PROPER VALUES OF THE SUBUNITARY PART OF A MATRIX

by

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A DISSERTATION

IN

MATHEMATICS

Submitted to the Graduate Faculty
of Texas Tech University in
Partial Fulfillment of
the Requirements for
the Degree of

DOCTOR OF PHILOSOPHY

Approved

Accepted

December, 1974

ACKNOWLEDGMENTS

I would like to thank Professor Ali R. Amir-Moéz for directing this dissertation. I shall always be grateful for the guidance and encouragement that he provided and also for his patience and genuine concern.

My appreciation is also extended to the other members of my committee, Professors Thomas Boullion, Charles Burford, Robert Hunt, and Thomas Newman, for their helpful suggestions and for the interest that they have shown in this endeavor.

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CHAPTER I

INTRODUCTION

There is an analogy between matrices with complex entries and the complex numbers in which a matrix A and its adjoint A^* play roles corresponding to a complex number a and its conjugate $\bar{a}$. Hermitian matrices correspond to real numbers while positive semidefinite matrices correspond to nonnegative real numbers.

A matrix A can be written uniquely in the form $A = H + iK$ where H and K are Hermitian matrices. Similarly a complex number z can always be written in the form $z = a + ib$ where a and b are real numbers.

The polar decomposition of a complex number z, which is written $z = |z|e^{i\theta}$, has a natural analogue in matrices since a matrix A may be written $A = H_1U_1$ or $A = U_2H_2$ with U_1 and U_2 unitary and H_1 and H_2 positive semidefinite. In case A is nonsingular, we have that $U_1 = U_2$.

Questions are posed by A. Amir-Moéz and A. Horn in [1] about the proper values of the unitary part of a matrix and also about the proper values of products of matrices.

Relationships between the arguments of the proper values of a nonsingular matrix and those of its unitary part have been established by A. Horn and R. Steinberg [6].

The authors of [2] exhibit a decomposition of a matrix into the product of a positive semidefinite matrix and a subunitary matrix.

This is known as the subpolar decomposition of a matrix. We use this form to establish relationships between the proper values of a matrix and those of its subunitary part.

The minimax principles of Horn and Steinberg given in [6] for unitary matrices are generalized to include normal matrices in [3]. Orthonormal bases of proper vectors were used to establish these theorems. In general, subunitary matrices do not have proper vectors with this property consequently no minimax principles are given for subunitary matrices in this paper.

In [6] Horn and Steinberg give conditions on two sets of numbers $\{ a_1 , \ldots , a_n \}$ and $\{ u_1 , \ldots , u_n \}$ which are sufficient to conclude that there exists a matrix A with $a_1 , \ldots , a_n$ as its proper values and $u_1 , \ldots , u_n$ as the proper values of the unitary part of A.

Halmos and McLaughlin [5] state conditions on a set of numbers $\{ s_1 , \ldots , s_k \}$ which are sufficient to enable the construction of a subunitary matrix with $\{ s_1 , \ldots , s_k \}$ as its set of proper values.

We use the ideas of [5] and [6] to construct a matrix having prescribed proper values which is the subunitary part of a matrix whose proper values are also predetermined.

The purpose of this paper is to present generalizations for some of the Horn-Steinberg theorems given in [6] by considering the case of a singular matrix and its subunitary part. We also investigate other properties of subunitary matrices.

Techniques of restricting linear transformations to invariant subspaces on which they are nonsingular are used and we discuss extension techniques which involve adding a supplement to a subunitary matrix and considering the resulting matrix which is unitary.

CHAPTER II

DEFINITIONS AND NOTATIONS

2.1 Notations: Let E_n be an n-dimensional unitary space. We denote vectors by Greek letters α, β, γ, . . . and scalars by a, b, c, ... The inner product of two vectors ξ and η will be denoted by (ξ,η). If $\xi_1, \ldots, \xi_n$ are vectors, then $[\xi_1, \ldots, \xi_n]$ will denote the subspace spanned by them. The adjoint of a linear transformation A on E_n will be denoted by A^* where $(A\xi,\eta) = (\xi,A^*\eta)$ for all ξ,η in E_n. Dim N will denote the dimension of the subspace N.

