QUASILINEAR ELLIPTIC SYSTEMS

by

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CHAPTER I

INTRODUCTION

We shall study a boundary value problem for the second order differential system

$$(1.1) \quad \sum_{\ell=1}^n \frac{\partial}{\partial x_{\ell}} \left\{ \sum_{j=1}^N \sum_{m=1}^n C_{ij}^{\ell m} [x, U(x)] \frac{\partial U^j(x)}{\partial x_m} + B_i^{\ell} [x, U(x)] \right\} \\ + F_i [x, U(x)] = 0, \quad 1 \leq i \leq N$$

where $x = (x_1, x_2, \dots, x_n)$ lies in a bounded open connected set Ω in E_n , and a solution $U = (U^1, U^2, \dots, U^N)$ maps Ω into E_N subject to the boundary conditions

$$(1.2) \quad U^i(x) = 0, \quad x \in S \equiv \text{boundary of } \Omega.$$

Formulations will be abbreviated when convenient through the combined use of summation convention on repeated indices, $U_{,m}$ for $\partial U / \partial x_m$, and suppression of arguments. Thus, the differential system (1.1) becomes

$$(1.3) \quad (C_{ij}^{\ell m} U_{,m}^j + B_i^{\ell})_{,\ell} + F_i = 0.$$

The Euclidean norm of appropriate dimension will be

denoted by $|\cdot|$. Thus, $|x| = \sum_{m=1}^n (x_m)^2$, $|U|^2 = \sum_{i=1}^N (U^i)^2$,

$$|B|^2 = \sum_{\ell=1}^n \sum_{i=1}^N (B_i^\ell)^2, \quad |U_x|^2 = \sum_{m=1}^n \sum_{i=1}^N (U_{,m}^i)^2, \text{ and}$$

$$|C|^2 = \sum_{\ell=1}^n \sum_{m=1}^n \sum_{i=1}^N \sum_{j=1}^N (C_{ij}^{\ell m})^2. \quad \text{Summation convention allows}$$

these norms to be respectively written as $|x|^2 = (x_m)^2$,

$$|U|^2 = (U^i)^2, \quad |B|^2 = (B_i^\ell)^2, \quad |U_x|^2 = (U_{,m}^i)^2, \text{ and}$$

$$|C|^2 = (C_{ij}^{\ell m})^2.$$

If $p \geq 1$, let $L_p(\Omega)$ be the Banach space of measurable functions from Ω to E_N with finite norm

$$(1.4) \quad ||U||_{p,\Omega}^p = \int_{\Omega} |U(x)|^p dx.$$

If $p = 2$, $(\cdot, \cdot)_{2,\Omega}$ will denote the natural inner product induced by (1.4), and we let $H^1(\Omega)$ denote the Hilbert space induced by

$$(1.5) \quad (U, V)_{0,\Omega} = (U, V)_{2,\Omega} + (U_{,m}, V_{,m})_{2,\Omega} \\ \equiv \int_{\Omega} U^i V^i dx + \int_{\Omega} U_{,m}^i V_{,m}^i dx.$$

Also, let $H_0^1(\Omega)$ be the completion of $C_0^\infty(\Omega)$ in the $H^1(\Omega)$ norm.

We shall say that the function U from Ω into E_N satisfies

a Hölder condition with exponent α , where $\alpha \in (0,1]$, and with Hölder constant M in the region $\bar{\Omega}$ (closure of Ω) if

$$(1.6) \quad |U^i|_{(\alpha),\Omega} \equiv \sup \rho^{-\alpha} \text{osc}\{U^i, K_\rho \cap \Omega\} \equiv M < \infty$$

for each $i = 1, \dots, N$ where K_ρ denotes an arbitrary open sphere of radius ρ in E_n and $\text{osc}\{U^i, K_\rho \cap \Omega\}$ is the difference between $\text{ess sup}\{U^i(x) : x \in K_\rho \cap \Omega\}$ and $\text{ess inf}\{U^i(x) : x \in K_\rho \cap \Omega\}$. The supremum in (1.6) is taken over $0 < \rho \leq \rho_0$.

$C_{0,\alpha}(\bar{\Omega})$ is the Banach space of elements consisting of all bounded functions U that are continuous in Ω such that $|U|_{(\alpha),\Omega}^2 \equiv \sum_{i=1}^N (|U^i|_{(\alpha),\Omega})^2$ is finite. The norm in $C_{0,\alpha}(\bar{\Omega})$ is defined as

$$(1.7) \quad |U|_{\alpha,\Omega} \equiv \sup_{x \in \Omega} |U(x)| + |U|_{(\alpha),\Omega}.$$

The domain of definition of the elements $C_{0,\alpha}(\bar{\Omega})$ can be thought of as extended to S in such a way as to be continuous from within.

A function belongs to $C_{0,\alpha}(\Omega)$ if it belongs to $C_{0,\alpha}(\bar{\Omega}^\Gamma)$ for all $\bar{\Omega}^\Gamma \subset \Omega$.

We denote by $B(\Omega, M, \gamma, \delta, 1/q)$ the class of real-valued functions V in $H^1(\Omega)$ with $\text{ess sup}_{x \in \Omega} |V(x)| \leq M$ such that, for both V and $-V$, the following inequalities are valid in an arbitrary open sphere $K_\rho \cap \Omega$ for arbitrary $\sigma \in (0,1)$:

$$(1.8) \quad \int_{A_{k,\rho-\sigma\rho}} |V_x|^2 dx \leq \gamma [\sigma^{-2} \rho^{-2(1-n/q)} \operatorname{ess\,sup}_{x \in A_{k,\rho}} [V(x) - k]^2 + 1] (\operatorname{mes} A_{k,\rho})^{1-2/q}$$

for $k \geq \operatorname{ess\,sup}_{x \in K_\rho} V(x) - \delta$ where $A_{k,\rho} \equiv \{x \in K_\rho \mid V(x) > k\}$,

$K_{\rho-\sigma\rho}$ is a sphere concentric with K_ρ , and $\operatorname{mes} G \equiv$ measure of G . We may assume the radii of the spheres ρ in (1.8) do not exceed some positive number $\bar{\rho}$. The parameters M , γ , and δ are arbitrary positive numbers and $2 \leq n < q < \infty$.

For applications, it will be useful to note the following: If, in an open sphere K_ρ , a function V satisfies the inequalities

$$(1.9) \quad \int_{A_{k,\rho}} |V_x|^2 \zeta^2 dx \leq \gamma' \left[\int_{A_{k,\rho}} (V - k)^2 |\zeta_x|^2 dx + (\operatorname{mes} A_{k,\rho})^{1-2/q} \right]$$

for $k \geq \operatorname{ess\,sup}_{x \in K_\rho} V(x) - \delta$ and for an arbitrary nonnegative

smooth function ζ that vanishes on the surface of K_ρ , then inequality (1.8) is valid for K_ρ and an arbitrary concentric sphere $K_{\rho-\sigma\rho}$ where $\sigma \in (0,1)$. (See [8, p. 82].)

We denote by $B(\bar{\Omega}, M, \gamma, \delta, 1/q)$ the class of real-valued functions V in $B(\Omega, M, \gamma, \delta, 1/q)$ that, together with their negatives, satisfy inequalities (1.8) for the spheres K_ρ

that intersect S for $k \geq \operatorname{ess\,sup}_{x \in K_\rho \cap \Omega} V(x) - \delta$ and

$k \geq \operatorname{ess\,sup}_{x \in K_\rho \cap S} V(x)$. The regions of intersection $A_{k,\rho}$ for the

spheres intersecting S in (1.8) are defined as the sets of points in $K_\rho \cap \Omega$ at which $V(x) > k$.

Finally, we shall say that the boundary S of a region Ω satisfies condition (A) if there exist two positive numbers a_0 and θ_0 such that, for an arbitrary sphere K_ρ with center on S of radius $\rho \leq a_0$ and for an arbitrary component $\hat{\Omega}_\rho$ of the intersection of the sphere K_ρ with Ω , the inequality

$$(1.10) \quad \operatorname{mes} \hat{\Omega}_\rho \leq (1 - \theta_0) \operatorname{mes} K_\rho$$

holds. We shall use condition (A) in Chapter VI only. For the other chapters, we simply assume that S is sufficiently smooth to ensure that integration by parts is valid.

Solution of (1.1) will be developed and considered in a weak sense. If $Z \in H_0^1(\Omega)$, multiply (1.1) by Z^i and obtain (sum by convention)

$$(1.11) \quad \int_{\Omega} Z^i (C_{ij}^{\ell m} U_{,m}^j + B_i^\ell)_{, \ell} dx + \int_{\Omega} Z^i F_i dx = 0.$$

This implies

$$(1.12) \quad \int_{\Omega} Z_{, \ell}^i C_{ij}^{\ell m} U_{,m}^j dx + \int_{\Omega} Z_{, \ell}^i B_i^\ell dx - \int_{\Omega} Z^i F_i dx = 0$$

as a motivation for a definition of weak solutions of (1.1) and (1.2).

DEFINITION 1.1. We shall say that $U \in H_0^1(\Omega)$ is a weak solution of (1.1) and (1.2) if (1.12) is valid whenever $Z \in H_0^1(\Omega)$.

The main result of this paper is a proof of the existence of a weak solution to (1.1) and (1.2) with minimal conditions on the auxiliary functions C , B , and F . Visik [13] studied existence of weak solutions as well as uniqueness for higher order (≥ 2) quasilinear strongly elliptic systems but required considerable smoothness conditions on the auxiliary functions. Browder [2,3] also considered existence of weak solutions and uniqueness for similar systems but imposed monotonicity constraints on the auxiliary functions with some growth rate restrictions.

We shall also consider uniqueness of our weak solution, but stronger restrictions on the auxiliary functions are necessarily needed as will be shown by an example. In addition, stability with respect to variations in C , B , and F will be studied. We shall conclude with a proof of the boundedness of the Hölder norms of the weak solutions under more restrictive conditions on C , B , and F than those needed for existence. This regularity argument uses essentially the same method as that of Ladyzhenskaya and Ural'tseva [8, Chapter 4] for a single equation.

CHAPTER II

AUXILIARY RESULTS

Certain known results and easily proved propositions that will be needed throughout the paper are collected in this chapter.

DEFINITION 2.1. Let Y be a Banach space. A set $D \subset Y$ is relatively compact in Y if $\bar{D}$ is compact in Y .

DEFINITION 2.2. Let X and Y be Banach spaces. A mapping $f : X \rightarrow Y$ is compact if it is continuous and maps bounded sets into relatively compact sets, i.e., if $B \subset X$ is bounded, then $f(B)$ is relatively compact in Y .

THEOREM 2.1. (Rellich) Every bounded set in $H_0^1(\Omega)$ is relatively compact in $L_2(\Omega)$.

PROOF. (See [1, p. 194; 9].)

THEOREM 2.2. (Schauder) Let A be a closed convex bounded set contained in the Banach space X . Every compact mapping $f : A \rightarrow A$ has a fixed point.

PROOF. (See [12, p. 96].)

THEOREM 2.3. Let V be an arbitrary real-valued function in the class $B(\Omega, M, \gamma, \delta, 1/q)$, and let K_{ρ_0} be a sphere of radius $\rho_0 \leq 1$ belonging to Ω . Then, for an arbitrary sphere

K_ρ , where $\rho \leq \rho_0$ that is concentric with K_{ρ_0} , the oscillation of V in K_ρ satisfies the inequality

$$(2.1) \quad \text{osc } \{V, K_\rho\} \leq c(\rho/\rho_0)^\alpha$$

where

$$(2.2) \quad \alpha = \min \{-\log_4(1 - 2^{1-s}), 1 - n/q\},$$

$$(2.3) \quad c = 4^\alpha \max \{2M, 2^s \rho_0^{1-n/q}\},$$

and the number $s \geq \log_2(2M/\delta) + 3$.

PROOF. (See [8, p. 90].)

THEOREM 2.4. If S satisfies condition (A) and if the function V in $B(\bar{\Omega}, M, \gamma, \delta, 1/q)$ satisfies on S a Hölder condition, more precisely, if

$$(2.4) \quad \text{osc } \{V, S \cap K_\rho\} \leq L\rho^\epsilon, \quad \epsilon > 0$$

for spheres K_ρ (where $\rho \leq a_0$) with centers on S , then V satisfies a Hölder condition in $\bar{\Omega}$; that is, for arbitrary K_ρ

$$(2.5) \quad \text{osc } \{V, K_\rho \cap \Omega\} \leq c\rho^\alpha.$$

Here, the constants c and α are determined only by the parameters M , δ , γ , and $1/q$ of the class $B(\bar{\Omega}, M, \gamma, \delta, 1/q)$, by L and ϵ in (2.4), and by the constants a_0 and θ_0 in condition (A) on S .

PROOF. (See [8, p. 92].)

LEMMA 2.1. Let U be an arbitrary function in $H_0^1(\Omega)$.

Then

$$(2.6) \quad ||U||_p \leq M_0 (\text{mes } \Omega)^{1/p - (q-2)/2q} ||U_x||_{2,\Omega}$$

where $1 \leq p \leq 2q/(q-2)$ with $q \geq n$ if $n \geq 3$, $q > 2$ if $n = 2$, and the constant M_0 is determined only by p and n .

PROOF. (See [8, p. 45].)

COROLLARY 2.1. The norms $||U||_{0,\Omega}$ and $||U_x||_{2,\Omega}$ are equivalent on $H_0^1(\Omega)$.

PROOF. From Lemma 2.1 with $p = 2$, we get the inequalities

$$(2.7) \quad ||U_x||_{2,\Omega}^2 \leq ||U||_{2,\Omega}^2 + ||U_x||_{2,\Omega}^2 \leq (1+b^2) ||U_x||_{2,\Omega}^2$$

where $b^2 \equiv M_0^2 (\text{mes } \Omega)^{2/n}$ with M_0 from Lemma 2.1 and the proof is complete.

LEMMA 2.2. Suppose a and c are nonnegative numbers and ε is an arbitrary positive number. If $0 < \lambda < 1$, then

$$(2.8) \quad a^\lambda c^{1-\lambda} \leq \varepsilon a + (1-\lambda) (\lambda/\varepsilon)^{\lambda/(1-\lambda)} c.$$

$$\text{PROOF.} \quad a^\lambda c^{1-\lambda} = (a\varepsilon/\lambda)^\lambda [(\lambda/\varepsilon)^{\lambda/(1-\lambda)} c]^{1-\lambda}$$

$$\leq \varepsilon a + (1-\lambda) (\lambda/\varepsilon)^{\lambda/(1-\lambda)} c$$

where the last inequality is from [11, p. 112], and the proof is complete.

Note that in the special case of $a = q^2$, $c = d^2$, and $\lambda = 1/2$, (2.8) reduces to

$$(2.9) \quad |qd| \leq \varepsilon q^2 + (1/4\varepsilon)d^2.$$

We conclude Chapter 2 with a theorem that is useful in showing boundedness of weak solutions.

THEOREM 2.5. Suppose that a real-valued function V in $H^1(\Omega)$ has bounded $\text{ess sup}_{x \in \Omega} V(x)$ and that for $k \geq \hat{k} \geq \text{ess sup}_{x \in S} V(x)$ satisfies the inequalities

$$(2.10) \quad \int_{A_k} |V_x| \, dx \leq \gamma \left[\int_{A_k} (V - k)^2 \, dx + \sum_{i=1}^{N_0} k^{\alpha_i} (\text{mes } A_k)^s \right]$$

where $s = 1 - 2/n + \varepsilon_i$. Also, γ , α_i , ε_i , and N_0 are positive constants and $A_k \equiv \{x \in \Omega \mid V(x) > k\}$. Suppose also that $2 \leq \alpha_i < 2(1 + \varepsilon_i)$. Then, $\text{ess sup}_{x \in \Omega} V(x)$ does not exceed some constant that depends only on γ , N_0 , α_i , ε_i , n , $\hat{k}$, $\text{mes } \Omega$, and the norm $\|V\|_{1, A_{\hat{k}}}$ with $i = 1, \dots, N_0$.

PROOF. (See [8, p. 73].)

CHAPTER III

EXISTENCE

We now impose conditions on the auxiliary functions of (1.1) in order to obtain the existence of a weak solution of (1.1) and (1.2). After proving existence under the conditions of Assumption 3.1, we will relax some of these restrictions and prove existence under the more general conditions.

