition 3.1 gives

$$\hat{x}_2 = u + \sum_{r=1}^2 A_r(u) E_{r,2}(u; \lambda) \quad (3.40)$$

and using the relation

$$\hat{x}_2 = \frac{\hat{t}_2}{\sigma} \quad (3.41)$$

(3.40) becomes

$$\hat{t}_2 = \sigma u + \sum_{r=1}^2 \sigma A_r(u) E_{r,2}(u; \lambda). \quad (3.42)$$

The corresponding inverse Cornish-Fisher expansion is

$$t_{c2} = u + \frac{1}{\lambda} \left(\frac{u^3 + u}{4} \right). \quad (3.43)$$

Table 3.3 compares $\hat{t}_2$ and t_{c2} with the exact quantile value.

For $n = 4$, Definition gives

$$\hat{x}_4 = u + \sum_{r=1}^4 A_r(u) E_{r,4}(u; \lambda) \quad (3.44)$$

and using the relation

$$\hat{x}_4 = \frac{\hat{t}_4}{\sigma} \quad (3.45)$$

(3.44) becomes

$$\hat{t}_4 = \sigma u + \sum_{r=1}^4 \sigma A_r(u) E_{r,4}(u; \lambda). \quad (3.46)$$

The corresponding inverse Cornish-Fisher expansion is

$$t_{c4} = u + \frac{1}{\lambda} \left(\frac{u^3 + u}{4} \right) + \frac{1}{\lambda^2} \left(\frac{5u^5 + 16u^3 + 3u}{96} \right). \quad (3.47)$$

Table 3.3 compares $\hat{t}_4$ and t_{c4} with the exact quantile value, t .

TABLE 3.3
QUANTILES OF STUDENT-t DISTRIBUTION

	λ	.7500	.9500	.9750	.9975	.9995
t		.700	1.812	2.228	3.581	4.587
t_{c2}	10	.699	1.797	2.197	3.430	4.264
$\hat{t}_2$		.715	1.823	2.248	3.395	4.131
t		.687	1.725	2.086	3.153	3.850
t_{c2}	20	.687	1.721	2.079	3.119	3.777
$\hat{t}_2$		.690	1.727	2.090	3.109	3.744
t		.681	1.684	2.021	2.971	3.551
t_{c2}	40	.681	1.683	2.019	2.963	3.534
$\hat{t}_2$		.681	1.684	2.022	2.960	3.526
t		.679	1.671	2.000	2.915	3.460
t_{c2}	60	.679	1.670	2.000	2.912	3.453
$\hat{t}_2$		.679	1.671	2.000	2.910	3.449

TABLE 3.3--continued
QUANTILES OF STUDENT-t DISTRIBUTION

	λ	.7500	.9500	.9750	.9975	.9995
t		.700	1.812	2.228	3.581	4.587
t_{C4}	10	.700	1.811	2.225	3.559	4.525
$\hat{t}_4$		.700	1.813	2.232	3.510	4.718
t		.687	1.725	2.086	3.153	3.850
t_{C4}	20	.687	1.725	2.086	3.151	3.842
$\hat{t}_4$		.687	1.725	2.086	3.144	3.861
t		.681	1.684	2.021	2.971	3.551
t_{C4}	40	.681	1.684	2.021	2.971	3.550
$\hat{t}_4$		.681	1.684	2.021	2.970	3.553
t		.679	1.671	2.000	2.915	3.460
t_{C4}	60	.679	1.671	2.000	2.914	3.460
$\hat{t}_4$		.679	1.671	2.000	2.914	3.461

It is obvious from observing Table 3.3 that $\hat{t}_n$ and the Cornish-Fisher t_{cn} are comparable approximations of the exact value of the Student-t quantile. In many instances they are accurate to at least three decimal places. They also do well in the extreme tails of the distributions.

Again we have two examples which indicate that $\hat{x}_n$, like x_{cn} , can be used with confidence. It is also important to note that $\hat{x}_n$ may be a very good alternative when the cumulants are difficult to evaluate but when the derivatives are easily calculated. Also, in some cases $\hat{x}_n$ may be more desirable even when the cumulants are easily evaluated.