In what follows, all projections will be orthogonal. If M is a subspace of E_n and P is the projection on M, then for a linear transformation A on E_n we define a linear transformation $A|M$ as follows: if ξ is a vector in M, we let $(A|M)\xi = PA\xi$. We note that $PAP|M = A|M$ .. If $A \geq 0$, then A has a unique square root $\sqrt{A}$, where $\sqrt{A} \geq 0$ and $(\sqrt{A})^2 = A$. One can easily show that for any linear transformation A on E_n the transformations A^*A and AA^* are nonnegative.

The range of a linear transformation A will be denoted by R(A), while the null space of A will be denoted by N(A).

In what follows, we consider matrices of linear transformations with respect to orthonormal bases. Sometimes we state propositions in terms of linear transformations and other times we use matrices.

2.2 Definition: Two linear transformations A and B on E_n are said to be congruent if there exists a nonsingular linear transformation X on E_n such that $B = X^*AX$.

2.3 Definition: Let A be a linear transformation on E_n . The (Moore-Penrose) generalized inverse of A dehoted by A^- is the transformation G which satisfies the following conditions:

$$(1)\ AGA = A$$

$$(2)\ GAG = G$$

$$(3)\ (AG)^* = AG$$

$$(4)\ (GA)^* = GA \quad .$$

It has been shown that the set of equations has a unique solution $G = A^-$. We also note that $AA^- = P$ and $A^-A = Q$, where P is the projection on R(A) and Q is the projection on R(A*).

2.4 Definition: A linear transformation S on E_n is called subunitary (partial isometry) if $S^- = S^*$.

2.5 Definition: Let A be a linear transformation on E_n. Then the decompositions $A = \sqrt{AA^*}\ S = S\sqrt{A^*A}$ are called the subpolar forms of A, where S is a unique subunitary transformation with $S^-S = A^-A$ ($SS^- = AA^-$). It is clear that when A is nonsingular, S will be unitary. In all cases, S is unique. [2, p. 53-54] .

2.6 Definition: Let $(a_1, \ldots, a_n)$ be an n-tuple of complex numbers of absolute value one. If two of the a_i's are of the form e^{ib} and e^{ic} with $0 < b-c < \Pi$ and $0 \leq d \leq \frac{1}{2}(b-c)$, then the operation of replacing e^{ib} and e^{ic} by $e^{i(b-d)}$ and $e^{i(c+d)}$ is called a pinch of $(a_1, \ldots, a_n)$.

2.7 Definition: If (a_i) and (b_i) are n-tuples of real numbers, and if (a_i') and (b_i') are their rearrangements in non-increasing order, then we write $(a_i) \prec (b_i)$ when

$$\sum_{i=1}^{r} a_i' \leq \sum_{i=1}^{r} b_i' \quad , \quad r = 1, 2, \ldots n-1$$

and

$$\sum_{i=1}^{n} a_i' \leq \sum_{i=1}^{n} b_i' \quad .$$

CHAPTER III

HORN-STEINBERG THEOREMS

In this chapter the main results of A. Horn and R. Steinberg [6] concerning relationships between the proper values of a nonsingular matrix and those of its unitary part are listed.

In later chapters some similar results are given for singular matrices and their subunitary parts.

3.1 <u>THEOREM</u>: Let A be a nonsingular matrix with proper values $a_1, \ldots, a_n$. Let the proper values of U, the unitary part of A, be $u_1, \ldots, u_n$. Then the equivalent conclusions (1) and (2) hold.

(1) The n-tuple $(u_1, \ldots, u_n)$ can be reduced to $(a_1/|a_1|, \ldots, a_n/|a_n|)$ by a finite sequence of pinches.

(2) $\prod_{i=1}^{n} u_i = \prod_{i=1}^{n} a_i/|a_i|$ and exactly one of the following statements below holds:

(a) There is a line through 0 (the origin of the complex plane) containing all the u_i's and $(a_1/|a_1|, \ldots, a_n/|a_n|)$ is a rearrangement of $(u_1, \ldots, u_n)$.

(b) There is no line through 0 containing all of the u_i's, but there is a closed half plane H with 0 on its boundary containing all of the u_i's, and, if we choose a branch of the argument function which is continuous in H - {0}, then $(\arg a_i) \prec (\arg u_i)$.