ASSUMPTION 3.1. There exists a positive constant λ such that

$$(3.1) \quad C_{ij}^{\ell m} y_\ell^i y_m^j \geq \lambda |y|^2, \quad y \in E_{nN}.$$

Moreover, M_C is a positive constant such that

$$(3.2) \quad |C_{ij}^{\ell m}[x, U]|^2 \leq M_C^2, \quad x \in \Omega, U \in E_N.$$

Also, there exist nonnegative constants ε_0 , ε_1 , M_F , and M_B such that

$$(3.3) \quad \varepsilon_0 + \varepsilon_1 < \lambda^2 [1 + b^2]^{-2},$$

$$(3.4) \quad |F[x, U]|^2 \leq \varepsilon_0 |U|^2 + M_F, \quad x \in \Omega, U \in E_N,$$

$$(3.5) \quad |B[x, U]|^2 \leq \varepsilon_1 |U|^2 + M_B, \quad x \in \Omega, U \in E_N.$$

Here, b is from (2.7). Finally, F_i , B_i^ℓ , and $C_{ij}^{\ell m}$ are continuous maps from $L_2(\Omega)$ into $L_2(\Omega)$; i.e., if

$h(V) \equiv F_i[\cdot, V(\cdot)]$, then we assume that the mapping

h is continuous.

We shall use the Schauder Fixed Point Theorem to prove the existence of a weak solution of (1.1) and (1.2). However, we first consider the problem of proving existence of a solution to a linear system.

Let V be a fixed element of $L_2(\Omega)$. We wish to prove the existence of a unique $W \in H_0^1(\Omega)$ such that

$$(3.6) \quad \int_{\Omega} z_{,\ell}^i C_{ij}^{\ell m}[x,V] w_{,m}^j dx + \int_{\Omega} z_{,\ell}^i B_i^{\ell}[x,V] dx = \int_{\Omega} z^i F_i[x,V] dx$$

for every $z \in H_0^1(\Omega)$. For this purpose we define the following functionals on $H_0^1(\Omega) \times H_0^1(\Omega)$ and $H_0^1(\Omega)$, respectively.

$$(3.7) \quad L[Z,W] \equiv \int_{\Omega} z_{,\ell}^i C_{ij}^{\ell m}[x,V] w_{,m}^j dx.$$

$$(3.8) \quad P[Z] \equiv - \int_{\Omega} z_{,\ell}^i B_i^{\ell}[x,V] dx + \int_{\Omega} z^i F_i[x,V] dx.$$

We now prove a series of propositions concerning the functionals P and L .

PROPOSITION 3.1. P is a bounded linear functional on $H_0^1(\Omega)$.

PROOF. Clearly, P is a linear functional on $H_0^1(\Omega)$, and we need only prove boundedness. From Schwarz's inequality, we get

$$(3.9) \quad \begin{aligned} |P[Z]| &\leq \int_{\Omega} |z_x| |B[x,V]| dx + \int_{\Omega} |z| |F[x,V]| dx \\ &\leq ||z_x||_{2,\Omega} ||B[\cdot, V(\cdot)]||_{2,\Omega} \\ &\quad + ||z||_{2,\Omega} ||F[\cdot, V(\cdot)]||_{2,\Omega} \end{aligned}$$

$$\begin{aligned}
&\leq (||B[\cdot, V(\cdot)]||_{2, \Omega} + ||F[\cdot, V(\cdot)]||_{2, \Omega}) \\
&\quad \times (||z||_{2, \Omega} + ||z_x||_{2, \Omega}) \\
&\equiv M(B, F, V) \{||z||_{2, \Omega} + ||z_x||_{2, \Omega}\} \\
&\leq 2 M(B, F, V) ||z||_{0, \Omega},
\end{aligned}$$

and the proof is complete.

PROPOSITION 3.2. L is a bilinear functional on $H_0^1(\Omega) \times H_0^1(\Omega)$, and there exist positive constants λ_0 and λ_∞ such that

$$(3.10) \quad |L[z, w]| \leq \lambda_\infty ||z||_{0, \Omega} ||w||_{0, \Omega}$$

and

$$(3.11) \quad \lambda_0 ||z||_{0, \Omega}^2 \leq L[z, z]$$

for all z and w in $H_0^1(\Omega)$.

PROOF. Clearly, L is a bilinear functional on $H_0^1(\Omega) \times H_0^1(\Omega)$, and we need only prove the existence of λ_0 and λ_∞ . From Schwarz's inequality, we have

$$\begin{aligned}
(3.12) \quad |L[z, w]| &\leq \int_{\Omega} |z_x| |C[x, V]| |w_x| dx \\
&\leq || |z_x| |C[x, V]| ||_{2, \Omega} ||w_x||_{2, \Omega} \\
&\leq M_C ||z||_{0, \Omega} ||w||_{0, \Omega}.
\end{aligned}$$

Thus, let $\lambda_\infty = M_C$ where M_C is from (3.2), and (3.10) is proved. For the proof of (3.11), we use (3.1) to get

$$(3.13) \quad \lambda \, ||z_x||_{2,\Omega}^2 \leq L[z,z]$$

for all z in $H_0^1(\Omega)$. However, (2.7) combined with (3.13) implies

$$(3.14) \quad \lambda (1 + b^2)^{-1} ||z||_{0,\Omega}^2 \leq \lambda ||z_x||_{2,\Omega}^2 \leq L[z,z]$$

Thus, define $\lambda_0 = \lambda(1 + b^2)^{-1}$ where b is from (2.7), and the proof is complete.

PROPOSITION 3.3. There exists a unique W in $H_0^1(\Omega)$ such that

$$(3.15) \quad L[Z,W] = P[Z]$$

for all Z in $H_0^1(\Omega)$. Thus, W is the unique solution of (3.6).

PROOF. Uniqueness is clear from inequality (3.11). For the existence, we observe that Proposition 3.1 and F. Riesz' representation theorem [14, p. 90] imply the existence of a unique f in $H_0^1(\Omega)$ such that

$$(3.16) \quad P[Z] = (Z,f)_{0,\Omega}$$

for all Z in $H_0^1(\Omega)$. Also, Proposition 3.2 and the Lax-Milgram Theorem [14, p. 92] imply the existence of a unique determined bounded linear operator $T : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ such that

$$(3.17) \quad L[Z,T(U)] = (Z,U)_{0,\Omega}$$

for all Z and U in $H_0^1(\Omega)$. In particular, (3.17) must be true for $U = f$, and this means

$$(3.18) \quad L[Z, T(f)] = (Z, f)_{0, \Omega} = P[Z]$$

for all Z in $H_0^1(\Omega)$. Thus, $W \equiv T(f)$ is the desired solution, and the proof is complete.

Now Proposition 3.3 enables us to use (3.6) to define a mapping $\Phi : L_2(\Omega) \rightarrow H_0^1(\Omega)$ by $W = \Phi(V)$. In order to prove the existence of a solution U to (1.1) and (1.2), we need only show that Φ has a fixed point. To accomplish this, we prove a series of lemmas concerning the mapping Φ .

LEMMA 3.1. The mapping $\Phi : L_2(\Omega) \rightarrow L_2(\Omega)$ is continuous whenever Assumption 3.1 is valid.

PROOF. Suppose $V_k \rightarrow V$ in $L_2(\Omega)$ as $k \rightarrow \infty$. It suffices to show $\|\Phi(V_k) - \Phi(V)\|_{2, \Omega} \rightarrow 0$ as $k \rightarrow \infty$ since then Lemma 2.1 implies $\Phi(V_k) \rightarrow \Phi(V)$ in $L_2(\Omega)$. With $W_k \equiv \Phi(V_k)$, $W \equiv \Phi(V)$, and the use of the corresponding integral identities involving W and W_k , we get

$$\begin{aligned} (3.19) \quad & \int_{\Omega} z_{, \ell}^i [C_{ij}^{\ell m}[x, V] W_{, m}^j - C_{ij}^{\ell m}[x, V_k] W_{k, m}^j] dx \\ & + \int_{\Omega} z_{, \ell}^i [B_i^{\ell}[x, V] - B_i^{\ell}[x, V_k]] dx \\ & = \int_{\Omega} z_{, \ell}^i [F_i[x, V] - F_i[x, V_k]] dx \end{aligned}$$

for each Z in $H_0^1(\Omega)$. This implies

$$(3.20) \quad \int_{\Omega} z_{, \ell}^i C_{ij}^{\ell m}[x, V_k] [W_{, m}^j - W_{k, m}^j] dx$$

$$\begin{aligned}
& + \int_{\Omega} z_{,\ell}^i [C_{ij}^{\ell m}[x,V] - C_{ij}^{\ell m}[x,V_k]] w_{,m}^j dx \\
& + \int_{\Omega} z_{,\ell}^i [B_i^{\ell}[x,V] - B_i^{\ell}[x,V_k]] dx \\
& = \int_{\Omega} z^i [F_i[x,V] - F_i[x,V_k]] dx.
\end{aligned}$$

In (3.20), we put $z = w - w_k$. Using Assumption 3.1, we get

$$\begin{aligned}
(3.21) \quad & \lambda \int_{\Omega} |z_x|^2 dx \\
& \leq \int_{\Omega} |z_x| |C[x,V] - C[x,V_k]| |w_x| dx \\
& + \int_{\Omega} |z_x| |B[x,V] - B[x,V_k]| dx \\
& + \int_{\Omega} |z| |F[x,V] - F[x,V_k]| dx.
\end{aligned}$$

We now give estimates for the integrals on the right hand side of (3.21). Using Lemma 2.2 with $\epsilon > 0$, we obtain ($\sum$ negates implied sum)

$$\begin{aligned}
(3.22) \quad & \int_{\Omega} |z_x| |B[x,V] - B[x,V_k]| dx \leq \epsilon \int_{\Omega} |z_x|^2 dx \\
& + (4\epsilon)^{-1} \sum_{\ell=1}^n \sum_{i=1}^N \int_{\Omega} |B_i^{\ell}[x,V] - B_i^{\ell}[x,V_k]|^2 dx
\end{aligned}$$

and

$$\begin{aligned}
(3.23) \quad & \int_{\Omega} |z| |F[x,V] - F[x,V_k]| dx \leq \epsilon \int_{\Omega} |z|^2 dx \\
& + (4\epsilon)^{-1} \sum_{i=1}^N \int_{\Omega} |F_i[x,V] - F_i[x,V_k]|^2 dx
\end{aligned}$$

$$\begin{aligned} &\leq b^2 \varepsilon \int_{\Omega} |z_x|^2 dx \\ &\quad + (4\varepsilon)^{-1} \sum_{i=1}^N \int_{\Omega} |F_i[x, V] - F_i[x, V_k]|^2 dx \end{aligned}$$

where b is from (2.7) in the last inequality. The other term in (3.21) yields the estimate

$$\begin{aligned} (3.24) \quad &\int_{\Omega} |z_x| |C[x, V] - C[x, V_k]| |w_x| dx \\ &\leq \varepsilon \int_{\Omega} |z_x|^2 dx \\ &\quad + (4\varepsilon)^{-1} \sum_{i=1}^N \sum_{j=1}^N \sum_{\ell=1}^n \sum_{m=1}^n \int_{\Omega} |C_{ij}^{\ell m}[x, V] - C_{ij}^{\ell m}[x, V_k]|^2 |w_x|^2 dx. \end{aligned}$$

Choose $\varepsilon = \lambda[2(2 + b^2)]^{-1}$ in the above inequalities. With this choice of ε , we may combine inequalities (3.21) through (3.24) to get

$$\begin{aligned} (3.25) \quad &||\Phi(V) - \Phi(V_k)||_{2, \Omega} \\ &\leq [(2 + b^2)/2\lambda]^{1/2} ||F[\cdot, V(\cdot)] - F[\cdot, V_k(\cdot)]||_{2, \Omega} \\ &\quad + ||B[\cdot, V(\cdot)] - B[\cdot, V_k(\cdot)]||_{2, \Omega} \\ &\quad + |||C[\cdot, V(\cdot)] - C[\cdot, V_k(\cdot)]|| |w_x| ||_{2, \Omega}. \end{aligned}$$

The first two terms on the right hand side of (3.25) converge to zero as k approaches ∞ by the continuity of B and F given in Assumption 3.1. Thus, it suffices to show that the last term in (3.25) converges to zero. To do this, we let $\varepsilon > 0$ be given and choose $U_{K(\varepsilon)} \in C_0^\infty(\Omega)$ such that

$||W_x - U_{K(\epsilon)_x}||_{2,\Omega} < \epsilon/(4 M_C)$ where M_C is from Assumption 3.1. This yields

$$\begin{aligned}
 (3.26) \quad & || |C[\cdot, V(\cdot)] - C[\cdot, V_k(\cdot)]| |W_x| ||_{2,\Omega} \\
 & \leq || |C[\cdot, V(\cdot)] - C[\cdot, V_k(\cdot)]| |W_x - U_{K(\epsilon)_x}| ||_{2,\Omega} \\
 & \quad + || |C[\cdot, V(\cdot)] - C[\cdot, V_k(\cdot)]| |U_{K(\epsilon)_x}| ||_{2,\Omega} \\
 & \leq 2 M_C ||W_x - U_{K(\epsilon)_x}||_{2,\Omega} \\
 & \quad + \sup_{x \in \Omega} |U_{K(\epsilon)_x}(x)| \\
 & \quad \times || |C[\cdot, V(\cdot)] - C[\cdot, V_k(\cdot)]| ||_{2,\Omega} \\
 & < \epsilon/2 + M_{K(\epsilon)} || |C[\cdot, V(\cdot)] - C[\cdot, V_k(\cdot)]| ||_{2,\Omega}
 \end{aligned}$$

where $M_{K(\epsilon)} \equiv \sup_{x \in \Omega} |U_{K(\epsilon)_x}(x)|$. Now, since C_{ij}^{lm} is

continuous from $L_2(\Omega)$ into $L_2(\Omega)$ from Assumption 3.1, there exists a positive integer $N(K(\epsilon))$ depending only on $K(\epsilon)$ such that $k > N(K(\epsilon))$ implies

$$(3.27) \quad M_{K(\epsilon)} || |C[\cdot, V(\cdot)] - C[\cdot, V_k(\cdot)]| ||_{2,\Omega} < \epsilon/2.$$

Combining (3.26) and (3.27), we obtain the desired result, and the proof is complete.

LEMMA 3.2. The mapping $\Phi : L_2(\Omega) \rightarrow L_2(\Omega)$ is a compact mapping whenever Assumption 3.1 is valid.

PROOF. Suppose A is a bounded set in $L_2(\Omega)$. Thus, there exists a constant M_1 such that A is contained in the

set $\{U \in L_2(\Omega) \mid ||U||_{2,\Omega} \leq M_1\}$. We wish to show that $\Phi(A)$ is bounded in $H_0^1(\Omega)$. For $V \in A$, we put $Z = W \equiv \Phi(V)$ in (3.6). With the use of Assumption 3.1, we get

$$(3.28) \quad \lambda \int_{\Omega} |W_x|^2 dx \leq \int_{\Omega} |W_x| |B[x,V]| dx + \int_{\Omega} |W| |F[x,V]| dx.$$

Lemma 2.2 now implies

$$(3.29) \quad \lambda \int_{\Omega} |W_x|^2 dx \leq \varepsilon \int_{\Omega} |W_x|^2 dx + (4\varepsilon)^{-1} \int_{\Omega} |B[x,V]|^2 dx \\ + \varepsilon \int_{\Omega} |W|^2 dx + (4\varepsilon)^{-1} \int_{\Omega} |F[x,V]|^2 dx$$

for any $\varepsilon > 0$. We now use (2.6) with $p = 2$ and Assumption 3.1 to obtain

$$(3.30) \quad \lambda \int_{\Omega} |W_x|^2 dx \leq \varepsilon(1 + b^2) \int_{\Omega} |W_x|^2 dx \\ + (4\varepsilon)^{-1} [(\varepsilon_0 + \varepsilon_1) \int_{\Omega} |V|^2 dx \\ + (M_F + M_B) \text{mes } \Omega]$$

where b is from (2.7). We now put $\varepsilon = \lambda[2(1 + b^2)]^{-1}$ in (3.30) and simplify to get

$$(3.31) \quad \int_{\Omega} |W_x|^2 dx \leq \lambda^{-2}(1 + b^2) [(\varepsilon_0 + \varepsilon_1) \int_{\Omega} |V|^2 dx \\ + (M_F + M_B) \text{mes } \Omega].$$

Corollary 2.1 and (3.31) now imply

$$(3.32) \quad ||W||_{0,\Omega}^2 = ||\Phi(V)||_{0,\Omega}^2 \leq (1 + b^2) ||W_x||_{2,\Omega}^2 \\ \leq \lambda^{-2}(1 + b^2)^2 [(\varepsilon_0 + \varepsilon_1) \int_{\Omega} |V|^2 dx$$

$$+ (M_F + M_B) \text{mes } \Omega].$$

Since $V \in A$, we have $||V||_{2,\Omega}^2 \leq M_1^2$. Substituting this into (3.32) yields

$$(3.33) \quad ||\Phi(V)||_{0,\Omega} \leq [\lambda^{-2}(1+b^2)^2 \{(\varepsilon_0 + \varepsilon_1)M_1^2 + (M_F + M_B) \text{mes } \Omega\}]^{1/2}.$$

Thus, $\Phi(A)$ is a bounded set in $H_0^1(\Omega)$ whenever A is a bounded set in $L_2(\Omega)$. Theorem 2.1 implies $\Phi(A)$ is relatively compact in $L_2(\Omega)$, and the proof is complete.

