FEEDBACK SYSTEM DESIGN: THE POLE PLACEMENT PROBLEM

by

ASHOK IYER, B.E., M.S.E.E.

A DISSERTATION

IN

ELECTRICAL ENGINEERING

Submitted to the Graduate Faculty
of Texas Tech University in
Partial Fulfillment of
the Requirements for
the Degree of

DOCTOR OF PHILOSOPHY

Approved

December, 1982

A.C.
801
73
1982
No. 87
Copied

ACKNOWLEDGEMENTS

I would like to express my sincere thanks to Horn Professor Richard E. Saeks for his expert guidance of this dissertation, Professors John Murray, Louis Hunt, William Portnoy and John Walkup for serving on my committee, Dr. Erol Emre for many interesting conversations and helpful comments and Mr. Chin-Long Wey for helping me with the multivariate example. It is a pleasure acknowledging their help.

Finally a special thanks to my parents, brother, sisters and friends for their encouragement and support during my entire education.

TABLE OF CONTENTS

ACKNOWLEDGEMENTS.	ii
LIST OF FIGURES	v
CHAPTER 1 INTRODUCTION	1
1.1 Fractional Representation Theory: Historial Background.	2
1.2 Fractional Representation Theory: Mathematical Preliminaries.	3
1.3 Stablization.	4
Theorem 1.1: Stabilization Theorem	8
CHAPTER 2 THE SINGLE VARIATE POLE PLACEMENT PROBLEM. . . .	9
2.1 The Single Variate Pole Placement Problem .	9
Theorem 2.1	11
Proof: Sufficiency.	11
Necessity.	14
Corollary 2.1	15
Proof.	15
2.2 Proper and Strictly Proper Plants	17
Case i.	18
Case ii	18
Case iii.	18
Case iv	19
2.3 Improper Plants	20
Case i.	20
Case ii	20

	Theorem 2.2: Pole Placement Theorem. . . .	21
	2.4 Parameterization of Solution Space.	21
	Corollary 2.2	27
CHAPTER 3	THE STABLE MULTIVARIATE POLE PLACEMENT PROBLEM .	28
	3.1 Multivariate Fractional Representation Theory.	28
	3.2 The Stable Pole Placement Problem	30
	Theorem 3.1: Stable Pole Placement Theorem	34
	Proof	34
CHAPTER 4	EXAMPLES	39
	Example 4.1: Unstable Proper Plant.	39
	Example 4.2: Stable Proper Plant.	42
	Example 4.3: Stable Multivariate Plant.	46
CHAPTER 5	CONCLUDING REMARKS	49
REFERENCES.		51

LIST OF FIGURES

Figure 1.1:	Basic Feedback System	4
Figure 4.1:	Unstable Plant Before Pole Placement and Stabilization	39
Figure 4.2:	Feedback System for Example 4.1 that Places the Poles of the System at Zeros of $q(s) = s^2 + 2s + 1$	42
Figure 4.3:	Plant Before Stabilization and Pole Placement	43
Figure 4.4:	Feedback System for Example 4.1 that Places Poles at Zeros of $q(s) = (s+2)(s+3)$	45

CHAPTER 1

INTRODUCTION

Ever since Meikle invented the first control system in 1750 which automatically steered windmills to face the direction of the wind, control systems have always been a great boon to man. Meikle's invention was followed by a spate of control system inventions during the Industrial Revolution. At about the same time Fourier and Laplace proposed their transform techniques which are extensively used in electrical engineering, especially in control theory.

Control theory began to evolve with the advent of Cauchy's mathematical work and the publication of two classical papers by Nyquist titled 'Regeneration Theory' in 1932 and Hazen's paper 'Theory of Servomechanism' in 1934. The biggest boost was received during World War II when the need for automatic control systems was acute.

The advent of the space programs in the sixties brought about the emergence of modern control theory. As a result of this rapid progress, the systematic design of control systems is an everyday affair now. Control theory is still a very active area of research, and has spread in various directions. One of the latest is the formulation of an algebraic fractional representation for the feedback system design problem. This dissertation hopes to make a modest contribution to the solution of the classical Pole Placement Problem [5],[6],[12],[25],[42],[53] in the algebraic fractional representation setting.

1.1 Fractional Representation Theory: Historical Background

In 1976 Youla, Bongiorno and Jabr published two classical papers [57], [58] in which a complete parameterization of a set of stabilizing compensators for a multivariate feedback system was obtained. In the ensuing years this work, often referred to as the YBJ theory, has lead to the development of an entirely new approach to feedback system design. This stabilization theory has been applied to some of the classical control problems such as:

- 1) Tracking and disturbance rejection [43],[45],[46]
- 2) Robust design algorithms [9],[45],[46]
- 3) Design with proper stable compensators [11],[45],[46]
- 4) Transfer function design [45],[46]
- 5) Simultaneous Design [9],[46]
- 6) Pole Placement [45],[46]

The pole placement problem has been extensively studied by the author and is the problem solved in this disseration.

The key to the YBJ theory is a three step design philosophy:

- 1) Stabilization
- 2) Achievement of design constraints
- 3) Optimization of system performance

First and foremost, a feedback system must be stable and hence the first step in the design process is the parameterization of all stabilizing compensators for the given plant. Although it might suffice to design a stable system, in practice stabilization is only the first step of the design process. We must characterize all compensators if we are to choose among the stabilizing

compensators to find one which also achieves the design constraints. Once the stabilizing compensators have been characterized, we must choose a subset of compensators which achieve a prescribed set of design constraints, in this case the subset of compensators which places poles for the given plant.

1.2 Fractional Representation Theory: Mathematical Preliminaries

The system will be described by a rational function

$$r(s) = \frac{p(s)}{q(s)} \quad (1.1)$$

Such a system is said to be stable if its poles lie in the strict left half-plane. Since the point at infinity is taken to lie on the imaginary axis, this implies that $r(s)$ is stable if and only if it is a proper rational function and $q(s)$ is a strictly Hurwitz polynomial.

A fractional representation of $r(s)$ is a factorization of the form

$$r(s) = \frac{n_r(s)}{d_r(s)} \quad (1.2)$$

where both $n_r(s)$ and $d_r(s)$ are stable rational functions and $d_r(s) \neq 0$. If $r(s)$ has a polynomial fractional representation as in (1.1) then we can take

$$n_r(s) = \frac{p(s)}{m(s)} \quad (1.3)$$

and

$$d_r(s) = \frac{q(s)}{m(s)} \quad (1.4)$$

where $m(s)$ is any Hurwitz polynomial such that the order of $m(s)$ equals the order of $r(s)$, verifying the existence of the required fractional representation [20].

The fractional representation (1.2) is said to be 'coprime' if there exists stable rational functions $u_r(s)$ and $v_r(s)$ such that

$$u_r(s)n_r(s) + v_r(s)d_r(s) = 1 \quad (1.5)$$

In the case of polynomials (1.5) is the equivalent to the requirement that $n_r(s)$ and $d_r(s)$ have no common zeros [2]. In the rational fractional representation, equation (1.5) implies that $n_r(s)$ and $d_r(s)$ have no common right half-plane zeros and conversely [7],[20]. One can justify this representation even though it deviates from the classical control theory, since only right half-plane zeros cause instabilities and this theory forbids the cancellation of right half-plane poles and zeros.

1.3 Stabilization: [11],[45]

The basic feedback system is shown in Figure 1.1

Figure 1.1: Basic Feedback System

Using the usual algebraic manipulation [11], the following system gains are obtained:

$$\begin{bmatrix} \epsilon_1(s) \\ \epsilon_2(s) \end{bmatrix} = \begin{bmatrix} h_{\epsilon_1\mu_1}(s) & h_{\epsilon_1\mu_2}(s) \\ h_{\epsilon_2\mu_1}(s) & h_{\epsilon_2\mu_2}(s) \end{bmatrix} \begin{bmatrix} \mu_1(s) \\ \mu_2(s) \end{bmatrix} \quad (1.6)$$

where

$$\begin{bmatrix} h_{\epsilon_1\mu_1}(s) & h_{\epsilon_1\mu_2}(s) \\ h_{\epsilon_2\mu_1}(s) & h_{\epsilon_2\mu_2}(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{1+p(s)c(s)} & \frac{-p(s)}{1+p(s)c(s)} \\ \frac{c(s)}{1+p(s)c(s)} & \frac{1}{1+p(s)c(s)} \end{bmatrix} \quad (1.7)$$

and

$$\begin{bmatrix} v_1(s) \\ v_2(s) \end{bmatrix} = \begin{bmatrix} h_{v_1\mu_1}(s) & h_{v_1\mu_2}(s) \\ h_{v_2\mu_1}(s) & h_{v_2\mu_2}(s) \end{bmatrix} \begin{bmatrix} \mu_1(s) \\ \mu_2(s) \end{bmatrix} \quad (1.8)$$

where

$$\begin{bmatrix} h_{v_1\mu_1}(s) & h_{v_1\mu_2}(s) \\ h_{v_2\mu_1}(s) & h_{v_2\mu_2}(s) \end{bmatrix} = \begin{bmatrix} h_{\epsilon_2\mu_1}(s) & h_{\epsilon_2\mu_2}(s) - 1 \\ 1 - h_{\epsilon_1\mu_1}(s) & -h_{\epsilon_1\mu_2}(s) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{c(s)}{1+p(s)c(s)} & \frac{-p(s)c(s)}{1+p(s)c(s)} \\ \frac{p(s)c(s)}{1+p(s)c(s)} & \frac{p(s)}{1+p(s)c(s)} \end{bmatrix} \quad (1.9)$$

The system is said to be stable if each of the eight feedback system gains of equations (1.7) through (1.9) are stable. Since the input/output gains, $h_{v\mu}$, are expressed in terms of the input/plant input gains, $h_{e\mu}$, via equation (1.9), we require the input/plant input gains are all stable.