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APPENDIX

I. Discussion of the Q's

In Table A, the Q's for $i = 1, 2, \dots, 8$ are exhibited. The Q's are defined by the table as

$$Q_i(x; \lambda) = \sum \frac{AB}{C}$$

where the summation is over the terms corresponding to i .

For example:

$$Q_1(x; \lambda) = \frac{-\beta_3}{3!} \psi^{(3)}(x) \quad \text{and}$$

$$Q_2(x; \lambda) = \frac{\beta_4}{4!} \psi^{(4)}(x) + \frac{\beta_3^2}{(3!)^2 2!} \psi^{(6)}(x)$$

and etc.

TABLE A.

i	A	B	C
1	$-\beta_3$	$\psi^{(3)}(x)$	$3!$
2	β_4	$\psi^{(4)}(x)$	$4!$
	β_3^2	$\psi^{(6)}(x)$	$(3!)^2 2!$
3	$-\beta_5$	$\psi^{(5)}(x)$	$5!$
	$-\beta_3 \beta_4$	$\psi^{(7)}(x)$	$3! 4!$
	$-\beta_3^3$	$\psi^{(9)}(x)$	$(3!)^3 3!$
4	β_6	$\psi^{(6)}(x)$	$6!$
	$\beta_3 \beta_5$	$\psi^{(8)}(x)$	$3! 5!$
	β_4^2	$\psi^{(8)}(x)$	$(4!)^2 2!$
	$\beta_3^2 \beta_4$	$\psi^{(10)}(x)$	$(3!)^2 4! 2!$
	β_3^4	$\psi^{(12)}(x)$	$(3!)^4 4!$
5	$-\beta_7$	$\psi^{(7)}(x)$	$7!$
	$-\beta_3 \beta_6$	$\psi^{(9)}(x)$	$3! 6!$
	$-\beta_4 \beta_5$	$\psi^{(9)}(x)$	$4! 5!$

Table A--continued,

	$-\beta_3^2$	$\psi^{(11)}(x)$	$(3!)^2 5! 2!$
	β_5		
	$-\beta_3 \beta_4^2$	$\psi^{(11)}(x)$	$3! (4!)^2 2!$
	$-\beta_3^3 \beta_4$	$\psi^{(13)}(x)$	$(3!)^3 4! 3!$
	$-\beta_3^5$	$\psi^{(15)}(x)$	$(3!)^5 5!$
6	β_8	$\psi^{(8)}(x)$	$8!$
	$\beta_3 \beta_7$	$\psi^{(10)}(x)$	$3! 7!$
	$\beta_4 \beta_6$	$\psi^{(10)}(x)$	$4! 6!$
	β_5^2	$\psi^{(10)}(x)$	$(5!)^2 2!$
	$\beta_3^2 \beta_6$	$\psi^{(12)}(x)$	$(3!)^2 6! 2!$
	$\beta_3 \beta_4 \beta_5$	$\psi^{(12)}(x)$	$3! 4! 5!$
	β_4^3	$\psi^{(12)}(x)$	$(4!)^3 3!$
	$\beta_3^3 \beta_5$	$\psi^{(14)}(x)$	$(3!)^3 5! 3!$
	$\beta_3^2 \beta_4^2$	$\psi^{(14)}(x)$	$(3!)^2 (4!)^2 (2!)^2$
	$\beta_3^4 \beta_4$	$\psi^{(16)}(x)$	$(3!)^4 4! 4!$
	β_3^6	$\psi^{(18)}(x)$	$(3!)^6 6!$

Table A--continued.