LEMMA 3.3. Suppose $A \equiv \{V \in L_2(\Omega) \mid ||V||_{2,\Omega} \leq M_2\}$ where

$$M_2^2 \equiv \frac{(M_F + M_B)(1+b^2)^2 \text{mes } \Omega}{\lambda^2 - (\varepsilon_0 + \varepsilon_1)(1+b^2)^2}, \quad b \text{ is from (2.7),}$$

and $\varepsilon_0, \varepsilon_1, M_F$, and M_B are from Assumption 3.1. Then $\Phi(A) \subset A$.

PROOF. Let V be an element of A . From (3.32), we get

$$(3.35) \quad ||\Phi(V)||_{0,\Omega}^2 \leq \lambda^{-2}(1+b^2)^2(\varepsilon_0 + \varepsilon_1) ||V||_{2,\Omega}^2 + \lambda^{-2}(1+b^2)^2(M_F + M_B) \text{mes } \Omega.$$

From the definition of A and (3.35), we obtain

$$(3.36) \quad ||\Phi(V)||_{0,\Omega}^2 \leq \lambda^{-2}(1+b^2)^2(\varepsilon_0 + \varepsilon_1) M_2^2 + \lambda^{-2}(1+b^2)^2(M_F + M_B) \text{mes } \Omega \leq M_2^2 [(\varepsilon_0 + \varepsilon_1)(1+b^2)^2 \lambda^{-2} + (M_F + M_B) \text{mes } \Omega].$$

$$\begin{aligned}
& + \lambda^{-2} \{ \lambda^2 - (\varepsilon_0 + \varepsilon_1) (1 + b^2)^2 \} \\
& = M_2^2,
\end{aligned}$$

and the proof is complete.

THEOREM 3.1. Suppose Assumption 3.1 is valid. Then the mapping $\Phi : L_2(\Omega) \rightarrow L_2(\Omega)$ has a fixed point, U , in $L_2(\Omega)$. Therefore, U is a solution of (1.1) and (1.2).

PROOF. Using Lemma 3.1 and Lemma 3.2, we have that the mapping $\Phi : L_2(\Omega) \rightarrow L_2(\Omega)$ is compact. By Lemma 3.3, there is a closed bounded convex set A in $L_2(\Omega)$ such that $\Phi(A)$ is contained in A . Theorem 2.2 implies Φ has a fixed point in $L_2(\Omega)$, and the proof is complete.

We now change the conditions on F and B in Assumption 3.1 to allow ε_0 to be arbitrary and place a monotonicity type of constraint on F .

ASSUMPTION 3.2. Suppose Assumption 3.1 is valid except for inequalities (3.3) and (3.4). We replace these inequalities with the following: There exists a constant η and nonnegative constants M and ε_1 such that

$$(3.37) \quad \lambda - b^2 \eta > 0$$

$$(3.38) \quad \varepsilon_1 < [b^{-1} \min\{\lambda, \lambda - b^2 \eta\}]^2$$

$$(3.39) \quad |F[x, U]|^2 \leq M [1 + |U|^2]$$

$$(3.40) \quad (U^i - V^i) (F_i[x, U] - F_i[x, V]) \leq \eta |U - V|^2$$

for all U and V in E_N and x in Ω .

In order to obtain a solution of (1.1) and (1.2) under Assumption 3.2, we first consider a different problem. We define a sequence of vector-valued functions $\{G^k\}_{k=1}^{\infty}$ from $\Omega \times E_N$ into E_N by $G_i^k[x, U] = (1 + |U|^2)^{-1/k} F_i[x, U]$ for $k = 1, 2, \dots$ and $i = 1, 2, \dots, N$. Let V be a fixed element of $L_2(\Omega)$ and k be an integer greater than 2. We consider the problem of finding $W_k \in H_0^1(\Omega)$ such that

$$(3.41) \quad \int_{\Omega} z_{,\ell}^i C_{ij}^{\ell m}[x, V] W_{k,m}^j dx + \int_{\Omega} z_{,\ell}^i B_i^{\ell}[x, V] dx \\ = \int_{\Omega} z^i G_i^k[x, W_k] dx$$

for all Z in $H_0^1(\Omega)$. In order to do this, we observe that for each U in $L_2(\Omega)$, there is a unique g_k in $H_0^1(\Omega)$ such that

$$(3.42) \quad \int_{\Omega} z_{,\ell}^i C_{ij}^{\ell m}[x, V] g_{k,m}^j dx + \int_{\Omega} z_{,\ell}^i B_i^{\ell}[x, V] dx \\ = \int_{\Omega} z^i G_i^k[x, U] dx$$

for all Z in $H_0^1(\Omega)$. The proof of this is identical to that of Proposition 3.3 and therefore omitted. This process defines a mapping $f_k : L_2(\Omega) \rightarrow H_0^1(\Omega)$ with $g_k = f_k(U)$. As in the previous existence argument, we shall show that f_k has a fixed point in $L_2(\Omega)$. This will yield a solution, W_k , to (3.41) for each V in $L_2(\Omega)$ and each $k = 3, 4, \dots$. We now prove a series of lemmas concerning the mapping f_k .

LEMMA 3.4. The mapping $f_k : L_2(\Omega) \rightarrow L_2(\Omega)$ is continuous whenever Assumption 3.2 is valid.

PROOF. Suppose $\{U_\alpha\}_{\alpha=1}^\infty$ is a sequence of functions in $L_2(\Omega)$ such that $U_\alpha \rightarrow U$ in $L_2(\Omega)$ as $\alpha \rightarrow \infty$. It suffices to show $||[f_k(U_\alpha) - f_k(U)]_x||_{2,\Omega} \rightarrow 0$ as $\alpha \rightarrow \infty$ because of Lemma 2.1. Using the corresponding forms of (3.42) for $f_k(U)$ and $f_k(U_\alpha)$, we get

$$(3.43) \quad \int_{\Omega} z_{,l}^i C_{ij}^{lm}[x,U] (f_k^i(U_\alpha) - f_k^i(U))_{,m} dx \\ = \int_{\Omega} z^i (G_i^k[x,U_\alpha] - G_i^k[x,U]) dx$$

for all z in $H_0^1(\Omega)$. Let $z = f_k(U_\alpha) - f_k(U)$ in (3.43) and use Assumption 3.2, Lemma 2.1, and Lemma 2.2 to obtain

$$(3.44) \quad \lambda \int_{\Omega} |z_x|^2 dx \leq \int_{\Omega} |z| |G^k[x,U_\alpha] - G^k[x,U]| dx \\ \leq \varepsilon \int_{\Omega} |z|^2 dx \\ + (4\varepsilon)^{-1} \sum_{i=1}^N \int_{\Omega} |G_i^k[x,U_\alpha] - G_i^k[x,U]|^2 dx \\ \leq \varepsilon b^2 \int_{\Omega} |z_x|^2 dx \\ + (4\varepsilon)^{-1} \sum_{i=1}^N \int_{\Omega} |G_i^k[x,U_\alpha] - G_i^k[x,U]|^2 dx$$

where b is from (2.7). In (3.44), let $\varepsilon = \lambda/2b^2$ and simplify to get

$$(3.45) \quad ||[f_k(U_\alpha) - f_k(U)]_x||_{2,\Omega} \\ \leq (b/\lambda) ||G^k[\cdot, U_\alpha(\cdot)] - G^k[\cdot, U(\cdot)]||_{2,\Omega}.$$

Since F is a continuous mapping from $L_2(\Omega)$ into $L_2(\Omega)$, it is clear that G^k also has this property. Thus, letting $\alpha \rightarrow \infty$, we get $\|G^k[\cdot, U_\alpha(\cdot)] - G^k[\cdot, U(\cdot)]\|_{2,\Omega} \rightarrow 0$, and the proof is complete.

LEMMA 3.5. The mapping $f_k : L_2(\Omega) \rightarrow L_2(\Omega)$ is a compact mapping whenever Assumption 3.2 is valid.

PROOF. In view of Theorem 2.1, it suffices to show that if A is a bounded set in $L_2(\Omega)$, then $f_k(A)$ is a bounded set in $H_0^1(\Omega)$. Thus, suppose $A \subseteq \{U \in L_2(\Omega) \mid \|U\|_{2,\Omega} \leq L\}$ for some positive number L . If $U \in A$, then letting $Z = f_k(U) = g_k$ in (3.42) and using Assumption 3.2 yields

$$(3.46) \quad \lambda \int_{\Omega} |Z_x|^2 dx \leq \int_{\Omega} |Z_x| |B[x, V]| dx \\ + \int_{\Omega} |Z| |G^k[x, U]| dx .$$

By using Lemma 2.1 and Lemma 2.2 as well as Assumption 3.2, we get

$$(3.47) \quad \lambda \int_{\Omega} |Z_x|^2 dx \leq \varepsilon \int_{\Omega} |Z_x|^2 dx + \varepsilon \int_{\Omega} |Z|^2 dx \\ + (4\varepsilon)^{-1} \int_{\Omega} |B[x, V]|^2 dx \\ + (4\varepsilon)^{-1} \sum_{i=1}^N \int_{\Omega} |G_i^k[x, U]|^2 dx \\ \leq \varepsilon (1 + b^2) \int_{\Omega} |Z_x|^2 dx \\ + M_{\varepsilon} + M(4\varepsilon)^{-1} \int_{\Omega} (1 + |U|^2)^{1-2/k} dx$$

where $M_\varepsilon \equiv (4\varepsilon)^{-1} \int_{\Omega} |B[x,V]|^2 dx$ and b is from (2.7). Let

$\varepsilon = \hat{\varepsilon} = \lambda [2(1 + b^2)]^{-1}$ in (3.47) and simplify to obtain

$$\begin{aligned}
 (3.48) \quad ||z_x||_{2,\Omega}^2 & \leq 2M_{\hat{\varepsilon}}/\lambda + M(1 + b^2)/(\lambda^2) \int_{\Omega} (1 + |U|^2)^{1-2/k} dx \\
 & \leq 2M_{\hat{\varepsilon}}/\lambda + M(1 + b^2)/(\lambda^2) \int_{\Omega} (1 + |U|^2) dx \\
 & \leq 2M_{\hat{\varepsilon}}/\lambda + [M(1 + b^2)/(\lambda^2)] (\text{mes } \Omega + L^2) \equiv \hat{L}.
 \end{aligned}$$

This inequality holds for all $U \in A$, and now Corollary 2.1 gives

$$\begin{aligned}
 (3.49) \quad ||z||_{0,\Omega} & \equiv ||f_k(U)||_{0,\Omega} \leq (1 + b^2)^{1/2} ||[f_k(U)]_x||_{2,\Omega} \\
 & \leq [(1 + b^2)\hat{L}]^{1/2}
 \end{aligned}$$

for all $U \in A$, and the proof is complete.

LEMMA 3.6. Suppose $A_k = \{U \in L_2(\Omega) \mid ||U||_{2,\Omega} \leq \tilde{M}_k\}$

where

$$\begin{aligned}
 \tilde{M}_k^2 & \equiv 2(1 + b^2) \{ 2\hat{M}/\lambda + (2P/k) [2(k-2)(1 + b^2)/k]^{(k-2)/2} \\
 & \quad + (\text{mes } \Omega)/[2(1 + b^2)] \}
 \end{aligned}$$

with $\hat{M} \equiv [(1 + b^2)/(2\lambda)] \int_{\Omega} |B[x,V]|^2 dx$, b is from (2.7), and

$P \equiv [M(1 + b^2)/(\lambda^2)]^{k/2}$. Then $f_k(A_k) \subset A_k$.

PROOF. Let $U \in A_k$. Using the first inequality of (3.48) and Lemma 2.2 (with $\lambda = 1 - 2/k$, $a = 1 + |U|^2$, $k > 2$) yields

$$\begin{aligned}
 (3.50) \quad ||[f_k(U)]_x||_{2,\Omega} &\leq \{2\hat{M}/\lambda + \epsilon \int_{\Omega} (1 + |U|^2) dx \\
 &\quad + (2/k) [(k-2)/\epsilon k]^{(k-2)/2} p\}^{1/2} \\
 &= \epsilon \int_{\Omega} |U|^2 dx + K(\epsilon, k)
 \end{aligned}$$

where $\epsilon > 0$, $\hat{M} = M_{\hat{\epsilon}}$ of (3.48), and

$$K(\epsilon, k) \equiv (2\hat{M}/\lambda) + (2/k) [(k-2)/\epsilon k]^{(k-2)/2} p + \epsilon (\text{mes } \Omega).$$

Thus, $U \in A_k$ implies

$$(3.51) \quad ||[f_k(U)]_x||_{2,\Omega} \leq [\epsilon \tilde{M}_k + K(\epsilon, k)]^{1/2}.$$

Corollary 2.1 along with (3.51) gives

$$(3.52) \quad ||f_k(U)||_{0,\Omega} \leq \{(1 + b^2) [\epsilon \tilde{M}_k + K(\epsilon, k)]\}^{1/2}.$$

Put $\epsilon = [2(1 + b^2)]^{-1}$ in (3.52) and get

$$\begin{aligned}
 (3.53) \quad ||f_k(U)||_{0,\Omega} &\leq \{(1 + b^2) [\{2(1 + b^2)\}^{-1} \tilde{M}_k^2 \\
 &\quad + [2(1 + b^2)]^{-1} \tilde{M}_k^2]\}^{1/2} \\
 &= \tilde{M}_k
 \end{aligned}$$

where we have used the identity $[K(\epsilon, k)]^2 = \tilde{M}_k^2 / [2(1 + b^2)]$ for $\epsilon = [2(1 + b^2)]^{-1}$, and the proof is complete.

THEOREM 3.2. Suppose Assumption 3.2 is valid. Then

the mapping $f_k : L_2(\Omega) \rightarrow L_2(\Omega)$ has a fixed point in $L_2(\Omega)$. Therefore, (3.41) has at least one solution for each $k \geq 3$ and each V in $L_2(\Omega)$.

PROOF. Lemma 3.4 and Lemma 3.5 imply the mapping $f_k : L_2(\Omega) \rightarrow L_2(\Omega)$ is compact. By Lemma 3.6, there is a closed convex bounded set A_k such that $f_k(A_k) \subset A_k$. Therefore, Theorem 2.2 implies f_k has a fixed point, and the proof is complete.

We now show that the sequence $\{W_k\}_{k=1}^{\infty}$ of solutions to (3.41) has a convergent subsequence in $H_0^1(\Omega)$. First, we need a preliminary lemma.

LEMMA 3.7. For each V in $L_2(\Omega)$, there is a positive constant L depending on V but independent of k such that $\|W_k\|_{0,\Omega} \leq L$ for all $k \geq 3$.