For the stabilization theory the coprime fractional representation of the plant is of the form

$$p(s) = \frac{n_p(s)}{d_p(s)} \quad (1.10)$$

where $n_p(s)$ and $d_p(s)$ are stable and $d_p(s)$ is not identically zero. Since $n_p(s)$ and $d_p(s)$ are coprime, there exists stable $u_p(s)$ and $v_p(s)$ such that

$$u_p(s)n_p(s) + v_p(s)d_p(s) = 1 \quad (1.11)$$

In the single variate case every plant admits such a representation and hence we can assume (1.10) and (1.11) without loss of generality. For stabilization, the goal is to characterize compensators, represented in the form

$$c(s) = \frac{n_c(s)}{d_c(s)} \quad (1.12)$$

where $n_c(s)$ and $d_c(s)$ are stable and $d_c(s)$ is not identically zero. In order to prevent pole-zero cancellations between $p(s)$ and $c(s)$ in the right half plane, a stronger condition namely

$$p(s)c(s) = \frac{n_p(s)n_c(s)}{d_p(s)d_c(s)} \quad (1.13)$$

be a coprime fractional representation is required.

Substituting the fractional representations (1.10) and (1.12) into (1.7) and (1.9), yields

$$\begin{bmatrix} h_{\varepsilon_1\mu_1}(s) & h_{\varepsilon_1\mu_2}(s) \\ h_{\varepsilon_2\mu_1}(s) & h_{\varepsilon_2\mu_2}(s) \end{bmatrix} = \begin{bmatrix} \frac{d_p(s)d_c(s)}{d_p(s)d_c(s)+n_p(s)n_c(s)} & \frac{-n_p(s)d_c(s)}{d_p(s)d_c(s)+n_p(s)n_c(s)} \\ \frac{d_p(s)n_c(s)}{d_p(s)d_c(s)+n_p(s)n_c(s)} & \frac{d_p(s)d_c(s)}{d_p(s)d_c(s)+n_p(s)n_c(s)} \end{bmatrix} \quad (1.14)$$

and

$$\begin{bmatrix} h_{v_1\mu_1}(s) & h_{v_1\mu_2}(s) \\ h_{v_2\mu_1}(s) & h_{v_2\mu_2}(s) \end{bmatrix} = \begin{bmatrix} \frac{d_p(s)n_c(s)}{d_p(s)d_c(s)+n_p(s)n_c(s)} & \frac{-n_p(s)n_c(s)}{d_p(s)d_c(s)+n_p(s)n_c(s)} \\ \frac{n_p(s)n_c(s)}{d_p(s)d_c(s)+n_p(s)n_c(s)} & \frac{n_p(s)d_c(s)}{d_p(s)d_c(s)+n_p(s)n_c(s)} \end{bmatrix} \quad (1.15)$$

Theorem 1.1: Stabilization Theorem [11],[45]

For the feedback system of Figure 1.1, let the plant have a coprime fractional representation given by equations (1.10) and (1.11). Then for any stable $w(s)$ such that $[w(s)n_p(s)+v_p(s)]$ is not identically zero, the compensator

$$c(s) = \frac{[-w(s)d_p(s)+u_p(s)]}{[w(s)n_p(s)+v_p(s)]} = \frac{n_c(s)}{d_c(s)} \quad (1.16)$$

stabilizes the feedback system and yields a coprime fractional representation $p(s)c(s) = [n_p(s)n_c(s)]/[d_p(s)d_c(s)]$. Conversely, every such stabilizing compensator is of this form for some stable $w(s)$. $\square$

The remarkable feature of this stabilizing theory is that the feedback system gains are actually affine in nature. The above theorem gives a complete parameterization of the stabilizing compensators for our feedback system, the only requirement being $[w(s)n_p(s)+v_p(s)]$ is not identically zero.

In the following chapter, the single-variate pole placement problem is stated and solved. The results are indeed very surprising. Chapter 3 deals with the pole placement problem for stable multivariate plants. Chapter 4 illustrates a few examples and the implementation of the pole placement problem. Finally, Chapter 5 summaries the dissertation.

CHAPTER 2

THE SINGLE VARIATE POLE PLACEMENT PROBLEM

Very often, establishing the closed loop system poles is of fundamental importance. These poles are prescribed by the designer - they may be the required cut off frequency for a filter or the correct pole location for an aging plant. In many cases one need not realize the exact transfer function while simultaneously stabilizing the system. The resultant criterion may often be too restrictive [4],[6] and hard to implement. One may prefer to work in a less restrictive pole placement problem.

2.1 The Single Variate Pole Placement Problem:

In the pole placement problem we desire to find a stabilizing compensator such that the input-output transfer function takes the form

$$h_{v_2\mu_1}(s) = \frac{r(s)}{q(s)} \quad (2.1)$$

where $q(s)$ is a prescribed Hurwitz polynomial whose roots are the poles that are to be placed and $r(s)$ is an arbitrary numerator polynomial which has no common factors with $q(s)$. In order to guarantee that $h_{v_2\mu_1}(s)$ has no poles at infinity it is required that the order of $r(s)$ be less than or equal to the order of $q(s)$.

Since $q(s)$ is Hurwitz to ensure stability, any common factors between $r(s)$ and $q(s)$ will be in the left half plane and must be characterized by a polynomial coprimeness condition.

Consistent with the polynomial coprimeness concept, given a plant $\bar{p}(s)$

$$p(s) = \frac{a^l(s)a^r(s)}{b^l(s)b^r(s)} \quad (2.2)$$

we can construct

$$n_p(s) = \frac{a^l(s)a^r(s)}{m(s)} \quad (2.3)$$

and

$$d_p(s) = \frac{b^l(s)b^r(s)}{m(s)} \quad (2.4)$$

where $m(s)$ is a Hurwitz polynomial characterizing the poles of $n_p(s)$ and $d_p(s)$; $a^l(s)$ and $b^l(s)$ are Hurwitz polynomial characterizing the strict left-half plane zeros of $n_p(s)$ and $d_p(s)$ respectively; while $a^r(s)$ and $b^r(s)$ are anti-Hurwitz polynomials characterizing the finite closed right half-plane zeros of $n_p(s)$ and $d_p(s)$ respectively. Since $n_p(s)$ and $d_p(s)$ have no poles at infinity

$$o(m) \geq o(a^l) + o(a^r) \quad (2.5)$$

and

$$o(m) \geq o(b^l) + o(b^r) \quad (2.6)$$

where $o(\)$ denotes the order of the appropriate polynomial. Since $n_p(s)$ and $d_p(s)$ are coprime, they cannot have a common zero at $s = \infty$ and hence at least one of the inequalities (2.5) or (2.6) must be satisfied with an equality. If we let

$$k = |o(a^r) + o(a^l) - o(b^r) - o(b^l)| \quad (2.7)$$

then in the case of a strictly proper plant,

$$o(m) = o(a^l) + o(a^r) + k = o(b^l) + o(b^r) \quad (2.8)$$

or in the case of an improper plant,

$$o(m) = o(a^l) + o(a^r) = o(b^l) + o(b^r) + k \quad (2.9)$$

Using the notation introduced, the following fundamental theorem has been stated and proved. This theorem was originally stated by Karmakolias [29], but later modified and proved by the author.

Theorem 2.1

Given $p(s)$ there exists a compensator for the feedback system of Figure 1.1 which stabilizes the system and yields

$$h_{v_2\mu_1}(s) = \frac{r(s)}{q(s)}$$

where $q(s)$ is a prescribed Hurwitz polynomial and $r(s)$ is an arbitrary polynomial such that $o(r) \leq o(q)$, if and only if the equation

$$\alpha(s)a^r(s) + \beta(s)b^r(s) = q(s) \quad (2.10)$$

admits polynomial solutions, $\alpha(s)$ and $\beta(s)$, such that

$$o \leq o(\alpha) \leq o(q) + o(a^l) - o(m) \quad (2.11)$$

$$o \leq o(\beta) \leq o(q) + o(b^l) - o(m) \quad (2.12)$$

Proof: Sufficiency

If there exist polynomials $\alpha(s)$ and $\beta(s)$ satisfying

$$\alpha(s)a^r(s) + \beta(s)b^r(s) = q(s)$$

and (2.11) and (2.12), define $w(s)$ by

$$w(s) = \frac{\beta(s)m(s)}{q(s)b^T(s)} u_p(s) - \frac{\alpha(s)m(s)}{q(s)a^T(s)} v_p(s) \quad (2.13)$$

Since $q(s)$ and $b^T(s)$ are both Hurwitz polynomials and

$$o(qb^T) \geq o(\beta m) \text{ via 2.12.}$$

This verifies that $\frac{\beta(s)m(s)}{q(s)b^T(s)}$ is a stable rational function. Similarly the fact that $q(s)$ and $a^T(s)$ are both Hurwitz together with (2.11) implies that $\frac{\alpha(s)m(s)}{q(s)a^T(s)}$ is a stable rational function. Hence, $w(s)$ is a stable rational function, since it is represented as the sum and products of stable rational functions.