7	$-\beta_9$	$\psi^{(9)}(x)$	$9!$
	$-\beta_3\beta_8$	$\psi^{(11)}(x)$	$3!8!$
	$-\beta_4\beta_7$	$\psi^{(11)}(x)$	$4!7!$
	$-\beta_5\beta_6$	$\psi^{(11)}(x)$	$5!6!$
	$-\beta_3^2\beta_7$	$\psi^{(13)}(x)$	$(3!)^27!2!$
	$-\beta_3\beta_4\beta_6$	$\psi^{(13)}(x)$	$3!4!6!$
	$-\beta_3\beta_5^2$	$\psi^{(13)}(x)$	$3!(5!)^22!$
	$-\beta_4^2\beta_5$	$\psi^{(13)}(x)$	$(4!)^25!2!$
	$-\beta_3^3\beta_6$	$\psi^{(15)}(x)$	$(3!)^36!3!$
	$-\beta_3^2\beta_4\beta_5$	$\psi^{(15)}(x)$	$(3!)^24!5!2!$
	$-\beta_3\beta_4^3$	$\psi^{(15)}(x)$	$3!(4!)^33!$
	$-\beta_3^4\beta_5$	$\psi^{(17)}(x)$	$(3!)^45!4!$
	$-\beta_3^3\beta_4^2$	$\psi^{(17)}(x)$	$(3!)^3(4!)^23!2!$
	$-\beta_3^5\beta_4$	$\psi^{(19)}(x)$	$(3!)^54!5!$
	$-\beta_3^7$	$\psi^{(21)}(x)$	$(3!)^77!$
8	β_{10}	$\psi^{(10)}(x)$	$10!$

Table A--continued.

8	$\beta_3 \beta_9$	$\psi^{(12)}(x)$	3191
	$\beta_4 \beta_8$	$\psi^{(12)}(x)$	4181
	$\beta_5 \beta_7$	$\psi^{(12)}(x)$	5171
	β_6^2	$\psi^{(12)}(x)$	$(6!)^2 2!$
	$\beta_3^2 \beta_8$	$\psi^{(14)}(x)$	$(3!)^2 8! 2!$
	$\beta_3 \beta_4 \beta_7$	$\psi^{(14)}(x)$	314171
	$\beta_3 \beta_5 \beta_6$	$\psi^{(14)}(x)$	315161
	$\beta_4^2 \beta_6$	$\psi^{(14)}(x)$	$(4!)^2 6! 2!$
	$\beta_4 \beta_5^2$	$\psi^{(14)}(x)$	$4! (5!)^2 2!$
	$\beta_3^3 \beta_7$	$\psi^{(16)}(x)$	$(3!)^3 7! 3!$
	$\beta_3^2 \beta_4 \beta_6$	$\psi^{(16)}(x)$	$(3!)^2 4! 6! 2!$
	$\beta_3^2 \beta_5^2$	$\psi^{(16)}(x)$	$(3!)^2 (5!)^2 2! 2!$
	$\beta_3 \beta_4^2 \beta_5$	$\psi^{(16)}(x)$	$3! (4!)^2 5! 2!$
	β_4^4	$\psi^{(16)}(x)$	$(4!)^4 4!$
	$\beta_3^4 \beta_6$	$\psi^{(18)}(x)$	$(3!)^4 6! 4!$
	$\beta_3^3 \beta_4 \beta_5$	$\psi^{(18)}(x)$	$(3!)^3 4! 5! 3!$

Table A-- continued,

8	$\beta_3^2 \beta_4^3$	$\psi^{(18)}(x)$	$(3!)^2 (4!)^3 2!3!$
	$\beta_3^5 \beta_5$	$\psi^{(20)}(x)$	$(3!)^5 5!5!$
	$\beta_3^4 \beta_4^2$	$\psi^{(20)}(x)$	$(3!)^4 (4!)^2 4!2!$
	$\beta_3^6 \beta_4$	$\psi^{(22)}(x)$	$(3!)^6 4!6!$
	β_3^8	$\psi^{(24)}(x)$	$(3!)^8 8!$

II. Derivatives of the Standardized Gamma Distribution

Let $F(; \lambda)$ be defined as in (2.27). Then the derivatives of $F(; \lambda)$ may be expressed by

$$F^{(m)}(x; \lambda) = \frac{\lambda^{\frac{m}{2}}}{\Gamma(\lambda)} e^{-(x\lambda^{\frac{1}{2}} + \lambda)} \sum_{i=0}^{m-1} (-1)^{m-1-i} \binom{m-1}{i} D_u^{(i)} u^{\lambda-1} \Big|_{u=x\lambda^{\frac{1}{2}} + \lambda}$$