PROOF. In (3.41), let $Z = W_k$ and use Assumption 3.2 to get

$$(3.54) \quad \lambda \int_{\Omega} |Z_x|^2 dx \leq \int_{\Omega} |Z_x| |B[x,V]| dx \\ + \sum_{i=1}^N \int_{\Omega} W_k^i G_i^k[x, W_k] dx.$$

With the use of Lemma 2.2, we have

$$(3.55) \quad \int_{\Omega} |Z_x| |B[x,V]| dx \leq \epsilon \int_{\Omega} |Z_x|^2 dx \\ + (4\epsilon)^{-1} \int_{\Omega} |B[x,V]|^2 dx.$$

If $\eta \geq 0$ in (3.37), inequality (3.40) (with $U = W_k$ and $V = 0$) implies

$$\begin{aligned}
 (3.56) \quad & \sum_{i=1}^N W_k^i G_i^k[x, W_k] = \sum_{i=1}^N W_k^i \left(1 + \sum_{j=1}^N |W_k^j|^2\right)^{-1/k} F_i[x, W_k] \\
 & \leq \left[1 + \sum_{j=1}^N |W_k^j|^2\right]^{-1/k} \left[\sum_{i=1}^N \eta |W_k^i|^2 + |W_k| |F[x, 0]|^2\right] \\
 & \leq \left[1 + \sum_{i=1}^N |W_k^i|^2\right]^{-1/k} \left[(\eta + \epsilon) \sum_{i=1}^N |W_k^i|^2 + (4\epsilon)^{-1} |F[x, 0]|^2\right] \\
 & \leq (\eta + \epsilon) \sum_{i=1}^N |W_k^i|^2 + (4\epsilon)^{-1} |F[x, 0]|^2 \\
 & \equiv (\eta + \epsilon) |Z|^2 + (4\epsilon)^{-1} |F[x, 0]|^2.
 \end{aligned}$$

Combining (3.54), (3.55), and (3.56) yields

$$\begin{aligned}
 (3.57) \quad & (\lambda - \epsilon) \int_{\Omega} |Z_x|^2 dx \leq (\eta + \epsilon) \int_{\Omega} |Z|^2 dx \\
 & + (4\epsilon)^{-1} \left[\int_{\Omega} |B[x, V]|^2 dx + \int_{\Omega} |F[x, 0]|^2 dx \right].
 \end{aligned}$$

Lemma 2.1 and (3.57) imply

$$\begin{aligned}
 (3.58) \quad & [\lambda - \epsilon - b^2(\eta + \epsilon)] \int_{\Omega} |Z_x|^2 dx \\
 & + (4\epsilon)^{-1} \left[\int_{\Omega} |B[x, V]|^2 dx + \int_{\Omega} |F[x, 0]|^2 dx \right].
 \end{aligned}$$

We now choose $\epsilon = (\lambda - b^2\eta) [2(1 + b^2)]^{-1}$ in (3.58). Simplifying (3.58) yields

$$(3.59) \quad \int_{\Omega} |z_x|^2 dx \leq (1 + b^2)/(\lambda - b^2\eta)^2 \left[\int_{\Omega} |B[x, V]|^2 dx + \int_{\Omega} |F[x, 0]|^2 dx \right] \equiv \hat{L}.$$

Corollary 2.1 and (3.59) now yield

$$(3.60) \quad ||w_k||_{0, \Omega} \equiv ||z||_{0, \Omega} \leq [(1 + b^2)\hat{L}]^{1/2}.$$

If $\eta < 0$ in (3.37), we have from (3.40) the inequality

$$\begin{aligned} (3.61) \quad \sum_{i=1}^N w_k^i G_i^k[x, w_k] &= \sum_{i=1}^N (1 + \sum_{j=1}^N |w_k^j|^2)^{-1/k} w_k^i F_i[x, w_k] \\ &\leq (1 + \sum_{j=1}^N |w_k^j|^2)^{-1/k} [\eta \sum_{i=1}^N |w_k^i|^2 + |w_k| |F[x, 0]|] \\ &\leq |w_k| |F[x, 0]| \\ &\leq \epsilon \sum_{i=1}^N |w_k^i|^2 + (4\epsilon)^{-1} |F[x, 0]|^2. \\ &\equiv \epsilon |z|^2 + (4\epsilon)^{-1} |F[x, 0]|^2. \end{aligned}$$

Combining (3.54), (3.55), and (3.61) yields

$$\begin{aligned} (3.62) \quad (\lambda - \epsilon) \int_{\Omega} |z_x|^2 dx &\leq \epsilon \int_{\Omega} |z|^2 dx \\ &+ (4\epsilon)^{-1} \left[\int_{\Omega} |B[x, V]|^2 dx + \int_{\Omega} |F[x, 0]|^2 dx \right]. \end{aligned}$$

The analysis now follows as for $\eta \geq 0$ in (3.58), (3.59),

and (3.60) as if $\eta = 0$ and $\varepsilon = \lambda[2(1 + b^2)]^{-1}$ to get

$$(3.63) \quad |||W_k|||_{0,\Omega} \equiv |||Z|||_{0,\Omega} \\ \leq [(1 + b^2)^{1/2}/\lambda] \left[\int_{\Omega} |B[x,V]|^2 dx + \int_{\Omega} |F[x,0]|^2 dx \right]^{1/2},$$

and the proof is complete.

LEMMA 3.8. The sequence $\{W_k\}_{k=3}^{\infty}$ has a subsequence $\{W_{k_v}\}_{v=1}^{\infty}$ which converges in $H_0^1(\Omega)$.

PROOF. Lemma 3.7 along with Theorem 2.1 implies the existence of a subsequence $\{W_{k_v}\}_{v=1}^{\infty}$ which converges in $L_2(\Omega)$. We now show that this subsequence is Cauchy in $H_0^1(\Omega)$. Thus suppose W_{k_v} and W_{k_α} are elements of the subsequence. With the use of (3.41), we get

$$(3.64) \quad \int_{\Omega} Z_{,\ell}^i C_{ij}^{\ell m}[x,V] [W_{k_v}^j - W_{k_\alpha}^j]_{,m} dx \\ = \int_{\Omega} Z_{,\ell}^i [G_i^{kv}[x,W_{k_v}] - G_i^{k\alpha}[x,W_{k_\alpha}]] dx$$

for all Z in $H_0^1(\Omega)$. We put $Z = W_{k_v} - W_{k_\alpha}$ in (3.64) and use Assumption 3.2 to obtain

$$(3.65) \quad \lambda \sum_{i=1}^N \int_{\Omega} |W_{k_v}^i - W_{k_\alpha}^i|_x|^2 dx \\ \leq \sum_{i=1}^N \int_{\Omega} (W_{k_v}^i - W_{k_\alpha}^i) (G_i^{kv}[x,W_{k_v}] - G_i^{k\alpha}[x,W_{k_\alpha}]) dx$$

$$\begin{aligned}
&\leq \sum_{i=1}^N \int_{\Omega} |W_{k_v}^i - W_{k_\alpha}^i|^2 dx \\
&\quad + \sum_{i=1}^N \int_{\Omega} |G_i^{k_v}[x, W_{k_v}] - G_i^{k_\alpha}[x, W_{k_\alpha}]|^2 dx.
\end{aligned}$$

Therefore, to show that the subsequence $\{W_{k_v}\}_{v=1}^\infty$ is Cauchy in $H_0^1(\Omega)$, it suffices to show that the last integral on the right hand side of (3.65) approaches zero as v and α approach ∞ . To do this, it suffices to show

$$(3.66) \quad \lim_{v \rightarrow \infty} \sum_{i=1}^N \int_{\Omega} |G_i^{k_v}[x, W_{k_v}] - F_i[x, W_{k_v}]|^2 dx = 0$$

since we already know $\{F[\cdot, W_{k_v}(\cdot)]\}_{v=1}^\infty$ is Cauchy in $L_2(\Omega)$. For simplicity in notation, we replace W_{k_v} with Y for the remainder of the proof. From the definition of G^k and Assumption 3.2, we obtain

$$\begin{aligned}
(3.67) \quad I &\equiv \sum_{i=1}^N \int_{\Omega} |G_i^{k_v}[x, Y] - F_i[x, Y]|^2 dx \\
&= \int_{\Omega} |[1 + |Y|^2]^{-1/k_v} F[x, Y] - F[x, Y]|^2 dx \\
&\leq \int_{\Omega} |F[x, Y]|^2 |1 - (1 + |Y|^2)^{-1/k_v}|^2 dx \\
&\leq M \int_{\Omega} (1 + |Y|^2) |1 - (1 + |Y|^2)^{-1/k_v}|^2 dx \\
&\leq M \int_{\Omega} (1 + |Y|)^2 [(1 + |Y|^2)^{1/k_v} - 1]^2 dx.
\end{aligned}$$

The last integral in (3.67) is finite by virtue of Lemma 2.1 whenever v is large enough that $2 + 4/k_v < 2n/(n - 2)$ whenever $n \geq 3$. If $n = 2$, it is finite for all v . Using that same lemma and Hölder's inequality in (3.67), we get

$$\begin{aligned}
 (3.68) \quad I &\leq M \left[\int_{\Omega} (1 + |Y|)^{2n/(n-2)} dx \right]^{(n-2)/n} \\
 &\quad \times \left(\int_{\Omega} [1 + |Y|^2]^{1/k_v} - 1 dx \right)^{2/n} \\
 &\leq 4M \left[\int_{\Omega} (1 + |Y|^{2n/(n-2)}) dx \right]^{(n-2)/n} \times Q \\
 &\leq 4M [\text{mes } \Omega + \|Y_x\|_{2,\Omega}^{2n/(n-2)}]^{(n-2)/n} \times Q \\
 &\leq 4M [\text{mes } \Omega + L^{n/(n-2)}]^{(n-2)/n} \times Q
 \end{aligned}$$

where L is the bound for $\|W_k\|_{0,\Omega}$ given in Lemma 3.7 and $Q \equiv \left(\int_{\Omega} [1 + |Y|^2]^{1/k_v} - 1 dx \right)^{2/n}$. It remains for us to show that the integral Q on the right hand side of (3.68) converges to zero as $v \rightarrow \infty$. To do this, we use the identity

$$\begin{aligned}
 (3.69) \quad &\int_{\Omega} \{ (1 + |Y|^2)^{1/k_v} - 1 \}^n dx \\
 &= \sum_{\beta=0}^n (-1)^{\beta} \binom{n}{\beta} \int_{\Omega} (1 + |Y|^2)^{(n-\beta)/k_v} dx,
 \end{aligned}$$

and show

$$\begin{aligned}
(3.70) \quad & \lim_{v \rightarrow \infty} \int_{\Omega} (1 + |Y|^2)^{(n-\beta)/k_v} dx \\
& \equiv \lim_{v \rightarrow \infty} \int_{\Omega} (1 + \sum_{i=1}^N |w_{k_v}^i|^2)^{(n-\beta)/k_v} dx = \text{mes } \Omega
\end{aligned}$$

for all $0 \leq \beta \leq n$. We observe that (3.70) is trivial for $\beta = n$ and consider only the cases $0 \leq \beta \leq n - 1$. Thus, for $k_v \geq n + 1$, (3.70) yields the estimate

$$\begin{aligned}
(3.71) \quad \text{mes } \Omega & \leq \int_{\Omega} (1 + |Y|^2)^{(n-\beta)/k_v} dx \\
& \leq \int_{\Omega} (1 + |Y|)^{2(n-\beta)/k_v} dx \\
& \leq \left[\int_{\Omega} 1 dx \right]^{1-[(n-\beta)/k_v]} \\
& \quad \times \left(\int_{\Omega} [1 + |Y|]^2 dx \right)^{(n-\beta)/k_v} \\
& \leq (\text{mes } \Omega)^{1-[(n-\beta)/k_v]} [2(\text{mes } \Omega + L^2)]^{(n-\beta)/k_v}
\end{aligned}$$

where L is from Lemma 3.7. Since the term on the right hand side of (3.71) approaches $\text{mes } \Omega$ as v approaches ∞ , we obtain (3.70). Now (3.70) and (3.69) yield ($Y \equiv w_{k_v}$)

$$\begin{aligned}
(3.72) \quad & \lim_{v \rightarrow \infty} \int_{\Omega} |1 - (1 + \sum_{j=1}^N |w_{k_v}^j|^2)^{1/k_v}|^n dx \\
& = \sum_{\beta=0}^n (-1)^\beta \binom{n}{\beta} \text{mes } \Omega = 0,
\end{aligned}$$

and the proof is complete.

THEOREM 3.3. Let $V \in L_2(\Omega)$, and suppose Assumption 3.2 is valid. Then there is a unique W in $H_0^1(\Omega)$ such that

$$(3.73) \quad \int_{\Omega} z_{,\ell}^i C_{ij}^{\ell m}[x,V] W_{,m}^j dx + \int_{\Omega} z_{,\ell}^i B_i^{\ell}[x,V] dx \\ = \int_{\Omega} z^i F_i[x,W] dx$$

for all z in $H_0^1(\Omega)$.

PROOF. From Theorem 3.2, there is a sequence $\{W_k\}_{k=3}^{\infty}$ contained in $H_0^1(\Omega)$ which satisfies the sequence of equations given by (3.41). Lemma 3.8 implies the existence of a subsequence $\{W_{k_v}\}_{v=1}^{\infty}$ of $\{W_k\}_{k=3}^{\infty}$ which converges to a function W in $H_0^1(\Omega)$. We observe that

$$(3.74) \quad \int_{\Omega} z_{,\ell}^i C_{ij}^{\ell m}[x,V] W_{k_v,m}^j dx \rightarrow \int_{\Omega} z_{,\ell}^i C_{ij}^{\ell m}[x,V] W_{,m}^j dx$$

as $v \rightarrow \infty$ since $W_{k_v,m} \rightarrow W_{,m}$ in $L_2(\Omega)$. Also, the continuity of F given in Assumption 3.2 along with (3.66) yields

$$(3.75) \quad \lim_{v \rightarrow \infty} \sum_{i=1}^N \int_{\Omega} |G_i^{k_v}[x, W_{k_v}] - F_i[x, W]|^2 dx = 0,$$

and this implies

$$(3.76) \quad \lim_{v \rightarrow \infty} \int_{\Omega} z^i G_i^{k_v}[x, W_{k_v}] dx = \int_{\Omega} z^i F_i[x, W] dx$$

for all z in $H_0^1(\Omega)$, and thus, W is a solution of (3.73).

For the uniqueness, suppose W and $\tilde{W}$ are solutions of (3.73) corresponding to some V in $L_2(\Omega)$. Then

$$\begin{aligned} (3.77) \quad \int_{\Omega} z^i_{,l} C^{\ell m}_{ij}[x,V] (W^j - \tilde{W}^j)_{,m} dx \\ = \int_{\Omega} z^i (F_i[x,W] - F_i[x,\tilde{W}]) dx \end{aligned}$$

for all Z in $H_0^1(\Omega)$. Put $Z = W - \tilde{W}$ in (3.77), and use Assumption 3.2 to get

$$(3.78) \quad \lambda \int_{\Omega} |[W - \tilde{W}]_x|^2 dx \leq \eta \int_{\Omega} |W - \tilde{W}|^2 dx.$$

If $\eta \leq 0$, then $||[W - \tilde{W}]_x||_{2,\Omega}^2 \leq 0$ and $W = \tilde{W}$. If $\eta > 0$, we apply Lemma 2.1 with $p = 2$ and use (3.78) to get

$$(3.79) \quad \lambda \int_{\Omega} |[W - \tilde{W}]_x|^2 dx \leq \eta b^2 \int_{\Omega} |[W - \tilde{W}]_x|^2 dx$$

which implies

$$(3.80) \quad (\lambda - b^2\eta) \int_{\Omega} |[W - \tilde{W}]_x|^2 dx \leq 0.$$

By virtue of (3.37), we obtain $W = \tilde{W}$, and the proof is complete.

With the use of Theorem 3.3, we can define a mapping $h : L_2(\Omega) \rightarrow H_0^1(\Omega)$ by $h(V) = W$ so that W is the solution of (3.73) whenever V is an element of $L_2(\Omega)$. We can now prove the existence of a solution to (1.1) and (1.2) under Assumption 3.2 by showing that the mapping h has a fixed point. Once again, we prove a series of lemmas concerning the mapping h .

LEMMA 3.9. The mapping $h : L_2(\Omega) \rightarrow L_2(\Omega)$ is continuous whenever Assumption 3.2 is valid.