Using the $w(s)$ defined by equation (2.13) to define a compensator for our system via the stabilization theorem we obtain the input/output transfer function $h_{v_{2\mu_1}}(s)$ from (1.15) and (1.16), as

$$h_{v_{2\mu_1}}(s) = -w(s)n_p(s)d_p(s) + u_p(s)n_p(s) \quad (2.14)$$

$$\begin{aligned} &= -\left[\frac{\beta(s)m(s)}{q(s)b^T(s)} u_p(s) - \frac{\alpha(s)m(s)}{q(s)a^T(s)} v_p(s) \right] n_p(s)d_p(s) \\ &\quad + u_p(s)n_p(s) \\ &= -\frac{\beta(s)m(s)n_p(s)d_p(s)u_p(s)}{q(s)b^T(s)} + \frac{\alpha(s)m(s)n_p(s)d_p(s)v_p(s)}{q(s)a^T(s)} \\ &\quad + u_p(s)n_p(s) \end{aligned}$$

Substituting (2.4) for $d_p(s)$ in the first term and (2.3) for $n_p(s)$ in the second term of the above equation we have

$$\begin{aligned}
 h_{v_2\mu_1}(s) &= - \frac{\beta(s)b^r(s)n_p(s)u_p(s)}{q(s)} + \frac{\alpha(s)a^r(s)d_p(s)v_p(s)}{q(s)} \\
 &\quad + u_p(s)n_p(s) \\
 &= [q(s) - \beta(s)b^r(s)] \frac{n_p(s)u_p(s)}{q(s)} + \frac{\alpha(s)a^r(s)v_p(s)d_p(s)}{q(s)}
 \end{aligned}$$

Substituting (2.9),

$$\begin{aligned}
 &= \alpha(s)a^r(s) \cdot \frac{n_p(s)u_p(s)}{q(s)} + \frac{\alpha(s)a^r(s)v_p(s)d_p(s)}{q(s)} \\
 &= \frac{\alpha(s)a^r(s)}{q(s)} [n_p(s)u_p(s) + v_p(s)d_p(s)]
 \end{aligned}$$

Using (1.11),

$$h_{v_2\mu_1}(s) = \frac{\alpha(s)a^r(s)}{q(s)} \quad (2.15)$$

To verify this is the required transfer function we must show that $o(r) \leq o(q)$. Using (2.5) and (2.11)

$$\begin{aligned}
 o(r) &= o(\alpha) + o(a^r) \\
 &\leq o(\alpha) + o(m) - o(a^1) \\
 &\leq o(\alpha) + o(a^1) - o(m) + o(m) - o(a^1) \\
 &= o(q)
 \end{aligned} \quad (2.16)$$

This verifies that $h_{v_2\mu_1}(s)$ has no poles at infinity, completing the sufficiency of our proof.

Necessity:

Assume there exists a stable $w(s)$ such that

$$h_{v_2\mu_1}(s) = -w(s)n_p(s)d_p(s) + u_p(s)n_p(s) = \frac{r(s)}{q(s)} \quad (2.17)$$

Substituting (2.3) into (2.17) implies that

$$u_p(s) - w(s)d_p(s) = \frac{r(s)m(s)}{q(s)a^r(s)a^l(s)} \quad (2.18)$$

Since $u_p(s) - w(s)d_p(s)$ is stable while $m(s)$, $q(s)$ and $a^l(s)$ are Hurwitz polynomials, the anti-Hurwitz polynomial, $a^r(s)$, in (2.18) must be cancelled by a corresponding factor in $r(s)$. Hence (2.18) implies there exists a polynomial $\alpha(s)$ such that

$$r(s) = \alpha(s)a^r(s) \quad (2.19)$$

Moreover, since $u_p(s) - w(s)d_p(s)$ is stable and given by

$$u_p(s) - w(s)d_p(s) = \frac{\alpha(s)m(s)}{q(s)a^l(s)} \quad (2.20)$$

it follows that,

$$o(\alpha) + o(m) \leq o(q) + o(a^l)$$

implying

$$o(\alpha) \leq o(q) + o(a^l) - o(m) \quad (2.21)$$

verifying (2.11).

Similiarly, it follows that equation (1.11) together with (2.17) (2.19) and (2.4) that

$$v_p(s) + w(s)n_p(s) = \frac{[q(s) - \alpha(s)a^r(s)]m(s)}{q(s)b^r(s)b^l(s)} \quad (2.22)$$

As before, since $v_p(s) + w(s)n_p(s)$ is stable while $q(s)$, $b^l(s)$ and $m(s)$ are Hurwitz it follows that there exists a polynomial $\beta(s)$, such that

$$[q(s) - \alpha(s)a^r(s)] = \beta(s)b^r(s) \quad (2.23)$$

which is the desired equality. Furthermore, since

$$v_p(s) + w(s)n_p(s) = \frac{\beta(s)m(s)}{q(s)b^l(s)} \quad (2.24)$$

is stable, it follows that

$$o(\beta) \leq o(q) + o(b^l) - o(m) \quad (2.25)$$

verifying (2.12). This completes the proof to Theorem 2.1. $\square$

Furthermore if it is required that $r(s)$ and $q(s)$ be polynomial coprime in Theorem 2.1, the following corollary is stated and proved.

Corollary 2.1

In Theroem 2.1, $q(s)$ and $r(s)$ are polynomial coprime if and only if $\alpha(s)$ and $\beta(s)$ are polynomial coprime.

Proof:

Assuming $\alpha(s)$ and $\beta(s)$ are polynomial coprime, if $r(s) = \alpha(s)a^r(s)$ and $q(s)$ are not polynomial coprime, they have a

common zero at $s = s_0$. Equation (2.10) implies that $\beta(s)b^r(s)$ also has a common zero at $s = s_0$. Since $q(s)$ is Hurwitz, $s = s_0$ must lie in the strict left-half plane, implying that it is actually a common zero of $\alpha(s)$ and $\beta(s)$, since the zeros of $a^r(s)$ and $b^r(s)$ lie in the right half plane. This is impossible since $\alpha(s)$ and $\beta(s)$ are assumed to be polynomial coprime. Hence $r(s)$ and $q(s)$ are polynomial coprime, completing the sufficiency of the proof.

To verify the necessity, assuming $r(s)$ and $q(s)$ are polynomial coprime, if $s = s_0$ is a common zero of $\alpha(s)$ and $\beta(s)$, it is also a common zero of $r(s) = \alpha(s)a^r(s)$ and $q(s) = \alpha(s)a^r(s) + \beta(s)b^r(s)$, which contradicts the initial assumption that $r(s)$ and $q(s)$ are polynomial coprime. Hence $\alpha(s)$ and $\beta(s)$ are polynomial coprime, completing the proof. $\square$

Consistent with Theorem 2.1 the solution to the pole placement problem reduces to the solution of the linear polynomial equation

$$\alpha(s)a^r(s) + \beta(s)b^r(s) = q(s) \quad (2.26)$$

subject to the constraints (2.11) and (2.12) which may be rewritten as

$$\text{i) } o(\alpha) \leq o(q) + o(a^1) - o(m) \quad (2.27)$$

$$\text{ii) } o(\beta) \leq o(q) + o(b^1) - o(m) \quad (2.28)$$

$$\text{iii) } o(m) \leq o(q) + o(a^1) \quad (2.29)$$

$$\text{iv) } o(m) \leq o(q) + o(b^1) \quad (2.30)$$

Equations (2.29) and (2.30) are obtained from the fact that $\alpha(s)$ and $\beta(s)$ are polynomials that is $o(\alpha) \geq 0$ and $o(\beta) \geq 0$.

Equation (2.25) represents $o(q) + 1$ equations in $o(\alpha) + o(\beta) + 2$ unknowns, and for a solution we require the generically necessary condition

$$o(q) + 1 \leq o(\alpha) + o(\beta) + 2 \quad (2.31)$$

2.2 Proper and Strictly Proper Plants:

Recall from equation (2.8), which states

$$o(m) = o(a^l) + o(a^r) + k = o(b^l) + o(b^r)$$

For a solution to the pole placement problem, we require the equations (2.31), (2.27), (2.28), (2.29), (2.30) and (2.8) to be satisfied simultaneously.

Substituting equations (2.27), (2.28) and (2.8) we have

$$\begin{aligned} o(q) + 1 &\leq [o(q) + o(a^l) - o(m) + o(q) + o(b^l) - o(m)] + 2 \\ &= 2o(q) - o(a^r) - o(b^r) - k + 2 \end{aligned}$$

or equivalently

$$o(q) \geq o(a^r) + o(b^r) + k - 1 \quad (2.32)$$

Now, equations (2.32) and equations (2.29) and (2.3) have to be satisfied simultaneously. Substituting equations (2.8) in (2.29) and (2.30) we have

$$o(q) \geq o(a^r) + k \quad (2.33)$$

and

$$o(q) \geq o(b^r) \quad (2.34)$$

Consider the following cases:

Case i: $o(b^r) \geq 1$ and $o(a^r) + k \geq 1$. Equation (2.32) states

$$o(q) \geq [o(a^r) + k] + o(b^r) - 1$$

Since $o(b^r) - 1 \geq 0$ because $o(b^r) \geq 1$, (2.32) implies

$o(q) \geq o(a^r) + k$. Hence (2.33) is satisfied. Since $o(a^r) + k \geq 1$, $o(a^r) + k - 1 \geq 0$, and (2.32) implies

$$o(q) \geq o(b^r)$$

and hence (2.34) is satisfied.

Hence under the conditions $o(b^r) \geq 1$ and $o(a^r) + k \geq 1$, the condition for the solution of the pole placement problem is (2.32).

Case ii: $o(b^r) = 0$ and $o(a^r) + k \geq 1$. In this case (2.32), (2.33) and (2.34) reduce to

$$i) \quad o(q) \geq o(a^r) + k - 1$$

$$ii) \quad o(q) \geq o(a^r) + k$$

$$iii) \quad o(q) \geq 0$$

respectively

iii) is satisfied by i), since $o(a^r) + k \geq 1$, for i) and ii) to be simultaneously satisfied $o(q) \geq o(a^r) + k$.