(c) There is no closed half plane with 0 on its boundary which contains all of the u_i's.

3.2 THEOREM: Let $(u_1, \ldots, u_n)$ be an n-tuple of complex numbers of absolute value one and $(a_1, \ldots, a_n)$ an n-tuple of nonzero complex numbers such that

$$\prod_{i=1}^{n} u_i = \prod_{i=1}^{n} a_i/|a_i|$$

and exactly one of the statements below holds:

(a) There is a line through 0 (the origin in the complex plane) containing all of the u_i's and $(a_1/|a_1|, \ldots, a_n/|a_n|)$ is a rearrangement of $(u_1, \ldots, u_n)$.

(b) There is no line through 0 containing all of the u_i's, but there is a closed half plane H with 0 on its boundary containing all of the u_i's, and, if we choose a branch of the argument function which is continuous in H - {0}, then $(\arg a_i) \prec (\arg u_i)$.

(c) There is no closed half plane with 0 on its boundary which contains all of the u_i's.

Then there exists a nonsingular matrix A with proper values $a_1, \ldots a_n$ and its unitary part has proper values $u_1, \ldots, u_n$.

CHAPTER IV

SOME PRELIMINARY THEOREMS

The following theorems play a role in the development of the theorems of the next chapter. Several of them are lemmas of [6] and they are listed without proof.

4.1 THEOREM: Let A be a nonsingular linear transformation on E_n with proper values $a_1, \ldots, a_n$. Let $u_1, \ldots, u_n$ be the proper values of the unitary part of A. Then there exists a matrix B with proper values $a_1, \ldots, a_n$ which is congruent to the diagonal matrix D with diagonal entries $u_1, \ldots, u_n$.

Proof. The matrix of A may be written with respect to an orthonormal basis so that $A = KU$ with K a positive definite matrix and U the unitary part of A. Now there exists a unitary matrix V such that $U = V^*DV$. A has the same proper values as B where $B = \sqrt{K}\, V^*DV\sqrt{K}$ and B is congruent to D.

4.2 THEOREM: Let $a_1, \ldots, a_n$ be proper values of a nonsingular linear transformation A on E_n and $u_1, \ldots, u_n$ be the proper values of the unitary part of A. Let $p_1, \ldots, p_n$ be positive numbers. Then there exists a linear transformation B on E_n whose proper values are $p_1a_1, \ldots, p_na_n$ and its unitary part has proper values $u_1, \ldots, u_n$. The proof is given in [6, p.542].

4.3 THEOREM: Let A be a nonsingular linear transformation on E_n such that $(A\xi,\xi) \neq 0$ and $\pi > \arg (A\xi,\xi) > 0$ for all nonzero ξ in E_n. Then A is congruent to a unitary transformation. The proof is given in [6, p.544].

Horn and Steinberg have shown that a knowledge of the proper values of the unitary part of a linear transformation A restricts the arguments of the proper values of A. The following theorem illustrates this point.

4.4 THEOREM: Let $a_1, \ldots, a_n$ be proper values of a nonsingular linear transformation on E_n and $u_1, \ldots, u_n$ be the proper values of the unitary part of A such that $\pi > \arg u_1 \geq \ldots \geq \arg u_n > 0$. Then $(\arg a_i) \prec (\arg u_i)$. The proof is given in [6, p.545].

4.5 THEOREM: If $a_1, \ldots, a_n$ and $u_1, \ldots, u_n$ are complex numbers with absolute value one which lie on a line through 0 and the a_i's are the proper values of a nonsingular linear transformation A and the u_i's are the proper values of the unitary part of A, then $(a_1, \ldots, a_n)$ is a rearrangement of $(u_1, \ldots, u_n)$. The proof is given in [6, p.546].

4.6 <u>THEOREM</u>: Suppose $(b_1, \ldots, b_n)$ and $(a_1, \ldots, a_n)$ are n-tuples of complex numbers of absolute value one such that

$$\prod_{i=1}^{n} b_i = \prod_{i=1}^{n} a_i .$$

Then there exist determinations of $(\arg a_i)$ and $(\arg b_i)$ for $i = 1, \ldots, n$ such that

$$\max \arg a_i - \min \arg a_i \leq 2\pi \quad \text{and}$$

$$(\arg b_i) \prec (\arg a_i) .$$

The proof may be found in [6, p. 546].