PROOF. Let $\{V_k\}_{k=1}^\infty$ be a sequence of functions in $L_2(\Omega)$ such that $V_k \rightarrow V$ in $L_2(\Omega)$ as $k \rightarrow \infty$. In view of Lemma 2.1, it suffices to show $\| [W_k - W]_x \|_{2,\Omega} \rightarrow 0$ where $W_k \equiv h(V_k)$ and $W \equiv h(V)$. Using the appropriate forms of (3.73) for W_k and W , we get

$$\begin{aligned}
 (3.81) \quad & \int_{\Omega} z^i_{,\ell} [C_{ij}^{\ell m}[x, V_k] W_{k,m}^j - C_{ij}^{\ell m}[x, V] W_{,m}^j] dx \\
 & + \int_{\Omega} z^i_{,\ell} [B_i^{\ell}[x, V_k] - B_i^{\ell}[x, V]] dx \\
 & = \int_{\Omega} z^i [F_i[x, W_k] - F_i[x, W]] dx
 \end{aligned}$$

for all z in $H_0^1(\Omega)$. This implies

$$\begin{aligned}
 (3.82) \quad & \int_{\Omega} z^i_{,\ell} C_{ij}^{\ell m}[x, V_k] [W_k^j - W^j]_{,m} dx \\
 & + \int_{\Omega} z^i_{,\ell} [C_{ij}^{\ell m}[x, V_k] - C_{ij}^{\ell m}[x, V]] W_{,m}^j dx \\
 & + \int_{\Omega} z_{,\ell} [B_i^{\ell}[x, V_k] - B_i^{\ell}[x, V]] dx \\
 & = \int_{\Omega} z^i [F_i[x, W_k] - F_i[x, W]] dx.
 \end{aligned}$$

In (3.82), we put $z = W_k - W$ and use Assumption 3.2 to get

$$\begin{aligned}
(3.83) \quad & \lambda \int_{\Omega} |z_x|^2 dx \\
& \leq \int_{\Omega} |z_x| |C[x, V_k] - C[x, V]| |w_x| dx \\
& + \int_{\Omega} |z_x| |B[x, V_k] - B[x, V]| dx \\
& + \int_{\Omega} z^i [F_i[x, W_k] - F_i[x, W]] dx.
\end{aligned}$$

We now apply Lemma 2.2 to the first two integrals on the right hand side of (3.83) to obtain

$$\begin{aligned}
(3.84) \quad & \lambda \int_{\Omega} |z_x|^2 dx \leq 2\epsilon \int_{\Omega} |z_x|^2 dx \\
& + (4\epsilon)^{-1} \left[\sum_{i,j=1}^N \sum_{\ell,m=1}^n \int_{\Omega} |C_{ij}^{\ell m}[x, V_k] - C_{ij}^{\ell m}[x, V]|^2 |w_x|^2 dx \right. \\
& + \left. \sum_{i=1}^N \sum_{\ell=1}^n \int_{\Omega} |B_i^{\ell}[x, V_k] - B_i^{\ell}[x, V]|^2 dx \right] \\
& + \int_{\Omega} z^i [F_i[x, W_k] - F_i[x, W]] dx.
\end{aligned}$$

With the use of Assumption 3.2, we have

$$(3.85) \quad \int_{\Omega} z^i [F_i[x, W_k] - F_i[x, W]] dx \leq \eta \int_{\Omega} |z|^2 dx.$$

Using (3.85) in (3.84) and simplifying, we get

$$\begin{aligned}
(3.86) \quad & (\lambda - 2\varepsilon) \int_{\Omega} |z_x|^2 dx \leq \eta \int_{\Omega} |z|^2 dx \\
& + (4\varepsilon)^{-1} \sum_{i,j=1}^N \sum_{\ell,m=1}^n \int_{\Omega} |C_{ij}^{\ell m}[x, V_k] - C_{ij}^{\ell m}[x, V]|^2 |w_x|^2 dx \\
& + (4\varepsilon)^{-1} \sum_{i=1}^N \sum_{\ell=1}^n \int_{\Omega} |B_i^{\ell}[x, V_k] - B_i^{\ell}[x, V]|^2 dx.
\end{aligned}$$

Let $\varepsilon = \min \{(\lambda - b^2\eta)/4, \lambda/4\}$ in (3.86). If $\eta \leq 0$, then (3.86) implies

$$(3.87) \quad \int_{\Omega} |z_x|^2 dx \leq (2/\lambda^2) [\Delta B + \Delta C]$$

where ΔC and ΔB are the last two integrals on the right hand side of (3.86), respectively. If $\eta \geq 0$, then Lemma 2.2 implies

$$(3.88) \quad \eta \int_{\Omega} |z|^2 dx \leq b^2\eta \int_{\Omega} |z_x|^2 dx.$$

Using (3.88) in (3.86) yields

$$(3.89) \quad \int_{\Omega} |z_x|^2 dx \leq [2/(\lambda - b^2\eta)^2] [\Delta B + \Delta C].$$

Combining (3.87) and (3.89), we get

$$\begin{aligned}
(3.90) \quad & \sum_{i=1}^N \int_{\Omega} |[w_k^i - w^i]_x|^2 dx \equiv \int_{\Omega} |z_x|^2 dx \\
& \leq \max \{2/\lambda^2, 2/(\lambda - b^2\eta)^2\} [\Delta B + \Delta C].
\end{aligned}$$

By the continuity of B given in Assumption 3.2, it is clear that ΔB of (3.90) converges to zero as $k \rightarrow \infty$. The proof that ΔC converges to zero is done in the proof of Lemma 3.1 beginning with inequality (3.26) and is therefore omitted

here. This completes the proof.

LEMMA 3.10. The mapping $h : L_2(\Omega) \rightarrow L_2(\Omega)$ is compact whenever Assumption 3.2 is valid.

PROOF. Since $h(V) \in H_0^1(\Omega)$ whenever $V \in L_2(\Omega)$, we need only show that $h(A)$ is bounded in $H_0^1(\Omega)$ whenever A is a bounded set in $L_2(\Omega)$. Theorem 2.1 then implies the lemma. Thus, suppose A is a bounded set in $L_2(\Omega)$, and K is a constant such that A is contained in the set

$\{V \in L_2(\Omega) \mid \|V\|_{2,\Omega} \leq K\}$. Let $V \in A$. We use Assumption 3.2, Lemma 2.2, and (3.73) with $Z = W = h(V)$ to obtain

$$\begin{aligned} (3.91) \quad \lambda \int_{\Omega} |W_x|^2 dx &\leq \int_{\Omega} |W_x| |B[x,V]| dx + \int_{\Omega} W^i F_i[x,W] dx \\ &\leq \varepsilon \int_{\Omega} |W_x|^2 dx + (4\varepsilon)^{-1} \int_{\Omega} |B[x,V]|^2 dx \\ &\quad + \eta \int_{\Omega} |W|^2 dx. \end{aligned}$$

We let $\varepsilon = \tilde{\varepsilon} \equiv \min \{\lambda/2, (\lambda - b^2\eta)/2\}$ in (3.91). If $\eta \leq 0$, then (3.91) yields

$$\begin{aligned} (3.92) \quad \|W_x\|_{2,\Omega}^2 &\leq (2\lambda\tilde{\varepsilon})^{-1} \|B[\cdot, V(\cdot)]\|_{2,\Omega}^2 \\ &\leq [\varepsilon_1/(2\lambda\tilde{\varepsilon})] \|V\|_{2,\Omega}^2 + (M_B \text{mes } \Omega)/(2\lambda\tilde{\varepsilon}). \end{aligned}$$

Corollary 2.1 and (3.92) imply

$$\begin{aligned} (3.93) \quad \|W\|_{0,\Omega}^2 &\leq (1 + b^2) [\varepsilon_1/(2\lambda\tilde{\varepsilon})] K^2 \\ &\quad + (M_B \text{mes } \Omega)/(2\lambda\tilde{\varepsilon}). \end{aligned}$$

If $\eta \geq 0$, then Lemma 2.1 along with (3.91) gives

$$\begin{aligned}
 (3.94) \quad & \lambda ||w_x||_{2,\Omega}^2 \leq (\tilde{\varepsilon} + b^2\eta) ||w_x||_{2,\Omega}^2 \\
 & + (4\tilde{\varepsilon})^{-1} ||B[\cdot, V(\cdot)]||_{2,\Omega}^2 \\
 & = [(\lambda + b^2\eta)/2] ||w_x||_{2,\Omega}^2 + (4\tilde{\varepsilon})^{-1} ||B[\cdot, V(\cdot)]||_{2,\Omega}^2.
 \end{aligned}$$

We now have from (3.94) and Assumption 3.2

$$\begin{aligned}
 (3.95) \quad & ||w_x||_{2,\Omega}^2 \leq [2\tilde{\varepsilon}(\lambda - b^2\eta)]^{-1} ||B[\cdot, V(\cdot)]||_{2,\Omega}^2 \\
 & \leq [\varepsilon_1/2\tilde{\varepsilon}(\lambda - b^2\eta)] ||V||_{2,\Omega}^2 \\
 & + (M_B \text{ mes } \Omega)/2\tilde{\varepsilon}(\lambda - b^2\eta).
 \end{aligned}$$

Corollary 2.1 and the definition of A imply

$$\begin{aligned}
 (3.96) \quad & ||W||_{0,\Omega}^2 \leq [\varepsilon_1/2\tilde{\varepsilon}(\lambda - b^2\eta)] K^2 \\
 & + (M_B \text{ mes } \Omega)/[2\tilde{\varepsilon}(\lambda - b^2\eta)],
 \end{aligned}$$

and the proof is complete.

LEMMA 3.11. Suppose Assumption 3.2 is valid and

$A \equiv \{V \in L_2(\Omega) \mid ||V||_{2,\Omega} \leq \hat{K}\}$ where

$\hat{K} \equiv \{(b^2 M_B \text{ mes } \Omega)/[(\min \{\lambda, \lambda - b^2\eta\})^2 - b^2\varepsilon_1]\}^{1/2}$. Then

$h(A) \subset A$.

PROOF. If $\eta \leq 0$, then (3.92) along with Lemma 2.1 implies $(W = h(V))$

$$\begin{aligned}
(3.97) \quad ||h(v)||_{2,\Omega}^2 &\leq b^2 ||h(v)_x||_{2,\Omega}^2 \\
&\leq [b^2/2\lambda\tilde{\epsilon}] [\epsilon_1 ||v||_{2,\Omega}^2 + M_B \text{mes } \Omega] \\
&\leq [b^2/2\lambda\tilde{\epsilon}] [\epsilon_1 \hat{K}^2 + M_B \text{mes } \Omega] \leq \hat{K}^2
\end{aligned}$$

If $\eta > 0$, then (3.95) along with Lemma 2.1 implies

$$\begin{aligned}
(3.98) \quad ||h(v)||_{2,\Omega}^2 &\leq b^2 ||h(v)_x||_{2,\Omega}^2 \\
&\leq [b^2/\{2\tilde{\epsilon}(\lambda - b^2\eta)\}] \{\epsilon_1 ||v||_{2,\Omega}^2 + M_B \text{mes } \Omega\} \\
&\leq [b^2/\{2\tilde{\epsilon}(\lambda - b^2\eta)\}] \{\epsilon_1 \hat{K}^2 + M_B \text{mes } \Omega\} \leq \hat{K}^2,
\end{aligned}$$

and the proof is complete.

THEOREM 3.4. Suppose Assumption 3.2 is valid. Then there is at least one solution to (1.1) and (1.2).

PROOF. Lemma 3.9 and Lemma 3.10 show that the mapping $h : L_2(\Omega) \rightarrow L_2(\Omega)$ is a compact map. Lemma 3.11 proves the existence of a closed bounded convex set A in $L_2(\Omega)$ such that $h(A) \subset A$. Theorem 2.2 implies h has a fixed point in $L_2(\Omega)$, and the proof is complete.

CHAPTER IV

UNIQUENESS AND STABILITY

We shall study uniqueness and stability of the weak solution of (1.1) and (1.2) under stronger assumptions on the auxiliary functions than those needed for existence. Strengthening of Assumption 3.1 and Assumption 3.2 is obviously needed as indicated by the following example:

$$(4.1) \quad \frac{d}{dx} \left(\frac{dU}{dx} - r(1 - 2x) U^{(r-1)/r} \right) = 0, \quad x \in (0, 1)$$

$$U(0) = U(1) = 0.$$

The auxiliary functions involved in (4.1) clearly satisfy Assumption 3.1 and Assumption 3.2, but the problem has two solutions, $U = 0$ and $U(x) = (x - x^2)^r$. Here, r is any fixed number greater than 1.

The following requirement imposed upon the auxiliary functions is motivated by a similar assumption used by Ford [5] in studying a two-point boundary value problem.

ASSUMPTION 4.1. Suppose there exist constants η and γ such that whenever $U, V, p^1, \dots, p^n, q^1, \dots, q^n \in E_N$ and $x \in \Omega$,

$$(4.2) \quad (p_\ell^i - q_\ell^i) (C_{ij}^{\ell m}[x, U] p_m^j - C_{ij}^{\ell m}[x, V] q_m^j + B_i^\ell[x, U] - B_i^\ell[x, V]) - (U^i - V^i) (F_i[x, U] - F_i[x, V])$$

$$\geq \gamma |p - q| - \eta |U - V|$$

where

$$(4.3) \quad \gamma \geq 0, \quad \gamma - b^2 \eta > 0,$$

and b is from (2.7).

When $n = N = 1$ the above assumption reduces to a special case of a condition given by Ford [5].

THEOREM 4.1. Suppose Assumption 4.1 is valid. Then there is at most one solution to (1.1) and (1.2).

PROOF. Suppose U and V are solutions of (1.1) and (1.2). This implies that U satisfies (1.12), and V satisfies a similar equation. Taking the difference between the two expressions yields

$$(4.4) \quad \int_{\Omega} W_{,\ell}^i [C_{ij}^{\ell m}[x, U] U_{,m}^j - C_{ij}^{\ell m}[x, V] V_{,m}^j + B_i^{\ell}[x, U] + B_i^{\ell}[x, V]] dx + \int_{\Omega} W^i [F_i[x, U] - F_i[x, V]] dx = 0$$

for all W in $H_0^1(\Omega)$. Let $W = U - V$ in (4.4), and use Assumption 4.1 to get

$$(4.5) \quad \lambda \int_{\Omega} |W_x|^2 dx - \eta \int_{\Omega} |W|^2 dx \leq 0.$$

If $\eta < 0$, then (4.3) and (4.5) imply

$$(4.6) \quad -\eta \int_{\Omega} |W|^2 dx \leq 0,$$

and this implies

$$(4.7) \quad \int_{\Omega} |W|^2 dx \leq 0.$$

This yields $W = 0$ a.e. If $\eta > 0$, then (4.5) gives

$$(4.8) \quad \gamma \int_{\Omega} |W_x|^2 dx \leq \eta \int_{\Omega} |W|^2 dx.$$

Lemma 2.1 with $p = 2$ yields the estimate

$$(4.9) \quad \eta \int_{\Omega} |W|^2 dx \leq \eta b^2 \int_{\Omega} |W_x|^2 dx.$$

Combining (4.8) and (4.9) produces

$$(4.10) \quad (\gamma - b^2\eta) \int_{\Omega} |W_x|^2 dx \leq 0.$$

Inequalities (4.3) and (4.10) imply

$$(4.11) \quad \int_{\Omega} |W_x|^2 dx \leq 0$$

which yields $W = 0$ a.e. and the proof is complete.

By requiring $\gamma > 0$ in Assumption 4.1, we can obtain a stability condition for the problem (1.1) and (1.2).

THEOREM 4.2. Suppose Assumption 4.1 is valid with $\gamma > 0$, and U is a solution of (1.1) and (1.2) with $|U_x| \in L_{2p}(\Omega)$, $p \geq 1$. If $\tilde{U}$ is a solution (1.1) and (1.2) with auxiliary functions $\tilde{C}$, $\tilde{B}$, and $\tilde{F}$ satisfying Assumption 4.1 ($\gamma > 0$) with all constants the same as for C , B , and F , then there exists a positive constant K_0 depending only on known constants such that

$$\begin{aligned}
(4.12) \quad & ||U - \tilde{U}||_{0,\Omega} \\
& \leq K_0 [|||U_x|||_{2p,\Omega} |||C[\cdot, U(\cdot)] - \tilde{C}[\cdot, U(\cdot)]|||_{2p',\Omega} \\
& \quad + |||B[\cdot, U(\cdot)] - \tilde{B}[\cdot, U(\cdot)]|||_{2,\Omega} \\
& \quad + |||F[\cdot, U(\cdot)] - \tilde{F}[\cdot, U(\cdot)]|||_{2,\Omega}]
\end{aligned}$$

where $(1/p) + (1/p') = 1$.