Hence under the condition $o(a^r) + k \geq 1$ and $o(b^r) = 0$, which refers to a stable plant, the condition to solve the pole placement problem is

$$o(q) \geq o(a^r) + k$$

Case iii: $o(b^r) \geq 1$ $o(a^r) + k = 0$. Since both $o(a^r)$ and k are positive integers greater than or equal to zero $o(a^r) + k = 0$ implies

$o(a^r) = 0$ and $k = 0$. In this case (2.32), (2.33) and (2.34) reduce to

$$i) \quad o(q) \geq o(b^r) - 1$$

$$ii) \quad o(q) \geq 0$$

$$iii) \quad o(q) \geq o(b^r).$$

Since $o(b^r) \geq 1$, ii) is satisfied by i). For i) and iii) to be satisfied simultaneously, $o(q) \geq o(b^r)$. Hence under the condition $o(b^r) \geq 1$ and $o(a^r) + k = 0$, which refers to a proper unstable plant with no right-half plane zero's, the condition to solve the pole placement problem is $o(q) \geq o(b^r)$.

Case iv: $o(b^r) = 0$ and $o(a^r) + k = 0$. Equations (2.32), (2.33) and (2.34) reduce to

$$o(q) \geq -1$$

$$o(q) \geq 0$$

$$o(q) \geq 0$$

respectively. Since $q(s)$ is polynomial, the solution to the pole placement problem for a stable miniphase plant is

$$o(q) \geq 0.$$

2.3 Improper Plants

For improper plants equation (2.9) states that

$$o(m) = o(a^l) + o(a^r) = o(b^l) + o(b^r) + k.$$

For the solution to the pole placement problem we require equations (2.32), (2.29), (2.30) and (2.9) to be satisfied simultaneously. Substituting (2.9) in (2.29) and (2.3) we have

$$o(q) \geq o(a^r) \quad (2.35)$$

$$o(q) \geq o(b^r) + k \quad (2.36)$$

Consider the following cases:

Case i: $o(a^r) \geq 1, o(b^r) + k \geq 1$.

By similar arguments of case i of the previous section, the condition for the solution of the pole placement problem is

$$o(q) \geq o(a^r) + o(b^r) + k - 1$$

Case ii: $o(a^r) = 0$ and $o(b^r) + k \geq 1$

Again using similar arguments of case ii in the previous section, the condition for the solution to the pole placement problem is

$$o(q) \geq o(b^r) + k.$$

The above cases for the proper and improper plants yields necessary and sufficient conditions for the linear equation (2.10) to be generically solvable. Indeed, the polynomial coprimeness of $a^r(s)$ and $b^r(s)$ implies that (2.10) will always admit coprime solutions when the above mentioned cases are satisfied. The equation (2.10) may also admit solutions which are not polynomial coprime. On the other hand if any one of the cases are not satisfied, the pole placement problem is generically unsolvable though it may admit solutions for certain $q(s)$ which lie in an appropriate lower dimension subspace.

Finally denote the integer $[o(a^r) + o(b^r) + k]$ by $\Pi(p)$ which represents the total number of closed right-half plane poles and

zeros of the plant. Indeed, $o(a^r)$ is precisely the number of finite closed right half plane zeros of $p(s)$, while $o(b^r)$ is the number of right-half plane poles of $p(s)$ and k represents the number of poles or zeros at infinity (depending on whether (2.8) or (2.9) holds). $\Pi(p)$ is a natural measure of the degree by which $p(s)$ fails to be miniphase. Consistent with the above, we can state the following theorem.

Theorem 2.2: Pole Placement Theorem

Given $p(s)$ there exists a compensator for the feedback system of Figure 1.1 which stabilizes the system and yields

$$h_{v_2\mu_1}(s) = \frac{r(s)}{q(s)}$$

where $q(s)$ is a prescribed Hurwitz polynomial, $r(s)$ is an arbitrary polynomial such that $o(r) \geq o(q)$, if

$$o(q) \leq \Pi(p) - \delta$$

where δ , the perturbation factor is either 0 or 1 as indicated by Table 2.1. Conversely if $o(q) < \Pi(p) - \delta$, the pole placement problem is generically unsolvable. $\square$

2.4 Parameterization of Solution Space

In this section the parameterization of the solution space for the pole placement problem is discussed. It follows from Theorem 2.1, that the space of compensators which resolve the pole placement problem for a given $q(s)$ take the form

TABLE 2.1: Perturbation factor ' δ ' for single variate pole placement problem.

	$\delta = 1$	$\delta = 0$
$p(s)$ Proper	$k + o(a^r) \geq 1$ and $o(b^r) \geq 1$	$k + o(a^r) = 0$ or $o(b^r) = 0$
$p(s)$ Improper	$k + o(b^r) \geq 1$ and $o(a^r) \geq 1$	$o(a^r) = 0$

$$c(s) = \frac{[-w(s)d_p(s) + u_p(s)]}{[w(s)n_p(s) + v_p(s)]} \quad (2.37)$$

where $w(s)$ is given by (2.13) with $\alpha(s)$ and $\beta(s)$ chosen to satisfy (2.11) and (2.12), together with the condition that $[w(s)n_p(s) + v_p(s)]$ is not identically zero.

If $\alpha^p(s)$ and $\beta^p(s)$ represent a particular solution and $\alpha^h(s)$ and $\beta^h(s)$ represent homogeneous solutions to equation (2.10), the entire solution space takes the form

$$\alpha(s) = \alpha^p(s) + \alpha^h(s) \quad (2.38)$$

$$\beta(s) = \beta^p(s) + \beta^h(s) \quad (2.39)$$

where $\alpha^p(s)$ and $\beta^p(s)$ satisfy

$$\alpha^p(s)a^r(s) + \beta^p(s)b^r(s) = q(s) \quad (2.40)$$

$\alpha^p(s)$ and $\beta^p(s)$ are particular solutions computed in Theorem 2.2.

$\alpha^h(s)$ and $\beta^h(s)$ satisfies

$$\alpha^h(s)a^r(s) + \beta^h(s)b^r(s) = 0 \quad (2.41)$$

If $j(s)$ is an arbitrary polynomial, we can pick

$$\alpha^h(s) = j(s)b^r(s) \quad (2.42)$$

and

$$\beta^h(s) = -j(s)a^r(s) \quad (2.43)$$

(2.42) and (2.43) are homogeneous solutions since they satisfy (2.41).

In order to show (2.42) and (2.43) characterize all the homogeneous solutions let $\tilde{\alpha}(s)$ and $\beta(s)$ be arbitrary homogeneous solutions. Since (2.42) and (2.43) are homogeneous solutions, they satisfy (2.41) namely

$$\tilde{\alpha}(s)a^r(s) + \beta(s)b^r(s) = 0. \quad (2.44)$$

picking $j(s) = \frac{\tilde{\alpha}(s)}{b^r(s)} \quad (2.45)$

$$-j(s)a^r(s) = -\frac{\tilde{\alpha}(s)}{b^r(s)} a^r(s)$$

$$= \frac{\tilde{\alpha}(s)}{b^r(s)} b^r(s)$$

$$= \beta(s) \quad (2.46)$$

which is in the form (2.43)

Since $a^r(s)$ and $b^r(s)$ are polynomial coprime, there exist polynomials $x(s)$ and $y(s)$ such that

$$a^r(s)x(s) + b^r(s)y(s) = 1. \quad (2.47)$$

Multiplying (2.45) by (2.47)

$$j(s) = \frac{\tilde{\alpha}(s)}{b^r(s)} [a^r(s)x(s) + b^r(s)y(s)]$$

substituting (2.44) and simplifying

$$j(s) = -\frac{\beta(s)b^r(s)x(s)}{b^r(s)} + y(s)\tilde{\alpha}(s)$$

$$= -\beta(s)x(s) + y(s)\tilde{\alpha}(s)$$

Hence $j(s)$ is polynomial, since it has been expressed as sums and products of polynomials.

Hence the above arguments conclude that the entire solution space of (2.10) takes the form

$$\alpha(s) = \alpha^p(s) + j(s)b^r(s) \quad (2.48)$$

and

$$\beta(s) = \beta^p(s) - j(s)a^r(s) \quad (2.49)$$

where $j(s)$ is an arbitrary polynomial. In the pole placement problem, however, the subset of the solution space that satisfies the constraints (2.11) and (2.12) of Theorem 2.1 is required. Assuming that $\alpha^p(s)$ and $\beta^p(s)$ satisfy (2.11) and (2.12), the constraints on the order of $j(s)$ satisfying (2.11) and (2.12) is required.

In the case of a proper plant recall (2.8),

$$o(m) = o(a^l) + o(a^r) + k = o(b^l) + o(b^r).$$

By substituting (2.8), $o(\alpha) = o(j) + o(b^r)$ and $o(\beta) = o(j) + o(a^r)$, (2.11) and (2.12) reduce to

$$o(j) \leq o(q) - o(a^r) - o(b^r) + k$$

$$o(j) \leq o(q) - o(a^r) - o(b^r).$$

and

$$o(j) \geq 0.$$

In order for the above three conditions simultaneously

$$o(j) \leq o(q) - o(a^r) - o(b^r) - k + \delta$$

Recall $o(a^r) + o(b^r) + k$ is the quantity $\pi(p)$, the above inequality may be rewritten as

$$o(j) \leq o(q) - \pi(p) + \delta, \quad (2.50)$$

where ' δ ' is the perturbation factor is either 0 or 1 and is given by Table 2.1.