III. Derivatives of the Standardized Chi-Square Distribution

Let $F(; \lambda)$ be defined as in (3.21). Then the derivatives of $F(; \lambda)$ may be expressed by

$$F^{(m)}(x; \lambda) = \frac{(2\lambda)^{\frac{m}{2}}}{\Gamma(\frac{\lambda}{2}) \cdot 2^{\frac{\lambda}{2}}} e^{-\frac{(2\lambda)^{1/2} x + \lambda}{2}} \cdot S_m$$

where

$$S_m = \sum_{i=0}^{m-1} \left(-\frac{1}{2}\right)^{m-1-i} \binom{m-1}{i} D_u^{(i)} u^{\frac{\lambda}{2}-1} \Big|_{u=(2\lambda)^{\frac{1}{2}} x + \lambda}$$

IV. Derivatives of the Standardized Student-t Distribution

Let $F(; \lambda)$ be defined as in (2.45). Then the derivatives of $F(; \lambda)$ may be expressed by

$$F^{(m)}(x; \lambda) = \frac{(\sigma)^m}{(\lambda\pi)^{\frac{1}{2}} \Gamma(\frac{\lambda}{2})} \frac{\left(\frac{1}{2}(\lambda+1)\right)}{1} D_u^{(m-1)} g(u) \Big|_{u=x\sigma}$$

where $\sigma = \sqrt{\frac{\lambda}{\lambda-2}}$ and g is defined by

$$g(u) = \left(1 + \frac{u^2}{\lambda}\right)^{-\frac{\lambda+1}{2}}.$$

Expressions of $D_u^{(m-1)}g(u)$ for $m = 2, 3, \dots, 9$ will now be obtained. Let us define T_k as

$$T_k(u) = \left(-\frac{1}{2}\lambda - 1/2\right) \cdot \left(-\frac{1}{2}\lambda - 3/2\right) \cdots \left(-\frac{1}{2}\lambda - \frac{k}{2}\right) \cdot \left(1 + \frac{u^2}{\lambda}\right)^{\left(-1/2\lambda - \frac{k+2}{2}\right)}$$

where $k = 2j+1$, $j = 0, 1, 2, \dots$. Then

$$D_u^{(1)}T_k(u) = T_{k+2}(u)$$

which can be used in obtaining the following

$$D_u^{(1)}g(u) = \frac{2u}{\lambda}T_1(u)$$

$$D_u^{(2)}g(u) = \left(\frac{4u^2}{\lambda^2}\right)T_3(u) + \left(\frac{2}{\lambda}\right)T_1(u)$$

$$D_u^{(3)}g(u) = \left(\frac{8u^3}{\lambda^3}\right)T_5(u) + \left(\frac{12u}{\lambda^2}\right)T_3(u)$$

$$D_u^{(4)}g(u) = \left(\frac{16u^4}{\lambda^4}\right)T_7(u) + \left(\frac{48u^2}{\lambda^3}\right)T_5(u) + \left(\frac{12}{\lambda^2}\right)T_3(u)$$

$$D_u^{(5)}g(u) = \left(\frac{32u^5}{\lambda^5}\right)T_9(u) + \left(\frac{160u^3}{\lambda^4}\right)T_7(u) + \left(\frac{120u}{\lambda^3}\right)T_5(u)$$

$$D_u^{(6)} g(u) = \left(\frac{64u^6}{\lambda^6}\right) T_{11}(u) + \left(\frac{480u^4}{\lambda^5}\right) T_9(u) + \left(\frac{720u^2}{\lambda^4}\right) T_7(u) + \left(\frac{120}{\lambda^3}\right) T_5(u).$$

$$D_u^{(7)} g(u) = \left(\frac{128u^7}{\lambda^7}\right) T_{13}(u) + \left(\frac{1344u^5}{\lambda^6}\right) T_{11}(u) + \left(\frac{3360u^3}{\lambda^5}\right) T_9(u)$$

$$+ \left(\frac{1680u}{\lambda^4}\right) T_7(u)$$

$$D_u^{(8)} g(u) = \left(\frac{256u^8}{\lambda^8}\right) T_{15}(u) + \left(\frac{3584u^6}{\lambda^7}\right) T_{13}(u) + \left(\frac{13440u^4}{\lambda^6}\right) T_{11}(u)$$

$$+ \left(\frac{13440u^2}{\lambda^5}\right) T_9(u) + \left(\frac{1680}{\lambda^4}\right) T_7(u).$$