PROOF. Using (1.12) and the corresponding form of (1.12) for $\tilde{U}$, we take the difference between the two equations to obtain

$$\begin{aligned}
(4.13) \quad & \int_{\Omega} z^i_{,\ell} [C^{\ell m}_{ij}[x, U] U^j_{,m} - \tilde{C}^{\ell m}_{ij}[x, \tilde{U}] \tilde{U}^j_{,m} + B^{\ell}_i[x, U] \\
& \quad - \tilde{B}^{\ell}_i[x, \tilde{U}]] dx - \int_{\Omega} z^i [F_i[x, U] - \tilde{F}_i[x, \tilde{U}]] dx = 0.
\end{aligned}$$

By adding and subtracting $\tilde{C}^{\ell m}_{ij}[x, U] U^j_{,m}$, $\tilde{B}^{\ell}_i[x, U]$, and $\tilde{F}_i[x, U]$ to the integrands and collecting terms, we get

$$\begin{aligned}
(4.14) \quad & \int_{\Omega} z^i_{,\ell} [\tilde{C}^{\ell m}_{ij}[x, U] U^j_{,m} - \tilde{C}^{\ell m}_{ij}[x, \tilde{U}] \tilde{U}^j_{,m} + \tilde{B}^{\ell}_i[x, U] \\
& \quad - \tilde{B}^{\ell}_i[x, \tilde{U}]] dx - \int_{\Omega} z^i [\tilde{F}_i[x, U] - \tilde{F}_i[x, \tilde{U}]] dx \\
& = \int_{\Omega} z^i_{,\ell} [\tilde{C}^{\ell m}_{ij}[x, U] U^j_{,m} - C^{\ell m}_{ij}[x, U] U^j_{,m}] dx \\
& \quad + \int_{\Omega} z^i_{,\ell} [\tilde{B}^{\ell}_i[x, U] - B^{\ell}_i[x, U]] dx
\end{aligned}$$

$$- \int_{\Omega} z^i [\tilde{F}_i[x, U] - F_i[x, U]] dx$$

for all z in $H_0^1(\Omega)$. Now let $z = U - \tilde{U}$, and apply Assumption 4.1 to the left hand side of (4.14) to obtain

$$\begin{aligned} (4.15) \quad & \gamma \int_{\Omega} |z_x|^2 dx - \eta \int_{\Omega} |z|^2 dx \\ & \leq \int_{\Omega} z_{,\ell}^i [\tilde{C}_{ij}^{\ell m}[x, U] - C_{ij}^{\ell m}[x, U]] U_{,m}^j dx \\ & + \int_{\Omega} z_{,\ell}^i [\tilde{B}_i^{\ell}[x, U] - B_i^{\ell}[x, U]] dx \\ & - \int_{\Omega} z^i [\tilde{F}_i[x, U] - F_i[x, U]] dx. \end{aligned}$$

We now use Lemma 2.2 to obtain the following estimates for the integrals on the right hand side of (4.15). For $\epsilon > 0$, we have

$$\begin{aligned} (4.16) \quad & \int_{\Omega} z_{,\ell}^i [\tilde{C}_{ij}^{\ell m}[x, U] - C_{ij}^{\ell m}[x, U]] U_{,m}^j dx \\ & \leq \int_{\Omega} |z_x|^2 dx + (4\epsilon)^{-1} \int_{\Omega} |\tilde{C}[x, U] - C[x, U]|^2 |U_x|^2 dx. \end{aligned}$$

$$\begin{aligned} (4.17) \quad & \int_{\Omega} z_{,\ell}^i [\tilde{B}_i^{\ell}[x, U] - B_i^{\ell}[x, U]] dx \leq \epsilon \int_{\Omega} |z_x|^2 dx \\ & + (4\epsilon)^{-1} \int_{\Omega} |\tilde{B}[x, U] - B[x, U]|^2 dx. \end{aligned}$$

$$(4.18) \quad \int_{\Omega} z^i [\tilde{F}_i[x, U] - F_i[x, U]] dx \leq \epsilon \int_{\Omega} |z|^2 dx$$

$$\begin{aligned}
& + (4\varepsilon)^{-1} \int_{\Omega} |\tilde{F}[x,U] - F[x,U]|^2 dx \\
& \leq \varepsilon b^2 \int_{\Omega} |z_x|^2 dx + (4\varepsilon)^{-1} \int_{\Omega} |\tilde{F}[x,U] - F[x,U]|^2 dx
\end{aligned}$$

where b is from (2.7). Combining (4.15) through (4.18) yields

$$\begin{aligned}
(4.19) \quad & \gamma \int_{\Omega} |z_x|^2 dx - \eta \int_{\Omega} |z|^2 dx \leq \varepsilon(2 + b^2) \int_{\Omega} |z_x|^2 dx \\
& + (4\varepsilon)^{-1} \left[\int_{\Omega} |\tilde{C}[x,U] - C[x,U]|^2 |U_x|^2 dx \right. \\
& \quad + \int_{\Omega} |\tilde{B}[x,U] - B[x,U]|^2 dx \\
& \quad \left. + \int_{\Omega} |\tilde{F}[x,U] - F[x,U]|^2 dx \right].
\end{aligned}$$

We also have from Hölder's inequality

$$\begin{aligned}
(4.20) \quad & \int_{\Omega} |\tilde{C}[x,U] - C[x,U]|^2 |U_x|^2 dx \\
& \leq \| |U_x| \|_{2p,\Omega}^2 \| |\tilde{C}[\cdot, U(\cdot)] - C[\cdot, U(\cdot)]| \|_{2p',\Omega}^2.
\end{aligned}$$

If $\eta \leq 0$, then

$$(4.21) \quad \gamma \int_{\Omega} |z_x|^2 dx - \eta \int_{\Omega} |z|^2 dx \geq \gamma \int_{\Omega} |z_x|^2 dx.$$

On the other hand, if $\eta > 0$, Lemma 2.1 with $p = 2$ implies

$$(4.22) \quad \gamma \int_{\Omega} |z_x|^2 dx - \eta \int_{\Omega} |z|^2 dx \geq (\gamma - b^2 \eta) \int_{\Omega} |z_x|^2 dx.$$

We define $\delta_0 \equiv \min \{\gamma, \gamma - b^2 \eta\}$. Then (4.21) and (4.22)

imply

$$(4.23) \quad \delta_0 \int_{\Omega} |z_x|^2 dx \leq \gamma \int_{\Omega} |z_x|^2 dx - \eta \int_{\Omega} |z|^2 dx.$$

Substituting (4.20) and (4.23) into (4.19) and collecting like terms, we get

$$(4.24) \quad [\delta_0 - \varepsilon(2 + b^2)] \int_{\Omega} |z_x|^2 dx \\ \leq (4\varepsilon)^{-1} [\| |u_x| \|_{2p, \Omega}^2 \| C[\cdot, U(\cdot)] - \tilde{C}[\cdot, U(\cdot)] \|_{2p', \Omega}^2 \\ + \| |B[\cdot, U(\cdot)] - \tilde{B}[\cdot, U(\cdot)] \|_{2, \Omega}^2 \\ + \| |F[\cdot, U(\cdot)] - \tilde{F}[\cdot, U(\cdot)] \|_{2, \Omega}^2].$$

Let $\varepsilon = \delta_0 [2(2 + b^2)]^{-1}$ in (4.24) and use Corollary 2.1 to obtain (4.12) with $K_0 = \{[(1 + b^2)(2 + b^2)]^{1/2}\} / \delta_0$. This completes the proof.

A slightly stronger stability result is now given in which no extra smoothness condition on any solution is required.

THEOREM 4.3. Suppose the sequences $\{C_{ijk}^{\ell m}\}_{k=1}^{\infty}$, $\{B_{ik}^{\ell}\}_{k=1}^{\infty}$, and $\{F_{ik}\}_{k=1}^{\infty}$ ($1 \leq i, j \leq N$ and $1 \leq \ell, m \leq n$) of functions from $\Omega \times E_N$ into E_1 satisfy the conditions of Assumption 4.1 and $\{C_{ijk}^{\ell m}\}_{k=1}^{\infty}$ satisfy inequality (3.2) with the constants the same as those for $C_{ij}^{\ell m}$, B_i^{ℓ} , and F_i . If $C_{ijk}^{\ell m}[\cdot, V(\cdot)]$, $B_{ik}^{\ell}[\cdot, V(\cdot)]$, and $F_{ik}[\cdot, V(\cdot)]$ converge to $C_{ij}^{\ell m}[\cdot, V(\cdot)]$, $B_i^{\ell}[\cdot, V(\cdot)]$, and $F_i[\cdot, V(\cdot)]$, respectively, in $L_2(\Omega)$ for each

V in $L_2(\Omega)$ as k approaches ∞ , then the sequence of weak solutions $\{U_k\}_{k=1}^{\infty}$ converges to U in $H_0^1(\Omega)$.

PROOF. The analysis is identical to that in the proof of Theorem 4.2 through inequality (4.19) with $\tilde{C}_{ij}^{\ell m}$, $\tilde{B}_i^{\ell}$, $\tilde{F}_i$, and $\tilde{U}$ replaced by $C_{ijk}^{\ell m}$, B_{ik}^{ℓ} , F_{ik} , and U_k , respectively. We then apply inequalities (4.19) and (4.23) to get

$$\begin{aligned}
 (4.25) \quad & ||U - U_k||_{0,\Omega} \\
 & \leq K_0 [|| |C[\cdot, U(\cdot)] - C_k[\cdot, U(\cdot)]| |U_x| ||_{2,\Omega} \\
 & \quad + || |B[\cdot, U(\cdot)] - B_k[\cdot, U(\cdot)]| ||_{2,\Omega} \\
 & \quad + || |F[\cdot, U(\cdot)] - F_k[\cdot, U(\cdot)]| ||_{2,\Omega}]
 \end{aligned}$$

where K_0 is from Theorem 4.2. From (4.25) and the hypothesis of the theorem, it suffices to show

$$(4.26) \quad \lim_{k \rightarrow \infty} || |C_k[\cdot, U(\cdot)] - C[\cdot, U(\cdot)]| |U_x| ||_{2,\Omega} = 0$$

since the last two terms in (4.25) clearly converge to zero as k approaches ∞ . To prove (4.26), we let $\delta > 0$ be given. We choose a sequence $\{W_j\}_{j=1}^{\infty} \subset C_0^{\infty}(\Omega)$ such that $\lim_{j \rightarrow \infty} ||W_j - U||_{0,\Omega} = 0$. Suppose A_j is a constant such that

$||W_{j_x}||_{\infty,\Omega} \leq A_j$, $j = 1, 2, \dots$. Choose a positive integer $J(\delta)$ depending only on δ such that $||W_{J(\delta)_x} - U_x||_{2,\Omega} < \delta/4M_C$ where M_C is the constant from (3.2). We now choose $K(J(\delta))$

depending only on $J(\delta)$ such that $k \geq K(J(\delta))$ implies

$$(4.27) \quad ||| C_k[\cdot, U(\cdot)] - C[\cdot, U(\cdot)] |||_{2, \Omega} < \delta / (2A_{J(\delta)}).$$

We now have the following sequence of inequalities:

$$\begin{aligned}
 (4.28) \quad & ||| |C_k[\cdot, U(\cdot)] - C[\cdot, U(\cdot)]| |U_x| |||_{2, \Omega} \\
 & \leq ||| |C_k[\cdot, U(\cdot)] - C[\cdot, U(\cdot)]| |U_x - W_{J(\delta)_x}| |||_{2, \Omega} \\
 & \quad + ||| |C_k[\cdot, U(\cdot)] - C[\cdot, U(\cdot)]| |W_{J(\delta)_x}| |||_{2, \Omega} \\
 & \leq 2M_C ||| U_x - W_{J(\delta)_x} |||_{2, \Omega} \\
 & \quad + A_{J(\delta)} ||| C_k[\cdot, U(\cdot)] - C[\cdot, U(\cdot)] |||_{2, \Omega} \\
 & < (\delta/2) + (\delta/2) = \delta
 \end{aligned}$$

whenever $k \geq K(J(\delta))$, and thus, (4.26) is proven. This completes the proof.

CHAPTER V

BOUNDEDNESS OF WEAK SOLUTIONS

We now show that under more stringent restrictions on the auxiliary functions than those used for existence all weak solutions of (1.1) and (1.2) are bounded (a.e.) by a constant depending only on the constants of Assumption 3.1 and other known constants.

ASSUMPTION 5.1. Suppose Assumption 3.1 is satisfied except for inequalities (3.3), (3.4), and (3.5). In place of these, we suppose there exist nonnegative constants M and α such that

$$(5.1) \quad |F[x, U]| + |B[x, U]| \leq M[1 + |U|^\alpha]$$

for all $x \in \Omega$ and $U \in E_N$ where $\alpha < \min \{1, 2/(n-2)\}$ for $n = 2, 3, 4$, or 5 and $\alpha = 1/2$ for $n \geq 6$. Furthermore, suppose

$$(5.2) \quad C_{ij}^{lm}[x, U] = 0 \quad \text{for } i \neq j, \quad x \in \Omega, \quad U \in E_N.$$

Before proving boundedness of the weak solutions in the $L_\infty(\Omega)$ norm, we consider some preliminary lemmas.

LEMMA 5.1. Suppose Assumption 3.1 is valid, and U is a solution of (1.1) and (1.2). Then there exists a constant M_0 depending only on known constants such that

$$(5.3) \quad ||U_x||_{2, \Omega} \leq M_0.$$

PROOF. In (1.12) put $Z = U$ and use Assumption 3.1 along with Lemma 2.2 to get

$$\begin{aligned}
 (5.4) \quad \lambda \int_{\Omega} |U_x|^2 dx &\leq \int_{\Omega} |U_x| |B| dx + \int_{\Omega} |U| |F| dx \\
 &\leq \varepsilon \int_{\Omega} |U_x|^2 dx + (4\varepsilon)^{-1} \int_{\Omega} |B|^2 dx \\
 &\quad + \varepsilon \int_{\Omega} |U|^2 dx + (4\varepsilon)^{-1} \int_{\Omega} |F|^2 dx.
 \end{aligned}$$

Assumption 3.1 and Lemma 2.1 imply

$$\begin{aligned}
 (5.5) \quad (4\varepsilon)^{-1} \int_{\Omega} [|B|^2 + |F|^2] dx \\
 &\leq (\varepsilon_0 + \varepsilon_1)/4\varepsilon \int_{\Omega} |U|^2 dx \\
 &\quad + [(M_F + M_B) \text{mes } \Omega]/4\varepsilon.
 \end{aligned}$$

Inequalities (5.4) and (5.5) combined with Lemma 2.1 yield

$$\begin{aligned}
 (5.6) \quad \lambda \int_{\Omega} |U_x|^2 dx \\
 &\leq [\varepsilon(1 + b^2) + [(\varepsilon_0 + \varepsilon_1)b^2/4\varepsilon]] \int_{\Omega} |U_x|^2 dx \\
 &\quad + [(M_F + M_B) \text{mes } \Omega]/4\varepsilon.
 \end{aligned}$$

In (5.6) we let $\varepsilon = \lambda[2(1 + b^2)]^{-1}$ and use (3.3) to get

$$\begin{aligned}
 (5.7) \quad \lambda \int_{\Omega} |U_x|^2 dx &\leq (\lambda/2) [(1 + 2b^2)/(1 + b^2)] \int_{\Omega} |U_x|^2 dx \\
 &\quad + [(M_F + M_B)(1 + b^2)/2\lambda] \text{mes } \Omega.
 \end{aligned}$$

Simplification of (5.7) yields (5.3), and the proof is complete.