Similarly in the case of an improper plant recall (2.9)

$$o(m) = o(a^l) + o(a^r) = o(b^l) + o(b^r) + k$$

Again substituting (2.9), $o(\alpha) = o(j) + o(b^r)$ and $o(\beta) = o(j) + o(a^r)$. (2.11) and (2.12) reduce to

$$o(j) \leq o(q) - o(a^r) - o(b^r)$$

$$o(j) \leq o(q) - o(a^r) - o(b^r) - k$$

and

$$o(j) \geq 0.$$

In order for the above three constraints to be satisfied simultaneously,

$$o(j) \leq o(q) - \pi(p) + \delta$$

where ' δ ' the perturbation factor is given by Table 2.1. We may parameterization the solution space for the pole placement problem via:

Corollary 2.2:

Let $p(s)$ and $q(s)$ be given and let $\alpha^p(s)$ and $\beta^p(s)$ be solutions of equation (2.10) such that $0 \leq o(\alpha^p) \leq o(q) + o(a^1) - o(m)$ and $0 \leq o(\beta^p) \leq o(q) + o(b^1) - o(m)$. Then the set of compensators which resolve the pole placement problem take the form

$$c(s) = \frac{[-w(s)d_p(s) + u_p(s)]}{[w(s)n_p(s) + v_p(s)]}$$

where

$$w(s) = \frac{[\beta^p(s) - j(s)a^r(s)]m(s)}{[q(s)b^1(s)]} u_p(s) - \frac{[\alpha^p(s) + j(s)b^r(s)]m(s)}{[q(s)a^1(s)]} v_p(s)$$

where $j(s)$ is an arbitrary polynomial such that

$$o(j) \leq o(q) - [\pi(p) - \delta]$$

and is chosen such that $[w(s)n_p(s) + v_p(s)]$ is not identically zero ' δ ' is the perturbation factor given by Table 2.1. $\square$

CHAPTER 3

THE STABLE MULTIVARIATE POLE PLACEMENT PROBLEM

In the first section of this chapter, the multivariate fractional representation theory [57],[58] will be formulated. In the ensuing sections the stable multivariate pole placement problem will be formulated and solved.

3.1 Multivariate Fractional Representation Theory

This algebraic theory [11] is set in a nest of rings, groups and multiplicative structures

$$G \supset H \supset I \supset J.$$

Here, G is a ring with identity which represents the feedback system and its subsystems and H is a subring of G containing the identity which models the systems which are stable in some sense. The compensator will be chosen so that the overall system will be represented by an operator in the subring H . Finally, I denotes the multiplicative set composed of elements of H which admit an inverse in G while J denotes the multiplicative subgroup of I made up of elements which are invertible in H . Detailed examples of the axiomatic structure, $\{G,H,I,J\}$ are given in reference [11] and will not be repeated here.

P has a right fractional representation in $\{G,H,I,J\}$ if

$$P = N_{pr} D_{pr}^{-1} \quad (3.1)$$

where $N_{pr} \in H$ and $D_{pr} \in I$. Furthermore, we say that this

is right coprime [28] if there exist U_{pr} and V_{pr} in H such that

$$U_{pr}N_{pr} + V_{pr}D_{pr} = I \quad (3.2)$$

This equality is equal to the classical coprimeness concept for rational functions and matrices while being well defined in our general ring theoretic setting. In particular, if G is the ring of rational functions and H is the ring of polynomials (3.2) implies that N_{pr} and D_{pr} have no common zeros; and if G is the ring of rational functions and H is the ring of exponentially stable rational functions, (1.2) implies that N_{pr} and D_{pr} have no common right-half plane zeros.

Since the ring G is, in general, non-commutative we also define a left fractional representation of P via the equality

$$P = D_{pl}^{-1}N_{pl} \quad (3.3)$$

where $D_{pl} \in H$ and $N_{pl} \in I$. Furthermore, this representation is left coprime [28] if there exist U_{pl} and V_{pl} in H such that

$$N_{pl}U_{pl} + D_{pl}V_{pl} = I \quad (3.4)$$

In the classical case of a rational functions or matrices these fractional representations are assured to exist. However, this is not the case in the general ring theoretic setting. Therefore, for distributed, time-varying and multidimensional systems we assure that our plant admits such a representation as a prerequisite to the theory.

Since H represents a ring of stable systems we say that the feedback system of Figure 1.1 is stable if all its internal and external gains are elements of H . By using the above described fractional representation theory a solution to the stabilization problem was obtained in reference [11]. In particular, if the plant P is given by $P = D_{p1}^{-1} N_{p1}$, the the compensator

$$C = (WN_{p1} + V_{pr})^{-1} (-WD_{p1} + U_{pr}) \quad (3.5)$$

stabilizes the feedback system for any $W \in H$ such that $(WN_{p1} + V_{pr}) \in I$; and every stabilizing compensator is given by (3.5) for some $W \in H$. Moreover, this compensator is used in the feedback system of all the usual gains of the closed loop system are linear (actually affine) in W . Specifically, the system's input/output gain is given by

$$H_{v_2 \mu_1} = 1 + N_{pr} W D_{p1} + N_{pr} U_{pr}$$

The remaining gains are also linear in W and are derived in reference [11].

3.2 The Stable Pole Placement Problem

In the stabilization theory discussed in the previous section it was assumed that the plant admitted both left and right coprime fractional representations

$$P(s) = N_{pr}(s) D_{pr}^{-1}(s) = D_{p1}^{-1}(s) N_{p1}(s) \quad (3.6)$$

for which there exist $U_{pr} \in H$ and $V_{pr} \in H$ such that

$$U_{pr}(s)N_{pr}(s)+V_{pr}(s)D_{pr}(s) = I \quad (3.7)$$

and $U_{p1} \in H$ and $V_{p1} \in H$ such that

$$N_{p1}(s)U_{p1}(s)+D_{p1}(s)V_{p1}(s) = I . \quad (3.8)$$

Since a doubly coprime fractional representation exists whenever separate left and right coprime fractional representations exist, we may work with a doubly coprime fractional representation without loss of generality. This, in turn, means that rather than three equalities (3.6) through (3.9), we have eight equalities with which to manipulate the feedback system equations. These eight equations can be written as follows:

$$\begin{bmatrix} D_{pr} & -U_{p1} \\ N_{pr} & V_{p1} \end{bmatrix} \begin{bmatrix} V_{pr} & U_{pr} \\ -N_{p1} & D_{p1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \quad (3.9)$$

and

$$\begin{bmatrix} V_{pr}(s) & U_{pr}(s) \\ -N_{p1}(s) & D_{p1}(s) \end{bmatrix} \begin{bmatrix} D_{pr}(s) & -U_{p1}(s) \\ N_{pr}(s) & V_{p1}(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \quad (3.10)$$

For the pole placement problem, we have considered a stable rational multivariate plant characterized by an n by n transfer function matrix $P(s)$. Since $P(s)$ is stable, it admits a trivial rational fractional representation with

$$N_{pr}(s) = N_{p1}(s) = P(s) \quad (3.11)$$

$$D_{pr}(s) = D_{p1}(s) = I \quad (3.12)$$

$$U_{pr}(s) = U_{p1}(s) = 0 \quad (3.13)$$

and

$$V_{pr}(s) = V_{p1}(s) = I \quad (3.14)$$

The stabilizing compensator given by (3.5) reduces to

$$C(s) = (W(s)P(s)+I)^{-1} (W(s)) \quad (3.15)$$

and the input/output gain given by equation (3.6) reduces to

$$H_{v_2\mu_1}(s) = P(s)W(s). \quad (3.16)$$

Here $W(s)$ is a stable rational matrix.

In this highly specialized multivariate case we formulate the pole placement problem by requiring that $W(s)$ be chosen to stabilize the system and yield an input/output gain of the form

$$H_{v_2\mu_1}(s) = \frac{1}{q(s)} R(s) \quad (3.17)$$

where $q(s)$ is the prescribed Hurwitz polynomial whose roots are the poles that are to be placed and $R(s)$ is an arbitrary n by n polynomial matrix such that

$$o(R) \leq o(q). \quad (3.18)$$

where the order of a matrix is defined to be the maximum of the orders of its elements.

Given the plant $P(s)$, a common Hurwitz denominator $b(s)$ is factored out. The numerator is then transformed into the Smith Canonical form [28]. After replacing the common denominator the resultant matrix form yields the Smith-Macmillan form [28] for $P(s)$ of (3.11).

$$P(s) = U(s)D(s)V(s) \quad (3.19)$$

where $U(s)$ and $V(s)$ are unimodular polynomial matrices and $D(s)$ is a diagonal matrix with rational entries $d_i(s)$ as in equation (3.20)

$$d_i(s) = \frac{a_i^r(s) a_i^l(s)}{b_i^l(s)}, \quad i = 1, 2, \dots, n. \quad (3.20)$$

In equation (3.20), $a_i^r(s)$ are polynomials whose roots are in the right-half plane (anti-Hurwitz) and $a_i^l(s)$ and $b_i^l(s)$ are polynomials that are Hurwitz. Since $D(s)$ is a diagonal matrix, it may further be written as the product of three diagonal matrices as follows:

$$D(s) = A_R(s)B_L^{-1}(s)A_L(s) \quad (3.21)$$

Here the diagonal entries of $A_R(s)$ are the polynomials $a_i^r(s)$, $i = 1, \dots, n$, $B_L^{-1}(s)$ are the polynomials $b_i^l(s)$, $i = 1, \dots, n$ and $A_L(s)$ are the polynomials $a_i^l(s)$, $i = 1, 2, \dots, n$. Hence, after all the above transformations the plant $P(s)$ can be written as

$$P(s) = U(s)A_R(s)B_L^{-1}(s)A_L(s)V(s) \quad (3.22)$$

Letting $V_i(s)$ denote the i th row of $V(s)$ we then obtain the following theorem.

Theorem 3.1: Stable Pole Placement Theorem.

i) If the stable multivariate pole placement problem admits a solution then

$$o(q) \geq o(b_i^1) - o(a_i^1) - o(V_i) \quad (3.22)$$

for all $i = 1, 2, \dots, n$.

ii) The stable multivariate pole placement problem admits a solution if

$$o(q) \geq o(b_i^1) - o(a_i^1) \quad (3.23)$$

for all $i = 1, 2, \dots, n$.

Proof:

i) Let there exist a stable $W(s)$ such that

$$H_{V_2 U_1}(s) = -P(s)W(s) = \frac{1}{q(s)} R(s)$$

Substituting for $P(s)$, using equation (3.22) we obtain

$$-U(s)A_R(s)B_L^{-1}(s)A_L(s)V(s)W(s) = \frac{1}{q(s)} R(s).$$

The above equation implies there exists a matrix $\alpha(s)$ such that

$$N_R(s) \alpha(s) = R(s) \quad (3.23)$$

where

$$N_R(s) = U(s)A_R(s) \quad (3.24)$$

and

$$\alpha(s) = -q(s)B_L^{-1}(s)A_L(s)V(s)W(s) \quad (3.25)$$

From equation (3.24) we can obtain

$$W(s) = -\frac{1}{q(s)} V^{-1}(s)A_L^{-1}(s)B_L(s)\alpha(s). \quad (3.26)$$

In equation (3.23) $R(s)$ is a polynomial matrix by definition and $N_R(s)$ is a polynomial matrix by construction. Hence, for the product $N(s)\alpha(s)$ to be a polynomial matrix, the poles of the elements of $\alpha(s)$ should cancel the zeros of the elements of $N_R(s)$. This is impossible as $N_R(s)$ has zeros in the right half plane where as $\alpha(s)$ has its poles in the left half plane, if it has any. Hence for the product $N_R(s)\alpha(s)$ to be polynomial, $\alpha(s)$ has to be polynomial.

Since $W(s)$ is stable, using equation (3.25),

$$o(\alpha_i) = o(q) - o(b_i^1) + o(a_i^1) + \max_j \{o(V_i^j) + o(W_j)\} \quad (3.27)$$

Here the superscript j refers to the j th column of the matrix and the subscript i or j refers to the respective rows. Since,

$$\max_j \{o(V_i^j) + o(W_j)\} \geq \max_j (V_i^j) + \max_j o(W_j) \geq o(V_i) + o(W_j) \quad (3.28)$$

and $W(s)$ is stable $o(W_j) \leq 0$ and hence (3.27) reduces to the inequality

$$o(\alpha_i) \leq o(q) - o(b_i^1) + o(a_i^1) + o(V_i)$$

Since $\alpha(s)$ is a polynomial matrix

$$o(\alpha_i) \geq 0 \text{ and hence}$$

$$0 \leq o(q) - o(b_i^1) + o(a_i^1) + o(v_i)$$

which implies

$$o(q) \geq o(b_i^1) - o(a_i^1) - o(v_i),$$

for all $i = 1, 2, \dots, n$.

Thus proving (i) and completing the necessity of the proof.

ii) It is given that

$$o(q) \geq o(b_i^1) - o(a_i^1), \quad i = 1, 2, \dots, n. \quad (3.23)$$

Let $\tilde{W}(s)$ be defined by the following equality

$$\tilde{W}(s) = -\frac{1}{q(s)} A_L^{-1}(s) B_L(s)$$

By the above definition of $\tilde{W}(s)$, $\tilde{W}(s)$ is a diagonal rational matrix whose elements are given by

$$\tilde{W}_i(s) = \frac{b_i^1(s)}{q(s)a_i^1(s)} = \frac{x_{m_i} s^{m_i} + s_{m_{i-1}} s^{m_{i-1}} + \dots + x_0}{y_{n_i} s^{n_i} + y_{n_{i-1}} s^{n_{i-1}} + \dots + y_0}$$

$$i = 1, 2, \dots, n \quad (3.29)$$

Let $K(s)$ be a diagonal polynomial matrix whose elements are given by

$$K_i(s) = \frac{y_{n_i}}{x_{m_i}} s^{n_i - m_i}, \quad i = 1, 2, \dots, n. \quad (3.30)$$

Defining $W(s)$ by

$$W(s) = V^{-1}(s) \tilde{W}(s) K(s) V(s) \quad (3.31)$$

not noting that the

$$\lim_{s \rightarrow \infty} W(s) = \lim_{s \rightarrow \infty} V^{-1}(s)K(s)W(s)V(s) = I \quad (3.32)$$

where I is the identity matrix. Using (3.23) and (3.32), $W(s)$ is guaranteed to be stable at $s = \infty$. Thus $W(s)$ is a stable rational matrix and is given by

$$W(s) = V^{-1}(s)W(s)K(s)V(s) = -V^{-1}(s) \frac{1}{q(s)} A_L^{-1}(s)B_L(s)K(s)V(s)$$

Now to check if the above stable $W(s)$ gives us the required input/output transfer function

$$\begin{aligned} H_{v_2 \mu_1}(s) &= -P(s)W(s) \\ &= -[U(s)A_R(s)B_L^{-1}(s)A_L(s)V(s)]^{-1} \\ &\quad [-V^{-1}(s)\frac{1}{q(s)}A_L^{-1}(s)B_L(s)K(s)V(s)] \\ &= \frac{1}{q(s)} U(s)A_R(s)K(s)V(s) \\ &= \frac{1}{q(s)} R(s) \end{aligned} \quad (3.33)$$

which is the required transfer function. $R(s) = U(s)A_R(s)K(s)V(s)$ is a polynomial matrix since $U(s)$, $A_R(s)$, $K(s)$ and $V(s)$ are all polynomial matrices by construction. This completes the proof. $\square$

The necessary and sufficient conditions of (3.22) and (3.23) do not coincide except in the single variate case where they shown in Theorem 2.2. The above theorem leaves many unanswered questions. In particular, since the Smith form is not unique,

the necessary condition can be tightened by varying the Smith form. These aspects will be discussed further in Chapter 5.

CHAPTER 4

EXAMPLES

In this chapter examples of both single variate and multivariate pole placement problems are presented. These examples highlight the simplicity and powerfulness of the fractional representation theory, especially the pole placement problem.

Example 4.1: Unstable Proper Plant

Figure 4.1: Unstable Plant Before Pole Placement and Stabilization.

For the system of Figure 4.1, the plant $p(s) = \frac{s+1}{s^2-4}$. It is required to find the compensator that places the poles of the system $h_{v\mu}$ at the zeros of $q(s) = s^2+s+1$, while simultaneously stabilizing the system. Since the formulation of the pole placement problem requires $n_p(s)$ and $d_p(s)$ have a common denominator, $m(s)$, $p(s)$ may be written, as in (2.3) and (2.4), as

$$p(s) = \frac{\frac{(s+1)}{(s+3)^2}}{\frac{(s^2-4)}{(s+3)^2}} = \frac{n_p(s)}{d_p(s)} \quad (4.1)$$

we can find

$$u_p(s) = \frac{(6s+13)}{(s+1)} \quad (4.2)$$

and

$$v_p(s) = 1. \quad (4.3)$$

Since $u_p(s)n_p(s) + v_p(s)d_p(s) = 1$, $n_p(s)$ and $d_p(s)$ are coprime.

Comparing (4.1) with (2.3) and (2.4),

$$a^1(s) = (s+1)$$

$$a^r(s) = 1$$

$$b^1(s) = (s+2)$$

$$b^r(s) = (s-2)$$

and

$$m(s) = (s+1)^2$$

The appropriate order of the above polynomials are

$$o(a^r) = 0$$

$$o(b^r) = 1$$

and

$$k = 1.$$

According to theorem 2.2, since $k + o(a^r) \geq 1$ and $o(b^r) \geq 1$, the perturbation factor $\delta = 1$ and

$$o(q) \geq \Pi(p) - \delta = 2 - 1 = 1.$$

Hence we may realize any Hurwitz denominator of order greater than or equal to 1 and hence, we can place the poles at zeros of $q(s) = s^2 + s + 1$.

In this case, using (2.11) and (2.12)

$$0 \leq o(\alpha) \leq o(q) + o(a^1) - o(m) = 1$$

$$0 \leq o(\beta) \leq o(q) + o(b^1) - o(m) = 1.$$

Equation (2.10), namely

$$\alpha(s)a^r(s) + \beta(s)b^r(s) = q(s)$$

is satisfied by

$$(3s+1)(1) + (s)(s-2) = s^2 + s + 1$$

Now using corollary 2.2 we can pick an arbitrary polynomial $j(s)$ such that

$$o(j) \leq o(q) - [\Pi(p) - \delta]$$

$$= 2 - [2 - 1] = 1$$

taking $j(s) = 0$ in Corollary 2.2,

$$w(s) = \left[\frac{(s)(s+3)^2}{(s^2+s+1)(s+2)} \right] \left[\frac{6s+13}{(s+1)} \right] - \left[\frac{(3s+1)(s+3)^2}{(s^2+s+1)(s+1)} \right] \quad [1]$$

$$= \left[\frac{(s+3)^2(3s^2+6s-2)}{(s^2+s+1)(s+1)(s+2)} \right]$$

$$c(s) = \left[\frac{(s+2)(3s^3+19s^2+33s+9)}{s(s+1)(s+3)^2} \right]$$

and

$$h_{v_2 \mu_1}(s) = \left[\frac{3s+1}{s^2+s+1} \right] = \frac{\alpha(s)a^r(s)}{q(s)}$$

as required.