LEMMA 5.2. Suppose Assumption 5.1 is valid with $n \geq 6$. Then there exists a constant $\overline{M}(s)$ depending only on known constants and the positive integer s such that

$$(5.8) \quad \int_{\Omega} |U|^{4s} dx \leq \overline{M}(s)$$

for each $s = 1, 2, 3, \dots$

PROOF. Let k be a fixed integer such that $1 \leq k \leq N$. Let $V = (0, \dots, 0, U^k, 0, \dots, 0)$ where the U^k appears in the k -th position of the vector V . The proof is by induction on s . Thus, suppose $s = 1$. In (1.12), let $Z(x) = V(x)\phi(x)$ where $\phi(x) \equiv \min \{|V(x)|^2, R\}$ and $R > 0$. Clearly, Z is an element of $H_0^1(\Omega)$. With this value of Z , we get

$$(5.9) \quad \int_{\Omega} \phi V_{,\ell}^i C_{ij}^{\ell m} U_{,m}^j dx + \int_{\Omega} \phi_{,\ell} C_{ij}^{\ell m} V^i U_{,m}^j dx \\ = - \int_{\Omega} V_{,\ell}^i \phi B_i^{\ell} dx - \int_{\Omega} V^i \phi_{,\ell} B_i^{\ell} dx + \int_{\Omega} V^i \phi F_i dx$$

Assumption 5.1 and the definition of ϕ applied to (5.9) give

$$(5.10) \quad \lambda \int_{\Omega} \phi |V_x|^2 dx + (\lambda/2) \int_{\Omega} |\phi_x|^2 dx \\ \leq \int_{\Omega} |V_x| |\phi| |B_k| dx + \int_{\Omega} |V| |\phi_x| |B_k| dx \\ + \int_{\Omega} |V| |\phi| |F_k| dx$$

where $|B_k| \equiv [\sum_{\ell=1}^n |B_k^\ell|^2]^{1/2}$. We now use Assumption 5.1 and Lemma 2.2 to estimate the integrals on the right hand side of (5.10).

$$\begin{aligned}
 (5.11) \quad & \int_{\Omega} |V_x| \Phi |B_k| \, dx \leq M \int_{\Omega} |V_x| \Phi \, dx \\
 & + M \int_{\Omega} |V_x| \Phi |U|^{1/2} \, dx \\
 & \leq 2\epsilon \int_{\Omega} |V_x|^2 \Phi \, dx + (M^2/4\epsilon) \int_{\Omega} \Phi \, dx \\
 & + (M^2/4\epsilon) \int_{\Omega} \Phi |U| \, dx \\
 & \leq 2\epsilon \int_{\Omega} |V_x|^2 \Phi \, dx + 2\epsilon \int_{\Omega} |\Phi|^2 \, dx \\
 & + [M^4/(64\epsilon^3)] [\text{mes } \Omega + \int_{\Omega} |U|^2 \, dx]
 \end{aligned}$$

where b is from (2.7).

$$\begin{aligned}
 (5.12) \quad & \int_{\Omega} |V| |\Phi_x| |B_k| \, dx \leq M \int_{\Omega} |V| |\Phi_x| \, dx \\
 & + M \int_{\Omega} |V| |\Phi_x| |U|^{1/2} \, dx \\
 & = M \int_{\Omega} (\Phi)^{1/2} |\Phi_x| \, dx + M \int_{\Omega} (\Phi)^{1/2} |\Phi_x| |U|^{1/2} \, dx \\
 & \leq 2\epsilon \int_{\Omega} |\Phi_x|^2 \, dx + (M^2/4\epsilon) \left[\int_{\Omega} \Phi \, dx \right. \\
 & \left. + \int_{\Omega} \Phi |U| \, dx \right]
 \end{aligned}$$

$$\begin{aligned}
&\leq 2\varepsilon \int_{\Omega} |\Phi_x|^2 dx + 2\varepsilon \int_{\Omega} |\Phi|^2 dx \\
&\quad + (M^4/64\varepsilon^3) [\text{mes } \Omega + \int_{\Omega} |U|^2 dx] \\
&\leq 2\varepsilon(1+b^2) \int_{\Omega} |\Phi_x|^2 dx \\
&\quad + (M^4/64\varepsilon^3) [\text{mes } \Omega + \int_{\Omega} |U|^2 dx].
\end{aligned}$$

We now use the inequality (no summation on m)

$$(5.13) \quad |[(|V|^2 \Phi)^{1/2}]_{,m}|^2 \leq 4 \Phi |V_{,m}|^2, \quad 1 \leq m \leq n,$$

from [8, p. 392] in estimating the last term in (5.10).

Thus, we obtain

$$\begin{aligned}
(5.14) \quad &\int_{\Omega} |V| \Phi |F_k| dx \leq M \int_{\Omega} |V| \Phi dx \\
&\quad + M \int_{\Omega} |V| \Phi |U|^{1/2} dx \\
&\leq 2\varepsilon \int_{\Omega} |V|^2 \Phi dx + (M^2/4\varepsilon) \left[\int_{\Omega} \Phi dx + \int_{\Omega} \Phi |U| dx \right] \\
&\leq \varepsilon \int_{\Omega} [(|V|^2 \Phi)^{1/2}]^2 dx + 2\varepsilon \int_{\Omega} |\Phi|^2 dx \\
&\quad + (M^4/64\varepsilon^3) [\text{mes } \Omega + \int_{\Omega} |U|^2 dx] \\
&\leq 2\varepsilon b^2 \int_{\Omega} |[(|V|^2 \Phi)^{1/2}]_x|^2 dx + 2\varepsilon b^2 \int_{\Omega} |\Phi_x|^2 dx \\
&\quad + (M^4/64\varepsilon^3) [\text{mes } \Omega + \int_{\Omega} |U|^2 dx]
\end{aligned}$$

$$\begin{aligned}
&\leq 8\epsilon b^2 \int_{\Omega} \Phi |v_x|^2 dx + 2\epsilon b^2 \int_{\Omega} |\phi_x|^2 dx \\
&\quad + (M^4/64\epsilon^3) [\text{mes } \Omega + \int_{\Omega} |U|^2 dx]
\end{aligned}$$

where we have used (5.13) to get the last inequality. We now may combine (5.10) through (5.14) to obtain

$$\begin{aligned}
(5.15) \quad &\lambda \int_{\Omega} \Phi |v_x|^2 dx + (\lambda/2) \int_{\Omega} |\phi_x|^2 dx \\
&\leq 2\epsilon [1 + 4b^2] \int_{\Omega} \Phi |v_x|^2 dx \\
&\quad + 2\epsilon [1 + 3b^2] \int_{\Omega} |\phi_x|^2 dx \\
&\quad + (3M^4/64\epsilon^3) [\text{mes } \Omega + \int_{\Omega} |U|^2 dx].
\end{aligned}$$

Choose $\epsilon = \lambda [8(1 + 4b^2)]^{-1}$ in (5.15) and collect like terms to get

$$\begin{aligned}
(5.16) \quad &(3\lambda/4) \int_{\Omega} \Phi |v_x|^2 dx + (\lambda/4) \int_{\Omega} |\phi_x|^2 dx \\
&\leq K_0(\lambda, \text{mes } \Omega, M, b, n) + K_1(\lambda, M) \int_{\Omega} |U|^2 dx \\
&\leq K_0 + K_1 M_0^2 b^2 \equiv M_1
\end{aligned}$$

where b is from (2.7), and M_0 is from (5.3). The constants K_0 and K_1 are obtained by simplifying (5.15) for the given ϵ . Lemma 2.1 with $p = 2$ combined with (5.16) implies

$$(5.17) \quad \int_{\Omega} |\Phi|^2 dx \leq b^2 \int_{\Omega} |\Phi_x|^2 dx \leq (4b^2/\lambda) M_1.$$

The right hand side of (5.17) is independent of R , and thus, by letting R approach ∞ , we obtain

$$(5.18) \quad \int_{\Omega} |V|^4 dx \leq (4b^2/\lambda) M_1.$$

Since $V^i = 0$ for $i \neq k$, $V^k = U^k$, and k was chosen arbitrarily, we have (5.18) true for all U^i , $1 \leq i \leq N$. Summing on i from 1 to N , we get

$$(5.19) \quad \int_{\Omega} |U|^4 dx \leq (4b^2/\lambda) M_1 N^3$$

where we have used the inequality

$$(5.20) \quad |U|^4 = \left[\sum_{i=1}^N |U^i|^2 \right]^2 \leq N^2 \sum_{i=1}^N |U^i|^4.$$

Now suppose (5.8) holds for $s = q \geq 1$. We must show that (5.8) is valid for $s = q + 1$. Thus, we define $Z = \Phi \psi V$ with $\Phi(x) \equiv \min \{ |V(x)|^{2(q+1)}, R \}$, $\psi(x) \equiv \min \{ |V(x)|^{2q}, R^q/(q+1) \}$ with $R > 0$, and V is defined as before. Clearly, Z is an element of $H_0^1(\Omega)$. With this value of Z in (1.12), we get

$$\begin{aligned} (5.21) \quad & \int_{\Omega} \Phi \psi V_{,\ell}^i C_{ij}^{\ell m} U_{,m}^j dx + \int_{\Omega} \Phi_{,\ell} \psi V^i C_{ij}^{\ell m} U_{,m}^j dx \\ & + \int_{\Omega} \Phi \psi_{,\ell} V^i C_{ij}^{\ell m} U_{,m}^j dx = - \int_{\Omega} \Phi \psi V_{,\ell}^i B_i^{\ell} dx \\ & + \int_{\Omega} \psi_{,\ell} \Phi V^i B_i^{\ell} dx + \int_{\Omega} \Phi_{,\ell} \psi V^i B_i^{\ell} dx \\ & + \int_{\Omega} \Phi \psi V^i F_i dx. \end{aligned}$$

Using Assumption 5.1, we obtain

$$\begin{aligned}
 (5.22) \quad & \lambda \int_{\Omega} \Phi \psi |V_x|^2 dx + \lambda [2(q+1)]^{-1} \int_{\Omega} |\Phi_x|^2 dx \\
 & + q\lambda [2(q+1)^2]^{-1} \int_{\Omega} |\Phi_x|^2 dx \\
 & \leq \int_{\Omega} \psi \Phi |V_x| |B_k| dx + \int_{\Omega} |\psi_x| \Phi |V| |B_k| dx \\
 & + \int_{\Omega} \psi |\Phi_x| |V| |B_k| dx + \int_{\Omega} \psi \Phi |V| |F_k| dx.
 \end{aligned}$$

We now give estimates analogous to those for the case $s = 1$ for the terms on the right hand side of (5.22). We shall use the fact that $|V| |\Phi_{,m}| = (\Phi)^{1/2(q+1)} |\Phi_{,m}| = (\psi)^{1/2q} |\Phi_{,m}|$ ($1 \leq m \leq n$) throughout the estimates.

$$\begin{aligned}
 (5.23) \quad & \int_{\Omega} |V| \psi |\Phi_x| |B_k| dx \leq M \int_{\Omega} |V| \psi |\Phi_x| dx \\
 & + M \int_{\Omega} |V| \psi |\Phi_x| |U|^{1/2} dx = M \int_{\Omega} \Phi^{\delta} |\Phi_x| dx \\
 & + M \int_{\Omega} \Phi^{\delta} |\Phi_x| |U|^{1/2} dx \\
 & \leq 2\epsilon \int_{\Omega} |\Phi_x|^2 dx + (M^2/4\epsilon) \left[\int_{\Omega} \Phi^{2\delta} dx \right. \\
 & \left. + \int_{\Omega} \Phi^{2\delta} |U| dx \right] \\
 & \leq 2\epsilon \int_{\Omega} |\Phi_x|^2 dx + 2\epsilon \int_{\Omega} |\Phi|^2 dx
 \end{aligned}$$

$$\begin{aligned}
& + [1/(2q+2)] [\delta/\varepsilon]^{2q+1} [M^2/4\varepsilon]^{2q+2} [\text{mes } \Omega \\
& \quad + \int_{\Omega} |U|^{2q+2} dx] \\
\leq & 2\varepsilon(1+b^2) \int_{\Omega} |\Phi_x|^2 dx \\
& + [1/(2q+2)] [\delta/\varepsilon]^{2q+1} [M^2/4\varepsilon]^{2q+2} [\text{mes } \Omega \\
& \quad + \int_{\Omega} |U|^{2q+2} dx]
\end{aligned}$$

with $\delta \equiv (2q+1)/(2q+2)$. Since $q \geq 1$, it is clear that $2q+2 \leq 4q$ which implies that the last integral in (5.23) converges by the induction hypothesis. Continuing with the estimates, we have

$$\begin{aligned}
(5.24) \quad & (2q+1) \int_{\Omega} \Phi \psi |V_x| |B_k| dx \leq (2q+1)M \left[\int_{\Omega} \Phi \psi |V_x| dx \right. \\
& \quad \left. + \int_{\Omega} \Phi \psi |V_x| |U|^{1/2} dx \right] \\
\leq & 2\varepsilon \int_{\Omega} \Phi \psi |V_x|^2 dx + (2q+1)^2 M^2/4\varepsilon \left[\int_{\Omega} \Phi \psi dx \right. \\
& \quad \left. + \int_{\Omega} \Phi \psi |U| dx \right] \\
\leq & 2\varepsilon \int_{\Omega} \Phi \psi |V_x|^2 dx + 2\varepsilon \int_{\Omega} |\Phi|^2 dx \\
& + [1/(2q+2)] [\delta/\varepsilon]^{2q+1} [(2q+1)^2 M^2/4\varepsilon]^{2q+2} [\text{mes } \Omega \\
& \quad + \int_{\Omega} |U|^{2q+2} dx]
\end{aligned}$$

$$\begin{aligned}
& \leq 2\varepsilon \int_{\Omega} \Phi \psi |V_x|^2 dx + 2\varepsilon b^2 \int_{\Omega} |\Phi_x|^2 dx \\
& + [1/(2q+2)] [\delta/\varepsilon]^{2q+1} [(2q+2)^2 M^2/4\varepsilon]^{2q+2} [\text{mes } \Omega \\
& + \int_{\Omega} |U|^{2q+2} dx].
\end{aligned}$$

Finally, the last term in (5.22) admits the estimate

$$\begin{aligned}
(5.25) \quad & \int_{\Omega} \psi \Phi |V| |F_k| dx \leq M \left[\int_{\Omega} \psi \Phi |V| dx \right. \\
& \left. + \int_{\Omega} \Phi \psi |V| |U|^{1/2} dx \right] \\
& = M \left[\int_{\Omega} \Phi^{2\delta} |V| dx + \int_{\Omega} \Phi^{2\delta} |V| |U|^{1/2} dx \right] \\
& \leq \varepsilon \int_{\Omega} |\Phi|^2 dx + \varepsilon \int_{\Omega} \Phi^{2\delta} |V|^2 dx \\
& + [1/(2q+2)] [\delta/\varepsilon]^{2q+1} M^{2q+2} \int_{\Omega} |V|^{2q+2} dx \\
& + (M^2/4\varepsilon) \int_{\Omega} \Phi^{2\delta} |U| dx \\
& \leq [1/(2q+2)] [\delta/\varepsilon]^{2q+1} \left\{ M^{2q+2} \int_{\Omega} |V|^{2q+2} dx \right. \\
& \left. + (M^2/4\varepsilon)^{2q+2} \int_{\Omega} |U|^{2q+2} dx \right\} + \varepsilon \int_{\Omega} |\Phi|^2 dx \\
& + \varepsilon \int_{\Omega} \Phi^{2\delta} |V|^2 dx.
\end{aligned}$$

To estimate the last term in (5.25), we use the inequality

$$\begin{aligned}
(5.26) \quad & |[(\phi^{2\delta} |v|^2)^{1/2}]_{,m}|^2 \leq 2(2q+1)^2 [\phi^{(2q-1)/(q+1)} \\
& + \phi^{2\delta}] |v_x|^2 \leq 2(2q+1)^2 [\phi^{2\delta} + A(q) + \phi\psi] |v_x|^2 \\
& = 2(2q+1)^2 [2\psi\phi + A(q)] |v_x|^2
\end{aligned}$$

where $A(q) = [2/(2q+1)] [(2q-1)/(2q+1)]^{(2q-1)/2}$. Combining (5.26) and (5.25), we get

$$\begin{aligned}
(5.27) \quad & \int_{\Omega} \psi \phi |v| |F_k| dx \leq 2\epsilon b^2 \int_{\Omega} |\phi_x|^2 dx \\
& + [1/(2q+2)] [\delta/\epsilon]^{2q+1} \{M^{2q+2} \int_{\Omega} |v|^{2q+2} dx \\
& + (M^2/4\epsilon)^{2q+2} \int_{\Omega} |u|^{2q+2} dx\} \\
& + 2b^2(2q+1)^2 A(q) \int_{\Omega} |v_x|^2 dx.
\end{aligned}$$

We may now combine (5.22) through (5.24) with (5.27) and choose $\epsilon = \epsilon' > 0$ sufficiently small and simplify to get

$$\begin{aligned}
(5.28) \quad & \int_{\Omega} \phi \psi |v_x|^2 dx + \int_{\Omega} |\phi_x|^2 dx \leq K_2(q, \epsilon', M, n, b, \Omega) \\
& + K_3(q, \epsilon', \Omega, n, b) \left[\int_{\Omega} |u|^{2q+2} dx + \int_{\Omega} |v_x|^2 + 1 \right].
\end{aligned}$$

However, by the induction hypothesis, the integrals on the right hand side of (5.28) are bounded by a constant $\tilde{M}(q)$ depending on q and known constants. Thus, we have a constant $\bar{M}(q)$ depending only on known constants and q such

that

$$(5.29) \quad \int_{\Omega} |\Phi_x|^2 dx \leq \bar{M}(q).$$

Lemma 2.1 now implies

$$(5.30) \quad \int_{\Omega} |\Phi|^2 dx \leq b^2 \bar{M}(q) \equiv \bar{\bar{M}}(q+1).$$

We now let R approach ∞ in (5.30) and obtain

$$(5.31) \quad \int_{\Omega} |V|^{4(q+1)} dx \leq \bar{\bar{M}}(q+1).$$

Since k was chosen arbitrarily between 1 and N , we have that (5.31) is valid for $U^i = V$, $1 \leq i \leq N$. By summing on i from 1 to N in (5.31), we obtain (5.8) for $s = q + 1$ whenever it is true for $s = q$. Thus, by induction, we have that (5.8) is true for all $s \geq 1$, and the proof is complete.