Figure 4.2 shows the entire system.

Figure 4.2: Feedback System for Example 4.1 that Places the Poles of the System at Zeros of $q(s) = s^2+s+1$.

Example 4.2: Stable Proper Plant.

Consider the plant in Figure 4.3.

Figure 4.3: Plant Before Stabilization and Pole Placement.

For the system of figure 4.3 it is required to find a compensator $c(s)$ that stabilizes the system and places poles at the zeros of $q(s) = (s+2)(s+3)$.

$n_p(s)$ and $d_p(s)$ are constructed as

$$n_p(s) = \frac{a^l(s)a^r(s)}{m(s)} = \frac{(1)(s-1)}{(s+4)} \quad (4.4)$$

$$d_p(s) = \frac{b^l(s)b^r(s)}{m(s)} = \frac{(s+1)(1)}{(s+4)} \quad (4.5)$$

Since $n_p(s)$ and $d_p(s)$ are coprime,

$$u_p(s)n_p(s) + v_p(s)d_p(s) = (-1.5)\frac{(s-1)}{(s+1)} + (2.5)\frac{(s+1)}{s+4} = 1$$

where

$$u_p(s) = -1.5 \quad (4.6)$$

and

$$n_p(s) = 2.5 \quad (4.7)$$

By the construction of $n_p(s)$ and $d_p(s)$ the orders of the following polynomials are

$$o(a^r) = 1$$

$$o(b^r) = 0$$

and

$$k = 0.$$

According to Theorem 2.2, since $k + o(a^r) \geq 1$ and $o(b^r) = 0$, the perturbation factor $\delta = 0$ and

$$o(q) \geq \Pi(p) - \delta = 1 - 0 = 1.$$

Hence we can realize any Hurwitz denominator of order greater than or equal to 1 and hence, we can place the poles at the zeros of $q(s) = (s+2)(s+3)$.

In this case, using (2.11) and (2.12)

$$0 \leq o(\alpha) \leq 2 + 0 - 1 = 1$$

and

$$0 \leq o(\beta) \leq 2 + 1 - 1 = 2$$

Equation (2.10) which requires

$$\alpha(s) a^r(s) + \beta(s) b^r(s) = q(s)$$

is satisfied by

$$\alpha(s) = (s+6)$$

and

$$\beta(s) = (12)$$

Now using Corollary 2.2, we can pick an arbitrary polynomial $j(s)$ such that

$$\begin{aligned} O(j) &\leq O(q) - [\Pi(p) - \delta] \\ &= 2 - [1 - 0] = 1. \end{aligned}$$

Taking $j(s) = (s+1)$ in Corollary 2.2

$$w(s) = \frac{6.5s^3 + 48.5s^2 + 93s - 8}{(s+1)(s+2)(s+3)} \quad (4.8)$$

and

$$c(s) = \frac{-w(s) \left(\frac{s+1}{s+4} \right) + (-1.5)}{w(s) \left(\frac{s-1}{s+4} \right) + (2.5)}$$

where $w(s)$ is given by (4.8).

Figure 4.4: Feedback System for Example 4.1 that places Poles at Zeros of $q(s) = (s+2)(s+3)$.

Example 4.3: Stable Multivariate Plant.

Consider a stable multivariate plant $P(s)$, whose transfer function is given by

$$P(s) = \frac{1}{(s+2)(s+3)} \begin{bmatrix} (s+1)(s+1) & (s+1)(s+4) & (s+2)(s+1) \\ (s+1)(s+2) & (s+1)(s-1) & (s+1)(s+3) \\ (s+1)(s+3) & (s+7)(s+1) & (s+1)(s+1) \end{bmatrix}$$

The Smith Macmillan form, for the above transfer function, as in equations (3.19) and (3.21) is

$$P(s) = U(s)A_R(s)B_L^{-1}(s)A_L(s)V(s)$$

where

$$U(s) = \frac{1}{13} \begin{bmatrix} 13(s+1) & (6s+9) & 13 \\ 13(s+2) & (6s+9) & 13 \\ 13(s+3) & (s+22) & 13 \end{bmatrix}$$

$$A_R(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B_L(s) = \begin{bmatrix} (s+2)(s+3) & 0 & 0 \\ 0 & (s+2)(s+3) & 0 \\ 0 & 0 & (s+2)(s+3) \end{bmatrix}$$

$$A_L(s) = \frac{1}{13} \begin{bmatrix} 13(s+1) & 0 & 0 \\ 0 & 169(s+1) & 0 \\ 0 & 0 & (s+1)(9s+20) \end{bmatrix}$$

and

$$V(s) = \begin{bmatrix} 1 & -5 & -\frac{2}{13} \\ 0 & 1 & -\frac{3}{13} \\ 0 & 0 & 1 \end{bmatrix}$$

Using Theorem 3.1 and equation (3.23), the stable multivariate problem admits a solution, if

$$o(q) \geq o(b_i^1) - o(a_i^1)$$

Therefore

$$\text{for } i = 1, \quad o(q) \geq 2 - 1 = 1$$

$$\text{for } i = 2, \quad o(q) \geq 2 - 1 = 1$$

$$\text{for } i = 3, \quad o(q) \geq 2 - 2 = 0.$$

The above three inequalities imply that, poles can be placed at the zeros of any polynomial that has an order greater than or equal to 1.

Hence, poles can be placed at $q(s) = (s+7)(s+8)(s+9)$. Using (3.30) and (3.33), the required input output transfer function takes the form

$$H_{v_2 \mu_1}(s) = \frac{1}{q(s)} R(s)$$

where

$$q(s) = (s+7)(s+8)(s+9)$$

and

$$R(s) = \frac{1}{13} \cdot \begin{bmatrix} 13s^2(s+1) & 13s^2(s+4) & -s^2(2s+29) \\ 13s^2(s+2) & 13s^2(s-1) & -s^2(2s+31) \\ 13s^2(s+3) & 13s^2(s+7) & -2s^2(s+72) \end{bmatrix}$$

CHAPTER 5

CONCLUDING REMARKS

A new approach, using the fractional representation theory, to the Pole Placement Problem was proposed. The starting point for the theory rests with the celebrated 1976 papers of Youla, Bongiorno and Jabr [57],[58] in which a complete parameterization of the set of stabilizing compensators for a general multivariate system was first formulated. Of course, such parameterizations have long been known in the single variate case, while several authors have several alternate formulations [1],[4],[8],[17],[38],[39],[51],[60]. Although the YBJ theory was formulated using a polynomial fractional representation [57],[58], it was soon discovered that the theory could be simplified and generalized by working with stable factors rather than polynomial factors. This in turn led to both distributed and ring theoretic formulations of the stabilization theorem. The YBJ theorem has also been formulated in an algebro-geometric setting [44].

In the single variate Pole Placement Problem the procedure of designing a compensator that stabilized the system and simultaneously placed poles at a desired location was developed. Unlike the case where one uses state feedback if one is to place the poles of the system via 'output feedback' which usually needs a dynamic compensator, there is a lower bound on the degree of the resultant feedback system. Indeed, as shown in Theorem 2.2, the number of poles that can be placed was related to the degree by which the plant failed to be miniphase. The further away the plant is from being miniphase, the higher the order of $q(s)$ should be, that is the greater the cost we must pay in order

to get the system to perform the required specifications. This is indeed a pleasing and intuitive result.

In the multivariate case, a highly specialized version of the Pole Placement Problem was formulated. Unfortunately the necessary and sufficient conditions of Theorem 3.1 do not coincide except in the single variate case. Needless to say the above theorem leaves many unanswered questions. In particular since the Smith form is not unique, the necessary conditions of the theorem can be conceivably tightened by simply varying the Smith canonical form.

Some future areas of research will be discussed now. As mentioned in the above paragraph a more general Pole Placement Theorem is required in the multivariate case. This could very well be an initial direction for future research. Secondly as the YBJ theory is very easy to implement, other classical control problems such as Input/Output Decoupling, Simultaneous Stabilization and Adaptive Control could be solved in the fractional representation setting. Finally, a Computer Aided Design package for the entire Feedback System Design in the above setting could be written.

REFERENCES

- [1] Antsaklis, P. J. and Pearson, J. B., "Stabilization and Regulation in Linear Multivariable Systems", IEEE Transactions on Automatic Control, Vol. AC-23, 1978, pp. 928-930.
- [2] Barnett, S., Matrices in Control Theory, London, Van Norstrand, 1971.
- [3] Barnett, S., "Matrices, Polynomials and Linear Time Invariant Systems", IEEE Transactions on Automatic Control, Vol. AC-18, 1973, pp. 1-10.
- [4] Bengtsson, G., "Output Regulation and Internal Modes-A Frequency Domain Approach", Automatica, Vol. 13, 1977, pp. 333-345.
- [5] Brasch, M. F. and Pearson, J. B., "Pole Placement Using Dynamic Compensators", IEEE Transactions on Automatic Control, Vol. AC-15, 1970, pp. 34-43.
- [6] Brockett, R. W. and Byrnes, C. I., "Multivariate Nyquist Criteria, Root Loci and Pole Placement: A Geometric Viewpoint", IEEE Transactions on Automatic Control, Vol. AC-26, 1981, pp. 271-283.
- [7] Callier, F. M. and Desoer, C. A., "Stabilization, Tracking and Disturbance Rejection in Linear Multivariate Distributed Systems", Proceedings of the 17th IEEE Conference on Decision and Control, San Diego, 1979, pp. 513-514.
- [8] Cheng, L. and Pearson, J. B., "Frequency Domain Synthesis of Multivariable Linear Regulators", IEEE Transactions on Automatic Control, Vol. AC-23, 1978, pp. 3-15.
- [9] Chua, O., M.S. Thesis, Texas Tech University, 1980.
- [10] Desoer, C. A., and Schulman, J. D., "Zeros and Poles of Matrix Transfer Functions and Their Dynamical Interpretations", IEEE Transactions on Circuits and Systems, Vol. CAS-21, 1974, pp. 3-8.
- [11] Desoer, C. A., Liu, R. W., Murray, J. and Saeks, R., "Feedback System Design: the fractional representation approach to analysis and synthesis", IEEE Transactions on Automatic Control Vol. AC-25, 1980, pp. 39-412.
- [12] Emre, E., "Pole Assignment by Dynamic Feedback", International Journal of Control, Vol. 33, 1981, pp. 311-321.

- [13] Emre, E., "The Polynomial Equation $QQ_c + RP_c = \Phi$: with applications to Dynamic Feedback", SIAM Journal of Control and Optimization, Vol. 18, 1980, pp. 611-620.
- [14] Emre, E., "Pole, Zero Cancellations in Dynamical Feedback Systems", Proceedings of the 20th IEEE Conference on Decision and Control, Orlando, 1982. (to appear)
- [15] Emre, E., "Further Results on Skew Prime Matrices for Linear Systems over Rings", Proceedings of the 20th IEEE Conference on Decision and Control, Orlando, 1982. (to appear)
- [16] Emre, E., and Silverman, L. M., "The equation $XR + QY = \Phi$: A Characterization of Solutions", SIAM Journal of Control and Optimization, Vol. 19, 1981, pp. 33-38.
- [17] Feintuch, A. and Saeks, R., System Theory: A Hilbert Space Approach, New York, Academic Press, 1982.
- [18] Francis, B., "The Multivariable Servomechanism Problem from the Input-Output Viewpoint", IEEE Transactions on Automatic Control, Vol. AC-22, 1977, pp. 322-328.
- [19] Francis, B. A., and Wonham, W. M., "The Role of Transmission Zeros in Linear Multivariate Regulators", International Journal of Control, Vol. 22, No. 5, 1975, pp. 657-681.
- [20] Francis, B., and Vidyasagar, M., "Algebraic and Topological Aspects of the Servo Problem for Lumped Linear Systems", Unpublished Notes, Yale University, 1980.
- [21] Gantmacher, F. R., "Applications of Theory of Matrices", New York, Interscience Publishers, 1959.
- [22] Gantmacher, F. R., "Theory of Matrices - Vol. 1", New York, Chelsea, 1959.
- [23] Gantmacher, F. R., "Theory of Matrices - Vol. II", New York, Chelsea, 1959.
- [24] Helton, J. W., "Orbit Structure of the Moebius Transformation Semigroup Acting on H ", in Topics in Functional Analysis, Advances in Mathematical Support Studies, Vol. 3, New York, Academic Press, 1978, pp. 129-157.
- [25] Howze, J. W., and Pearson, J. B., "Decoupling and Arbitrary Pole Placement in Linear Systems using Output Feedback", IEEE Transactions on Automatic Control, Vol. AC-15, 1970, pp. 660-663.

- [26] Hung, T.N. and Anderson, B.D O., "Triangularization Techniques for the Design of Multivariate Control Systems", IEEE Transactions on Automatic Control, Vol. AC-24, 1979, pp. 455-460.
- [27] Jury, E. I., "Inners and Stability of Dynamic Systems", New York, John Wiley and Sons, 1974.
- [28] Kailath, T., Linear Systems, New Jersey, Prentice Hall, Inc., 1980.
- [29] Karmkolias, C., "Adaptive Control in the Fractional Representation Approach", Unpublished Notes, Texas Tech University, 1979.
- [30] Khargonekar, P. P., and Ozguler, B. A., "System Theoretic and Algebraic Aspects of the Rings of Stable and Causal Stable Rational Functions", Unpublished Notes, Center for Mathematical System Theory, University of Florida, Gainesville, 1981.
- [31] Kourvaritakis, B., and MacFarlane, A. G. J., "Geometric Approach to analysis and synthesis of system zeros, Part I and II", International Journal of Control, Vol. 23, 1976, pp. 149-166 and pp. 167-181.
- [32] Kranc, G.M. "Input Output Analysis of Multivariate Feedback Systems", IEEE Transactions on Automatic Control, Vol. AC-3, 1957, pp. 21-28.
- [33] MacDuffee, C. C., "The Theory of Matrices", Berlin, Verlag Von Julius Springer, 1933.
- [34] MacFarlane, A. G. J., "The Role of Poles and Zeros in Multivariate Feedback Theory", Directions in Large Scale Systems, New York, Plenum Press, 1975, pp. 325-338.
- [35] MacFarlane, A. G. J., "The Development of Frequency-Response Methods in Automatic Control", IEEE Transactions on Automatic Control, Vol. AC-24.
- [36] MacFarlane, A. G. J., Frequency Response Methods in Control Systems, New York, IEEE Press, 1979.
- [37] Newman, M., Integral Matrices, New York, Academic Press, 1972.
- [38] Pernbo, L., Ph.D. Thesis, Lund Institute of Technology, 1978.
- [39] Pernbo, L., "An Algebraic Theory for Design of Controllers for Linear Multivariable Systems; Parts I and II", IEEE Transactions on Automatic Control, Vol. AC-26, 1981, pp. 171-182 and pp. 183-194.

- [40] Popov, V. M., "Some Properties of the Control Systems with Irreducible Matrix - Transfer Functions", Lecture Notes in Mathematics #144, Seminar on Differential Equations and Dynamical System, Springer, 1970, pp. 169-180.
- [41] Rosenbrock, H. H., State-Space and Multivariate Theory, New York, John Wiley and Sons, 1970.
- [42] Rosenbrock, H. H., and Hayton, G. E., "The General Problem of Pole Assignment", International Journal of Control, Vol. 27, 1978, pp. 837-852.
- [43] Sacks, R., and Murray, J., "Feedback System Design: the tracking and disturbance rejection problems", IEEE Transactions on Automatic Control, Vol. AC-26, 1981, pp. 203-217.
- [44] Sacks, R., and Murray, J., "Fractional Representation, Algebraic Geometry and Simultaneous Stabilization Problem", IEEE Transactions on Automatic Control, Vol. AC-27, 1982, pp. 895-903.
- [45] Sacks, R., Murray, J., Chua, O., Karmokolias, C., and Iyer, A., "Feedback System Design: The single variate case, Part I", Circuits, Systems and Signal Processing, Vol. 1, No. 2, 1982, pp. 138-169.
- [46] Sacks, R., Murray, J., Chua, O., Karmokolias, C., and Iyer, A., "Feedback System Design: The single variate case - Part II", Circuits Systems and Signal Processing, 1982 (to appear)
- [47] Sain, K. M., Peczkowski, J. L. and Melsa, J. L., Alternatives for Linear Multivariate Control, Chicago, National Engineering Consortium, Inc., 1978.
- [48] Vidyasagar, M., Schneider, H. and Francis B., "Algebraic and Topological Aspects of Feedback System Stabilization", Technical Report 80-09, Department of Electrical Engineering, University of Waterloo, 1980.
- [49] Vidyasagar, M., and Vishwanadham, M., "Algebraic Design Techniques for Reliable Stabilization", Technical Report 81-02, Department of Electrical Engineering, University of Waterloo, 1980.
- [50] Wolovich, W. A., "Skew Prime Polynomial Matrices", IEEE Transactions on Automatic Control, Vol. AC-23, 1975, pp. 880-887.
- [51] Wolovich, W. A., "Multivariable System Synthesis with Step Disturbance Rejection", IEEE Transactions on Automatic Control, Vol. AC-19, 1974, pp. 127-130.

- [52] Wolovich, W.A., Linear Multivariable Systems, New York, Springer-Verlag, 1974.
- [53] Wonham, W. M., "One Pole Assignment in Multi-input Controllable Linear Systems", IEEE Transactions on Automatic Control, Vol. AC-12, 1967, pp. 660-665.
- [54] Wonham, W. M., Linear Multivariable Control: A Geometric Approach, Lecture Notes in Mathematical Systems No. 101, New York, Springer-Verlag, 1974.
- [55] Wonham, W. M., "On pole Assignment in Multi-dimensional Linear Systems", Correspondence, IEEE Transactions on Automatic Control, Vol. AC-13, 1968, pp. 747-748.
- [56] Youla, D. C., "Interpolary Multichannel Spectral Estimation", Unpublished Notes, Polytechnic Institute of New York, 1979.
- [57] Youla, D. C., Bongiorno, J. J., and Jabr, H. A., "Modern Wiener-Hopf Design of Optimal Controllers, Part I", IEEE Transactions on Automatic Control, Vol. AC-21, 1976, pp. 3-15.
- [58] Youla, D. C., Bongiorno, J. J., and Jabr, H. A., "Modern Wiener-Hopf Design of Optimal Controllers", IEEE Transactions on Automatic Control, Vol. AC-21, 1976, pp. 319-338.
- [59] Youla, D. C., Bongiorno, J. J., and Lu, C. N., "Single Loop Stabilization of Linear Multivariable Dynamical Plants", Automatica, Vol. 10, 1974, pp. 159-173.
- [60] Zames, G., "Feedback and Optimal Sensitivity: Model reference transformations weighted seminorms and approximate inverses", IEEE Transactions on Automatic Control, Vol. AC-26, 1981, pp. 301-320.