THEOREM 5.1. Suppose Assumption 5.1 is valid. Then there exists a constant K depending only on known constants such that

$$(5.32) \quad \operatorname{ess\,sup}_{x \in \Omega} |U(x)| \leq K.$$

PROOF. Let r be an arbitrary integer between 1 and N and put $V = U^r$. In (1.12) let $Z^i(x) = 0$ if $i \neq r$ and $Z^r(x) = \max \{V(x) - k, 0\}$ where $k \geq 1$. From [8, p. 53], we have Z in $H_0^1(\Omega)$. Substituting Z into (1.12) yields

$$(5.33) \quad \int_A Z_{,\ell}^i C_{ij}^{\ell m} U_{,m}^j dx + \int_A Z_{,\ell}^i B_i^{\ell} dx = \int_A Z^i F_i dx$$

where $A \equiv A_k^r \equiv \{x \in \Omega \mid V(x) > k\}$. Assumption 5.1 and ele-

mentary estimates using Lemma 2.2 yield

$$\begin{aligned}
 (5.34) \quad \lambda \int_A |V_x|^2 dx &\leq \int_A |V_x| |B| dx + \int_A (V - k) |F| dx \\
 &\leq \varepsilon \int_A |V_x|^2 dx + (4\varepsilon)^{-1} \int_A |B|^2 dx \\
 &\quad + \int_A |F|^2 dx + \int_A (V - k)^2 dx.
 \end{aligned}$$

We use Assumption 5.1 to get the following estimates:

$$\begin{aligned}
 (5.35) \quad (4\varepsilon)^{-1} \int_A |B|^2 dx &\leq (M^2/2\varepsilon) \int_A [1 + |U|^{2\alpha}] dx \\
 &= (M^2/2\varepsilon) [\int_A |U|^{2\alpha} dx + \text{mes } A].
 \end{aligned}$$

$$\begin{aligned}
 (5.36) \quad \int_A |F|^2 dx &\leq 2M^2 \int_A [1 + |U|^{2\alpha}] dx \\
 &= 2M^2 [\int_A |U|^{2\alpha} dx + \text{mes } A].
 \end{aligned}$$

In (5.34) put $\varepsilon = \lambda/2$ and use (5.35) and (5.36) to get

$$\begin{aligned}
 (5.37) \quad \int_A |V_x|^2 dx &\leq (2/\lambda) \int_A (V - k)^2 dx \\
 &\quad + [4M^2(\lambda+1)/\lambda^2] [\text{mes } A + \int_A |U|^{2\alpha} dx]
 \end{aligned}$$

where M is from (5.1). We now use Lemma 5.1 and Lemma 5.2 to estimate the last integral in (5.37). First, we consider the cases $n = 3, 4$, or 5 . Lemma 2.1 and Lemma 5.1 imply the existence of a constant M_1 depending only upon known constants such that

$$(5.38) \quad \int_{\Omega} |V|^p dx \leq \int_{\Omega} |U|^p dx \leq M_1$$

for $1 \leq p \leq 2n/(n-2)$. Thus, with the use of Hölder's inequality, we obtain

$$(5.39) \quad \int_A |V|^{2\alpha} dx \leq \left[\int_A |V|^{2\alpha q} dx \right]^{1/q} [\text{mes } A]^{1-(1/q)}$$

where $q = n/[\alpha(n-2)]$ for $n = 3, 4$, or 5 . Using (5.38) in (5.39), we have

$$(5.40) \quad \int_A |V|^{2\alpha} dx \leq M_1^{\alpha(n-2)/n} [\text{mes } A]^{1-[\alpha(n-2)/n]}$$

From Assumption 5.1, we get $-\alpha(n-2)/n > -2/n$ which implies the existence of $\varepsilon_0 > 0$ such that $1-[\alpha(n-2)/n] = 1-(2/n)+\varepsilon_0$. Combining this with (5.39) and (5.40), we have

$$\begin{aligned} (5.41) \quad \int_A |V_x|^2 dx &\leq (2/\lambda) \int_A (V - k)^2 dx \\ &\quad + [4M^2(\lambda+1)/\lambda^2] \{\text{mes } A \\ &\quad + M_1^{[\alpha(n-2)/n]} [\text{mes } A]^{1-(2/n)+\varepsilon_0}\} \\ &\leq \{ [4M^2(\lambda+1)/\lambda^2] + M_1^{\alpha(n-2)/n} + 2/\lambda \} \left\{ \int_A (V - k)^2 dx \right. \\ &\quad \left. + k^2(\text{mes } A) + k^2(\text{mes } A)^{1-(2/n)+\varepsilon_0} \right\}. \end{aligned}$$

From the definition of V and A , we may conclude from Theorem 2.5 that there exists a constant K_0 depending only on known constants such that

$$(5.42) \quad \text{ess sup}_{x \in \Omega} U^r(x) = \text{ess sup}_{x \in \Omega} V(x) \leq K_0.$$

To obtain a bound for $\text{ess sup}_x (-U^r(x))$, we multiply (1.12) by -1 to get

$$(5.43) \quad \int_{\Omega} z^i_{,\ell} C^{\ell m}_{ij} (-U^j)_{,m} dx - \int_{\Omega} z^i_{,\ell} B^{\ell}_i dx + \int_{\Omega} z^i F_i dx = 0.$$

We now put $z^i = 0$ for $i \neq r$ and $z^r(x) = \max \{-V(x) - k, 0\}$ with $k \geq 1$. By repeating the analysis above with V replaced with $-V$, we obtain a constant K_1 depending only on known constants such that

$$(5.44) \quad \text{ess sup}_{x \in \Omega} (-U^r) = \text{ess sup}_{x \in \Omega} (-V(x)) \leq K_1.$$

Since r was chosen arbitrarily between 1 and N , we have (5.44) true for all such r . Therefore, (5.32) is proven for $n = 3, 4$, or 5 . If $n = 2$, choose $q = 1/\alpha$ in (5.39), and (5.41) becomes

$$(5.45) \quad \int_A |v_x|^2 dx \leq \{ [4M^2(\lambda+1)/\lambda^2] + M_1^\alpha + 2/\lambda \} \\ \times \left[\int_A (V - k)^2 dx + k^2(\text{mes } A) + k^2(\text{mes } A)^{1-\alpha} \right],$$

and the proof continues as in the first case. For $n \geq 6$, choose $q = 4s$ where s is the first integer greater than $n/8$. Thus, $q = 4s > n/2$, and there exists an $\epsilon_0 > 0$ such that $1 - (1/q) = 1 - (2/n) + \epsilon_0$, and (5.40) now becomes

$$(5.46) \quad \int_A |v|^{2\alpha} dx = \int_A |v| dx \leq [\bar{M}(s)]^{1/4s} (\text{mes } A)^{1 - (2/n) + \epsilon_0}$$

where $\bar{M}(s)$ is from (5.8). With this inequality, (5.41) now

has the form

$$(5.47) \quad \int_A |v_x|^2 dx \leq \{ [4M^2(\lambda+1)/\lambda^2] + [\bar{M}(s)]^{1/4s} + 2/\lambda \} \\ \times \{ \int_A (V - k)^2 dx + k^2(\text{mes } A) \\ + k^2(\text{mes } A)^{1-(2/n)+\varepsilon_0} \},$$

and the analysis continues as in the two previous cases.

This completes the proof.

The condition $\alpha < 1$ in Assumption 5.1 cannot in general be extended to $\alpha = 1$. Consider the following examples:

EXAMPLE 1. Suppose $N = 1$, $B_1^\ell[x, U] = 0$ for all $\ell = 1, 2, \dots, n$, $F_1[x, U] = nU$, $C_{11}^{\ell m}[x, U] = \delta_m^\ell \equiv$ Kronecker delta, and $\Omega = (0, \pi) \times (0, \pi) \times \dots \times (0, \pi)$ with the product taken over n intervals. Then, for any constant K , $U(x) = K(\sin(x_1))(\sin(x_2)) \dots (\sin(x_n))$ is a solution of (1.1) and (1.2).

EXAMPLE 2. Suppose $n = 1$, $N = 2$, $B_1^1[x, U] = -U_2$, $B_2^1[x, U] = U_1$, $F_i[x, U] = 0$, $C_{ij}^{11}[x, U] = \delta_j^i$ for $i, j = 1, 2$, and Ω is $(0, 2\pi)$. Then, for any constant K , $U(x) = K \begin{pmatrix} \sin(x) \\ \cos(x) - 1 \end{pmatrix}$ is a solution of (1.1) and (1.2).

CHAPTER VI

BOUNDEDNESS OF THE HÖLDER NORM

We can now find a bound for the Hölder norm of weak solutions of (1.1) and (1.2) under the restrictions of Assumption 5.1. The proof follows closely that of Ladyzhenskaya and Ural'tseva [8, pp. 249-251] for a single equation.

THEOREM 6.1. Suppose Assumption 5.1 is valid. Then an arbitrary weak solution U of (1.1) and (1.2) belongs to $C_{0,\beta}(\bar{\Omega})$ with exponent $\beta > 0$ depending on $M_0 \equiv \operatorname{ess\,sup}_{x \in \Omega} |U(x)|$ and known constants. The quantity $|U|_{\beta,\Omega}$ is bounded above by an expression determined by λ and M_c from (3.2) and the constants a_0 and θ_0 in the definition of condition (A).

PROOF. We shall show that U^r belongs to some class $B(\bar{\Omega}, M_0, \gamma, \delta, 0)$ for arbitrary r between 1 and N . In (1.12) let $Z^i(x) = 0$ for $i \neq r$ and $Z^r(x) = [\zeta(x)]^2 \max \{V(x) - k, 0\}$ where $V = U^r$, $k \geq 0$, and ζ is a smooth function of compact support with range in $[0,1]$ on some sphere K_ρ .

Observe that Z^r is nonzero only on the set $A \equiv A_{k,\rho}^r \equiv \{x \in K_\rho \cap \Omega \mid V(x) > k\}$, and thus, (1.12) becomes

$$\begin{aligned}
 (6.1) \quad & \int_A \zeta^2 C_{rr}^{\ell m} V_{,\ell} V_{,m} dx + \int_A 2 \zeta \zeta_{,\ell} C_{rr}^{\ell m} V_{,m} (V - k) dx \\
 & = - \int_A \zeta^2 V_{,\ell} B_r^\ell dx - \int_A 2 \zeta \zeta_{,\ell} (V - k) B_r^\ell dx \\
 & \quad + \int_A \zeta^2 (V - k) F_r dx.
 \end{aligned}$$

Using Assumption 5.1 and the boundedness of $|U|$ from Theorem 5.1, we get

$$\begin{aligned}
 (6.2) \quad \lambda \int_A \zeta^2 |v_x|^2 dx &\leq 2M_C \int_A \zeta |\zeta_x| (V - k) |v_x| dx \\
 &\quad + M[1 + M_O^\alpha] \int_A \zeta^2 |v_x| dx \\
 &\quad + 2M[1 + M_O^\alpha] \int_A \zeta |\zeta_x| (V - k) dx \\
 &\quad + M[1 + M_O^\alpha] \int_A \zeta^2 (V - k) dx.
 \end{aligned}$$

With the use of (2.8), we get the following estimates for the above integrals.

$$\begin{aligned}
 (6.3) \quad 2M_C \int_A \zeta |\zeta_x| (V - k) |v_x| dx \\
 \leq \varepsilon \int_A \zeta^2 |v_x|^2 dx + (M_C^2/\varepsilon) \int_A |\zeta_x|^2 (V - k)^2 dx.
 \end{aligned}$$

$$\begin{aligned}
 (6.4) \quad M[1 + M_O^\alpha] \int_A \zeta^2 |v_x| dx \\
 \leq \varepsilon \int_A \zeta^2 |v_x|^2 dx + \{M^2[1 + M_O^\alpha]^2/4\varepsilon\} \int_A \zeta^2 dx.
 \end{aligned}$$

$$\begin{aligned}
 (6.5) \quad 2M[1 + M_O^\alpha] \int_A \zeta |\zeta_x| (V - k) dx \\
 \leq (1/2) \int_A |\zeta_x|^2 (V - k)^2 dx + 2M^2[1 + M_O^\alpha]^2 \int_A \zeta^2 dx.
 \end{aligned}$$

Using (6.2) through (6.5), we get

$$\begin{aligned}
(6.6) \quad & \lambda \int_A \zeta^2 |V_x|^2 dx \leq 2\varepsilon \int_A \zeta^2 |V_x|^2 dx \\
& + [(M_C^2/\varepsilon) + (1/2)] \int_A |\zeta_x|^2 (V - k)^2 dx \\
& + [1 + M_O^\alpha]^2 M^2 [(1/4\varepsilon) + 2] \int_A \zeta^2 dx \\
& + M[1 + M_O^\alpha] \int_A \zeta^2 (V - k) dx
\end{aligned}$$

For values of k such that $\sup_{x \in K_\rho \cap \Omega} V(x) - k \leq \delta \equiv 1$, and $\varepsilon = \lambda/4$, we get

$$\begin{aligned}
(6.7) \quad & \int_A \zeta^2 |V_x|^2 dx \leq K(\alpha, M_O, M, \lambda, M_C) \left[\int_A |\zeta_x|^2 (V - k)^2 dx \right. \\
& \left. + \int_A \zeta^2 dx \right]
\end{aligned}$$

where $K(\alpha, M_O, M, \lambda, M_C) \equiv (2/\lambda^2) \{4M_C^2 + (\lambda/2) + \lambda[1+M_O^\alpha]^2 M_O^2 [1+2\lambda] + \lambda M[1+M_O^\alpha]\}$. Now, if we take the function ζ equal to unity in a sphere $K_{\rho-\sigma\rho}$ where $\sigma \in (0,1)$, that is concentric with the sphere K_ρ in such a way that (See [8, p. 250].)

$$(6.8) \quad |\zeta_x| \leq E/\sigma\rho$$

for some constant E , then (6.7) yields the series of inequalities

$$\begin{aligned}
(6.9) \quad & \int_{A_{k,\rho-\sigma\rho}^r} |V_x|^2 dx \leq K(\alpha, M_O, M, \lambda, M_C) \{ [1/(\sigma\rho)^2] \sup_{x \in A_{k,\rho}^r} \{V(x) - k\}^2 \\
& + 1 \} \text{mes } A_{k,\ell}^r
\end{aligned}$$

that determine the class $B(\overline{\Omega}, M_O, \gamma, \delta, 0)$.

We obtain analogous inequalities for the function $-V$ if we put $z^r(x) = [\zeta(x)]^2 \max \{-V(x) - k, 0\}$ where the values of k are such that $\sup_{x \in K_\rho \cap \Omega} (-V(x)) - k \leq \delta \equiv 1$ and $k \geq 0$.

Since r was chosen arbitrarily between 1 and N , we have U^i in $B(\bar{\Omega}, M_0, \gamma, \delta, 0)$ for each $i = 1, 2, \dots, N$. Therefore, Theorem 2.4 implies Theorem 6.1, and the proof is complete.

CHAPTER VII

CONCLUSION

The principal result of Chapter III is Theorem 3.4. The restrictions on C , B , and F needed to achieve this existence theorem seem quite weak with the possible exception of (3.38) limiting the size of ε_1 . However, to allow ε_1 to be arbitrary, it is likely that the method used here would have to be altered significantly.

The results of Chapter IV seem adequate since, in the case of $n = N = 1$, it reduces to the restrictions involved in the two-point boundary value problem studied by Ford [5].

In Chapter V and Chapter VI, we have used the methods that Ladyzhenskaya and Ural'tseva [8] developed for a single quasilinear equation and for linear systems to obtain some regularity results for our weak solutions. Although the necessity of having $\alpha < 1$ in Assumption 5.1 is demonstrated by an example, the condition that α decrease as n increases up to $n = 5$ and then equal $1/2$ for $n \geq 6$ seems unnatural. It would also be desirable to prove analogous results for a more general form for $C_{ij}^{\ell m}$.

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