For linear transformations A and B on E_n, the property that AB and BA have the same nonzero proper values is used in the proof of several theorems in the following chapter.

The next two theorems assert that the algebraic multiplicities of those proper values are the same and the geometric multiplicities are also the same.

4.7 <u>THEOREM</u>: Let A and B be linear transformations on E_n. Let m be a nonzero proper value of AB with algebraic multiplicity k. Then m is also a proper value of BA with the same algebraic multiplicity. The proof is given in [7, p.906].

4.8 <u>THEOREM</u>: Let A and B be linear transformations on E_n. Let m be a nonzero proper value of AB with geometric multiplicity r. Then m as a proper value of BA has the same geometric multiplicity r.

<u>Proof</u>. Let $M_1 = \{ \xi \mid AB\xi = m\xi \}$ and $M_2 = \{ \eta \mid BA\eta = m\eta \}$. It is clear that M_1 and M_2 are subspaces of E_n. Let dim $M_1 = r$ and $\{ \xi_1, \ldots, \xi_r \}$ be a basis in M_1. It is clear that $AB\xi_i = m\xi_i$ $i = 1, \ldots, r$. One observes that $BAB\xi_i = mB\xi_i$, $i = 1, \ldots, r$. Now consider $\{ B\xi_1, \ldots, B\xi_r \}$ and let $x_1B\xi_1 + \ldots + x_rB\xi_r = \vec{0}$. Then $AB(x_1\xi_1 + \ldots + x_r\xi_r) = x_1m\xi_1 + \ldots + x_rm\xi_r = \vec{0}$. Since $m \neq 0$, it follows that $x_1\xi_1 + \ldots + x_r\xi_r = \vec{0}$ which implies $x_i = 0$ for $i = 1, \ldots, r$. Therefore $\{ B\xi_1, \ldots, B\xi_r \}$ is linearly independent. This implies that dim $M_2 \geq$ dim $M_1 = r$.

Similarly starting with m as a proper value of BA one obtains dim $M_1 \geq$ dim M_2 which proves the theorem.

CHAPTER V

PROPER VALUES OF THE SUBUNITARY PART OF A MATRIX

In this chapter relationships between the proper values of a matrix and those of its subunitary part are established by restricting the subunitary part to various invariant subspaces.

Many relationships which have been established between the proper values of a nonsingular matrix and those of its unitary part make use of the orthonormal basis of proper vectors of the unitary part. In general, no such basis exists using the proper vectors of the subunitary part. Consequently other restrictions on S are sometimes required and some results concerning the subunitary part are weaker than the analogous ones for the unitary part.

We begin by considering a linear transformation restricted to its range and then examine a transformation restricted to the subspace generated by its proper vectors.

5.1 <u>THEOREM</u>: Let A be a linear transformation on E_n and let P be the projection on R(A). Then a is a nonzero proper value of A and ξ a corresponding proper vector if and only if a is a proper value of $PAP|R(A)$ with ξ as its corresponding proper vector.

<u>Proof</u>. It is clear that PA = A, and by 4.8 the nonzero proper values of AP = PAP are the same as the ones of PA = A with equal geometric multiplicity.

Now let $A\xi = a\xi$, $a \neq 0$, $\xi \neq \vec{0}$.

Since $\xi \varepsilon R(A)$, it follows that $P\xi = \xi$ and $PAP\xi = PA\xi = A\xi = a\xi$. Thus the theorem is proved.

5.2 THEOREM: Let A be a linear transformation on E_n . Let $a_1, \ldots, a_h$ be distinct nonzero proper values of A. Let E_j be the subspace spanned by the set of proper vectors corresponding to a_j , $j = 1, \ldots, h$. If $N = E_1 \oplus \ldots \oplus E_h$ and E is the projection on N, then $EAE|N$ is nonsingular.

Proof. Let $\xi \varepsilon N$ with $\xi \neq \vec{0}$. We can write $\xi = \xi_1 + \ldots + \xi_h$ where $\xi_j \varepsilon E_j$. Since $\xi \varepsilon N$, $E\xi = \xi$. Then $EAE\xi = EA\xi = E(a_1\xi_1 + \ldots + a_h\xi_h) = a_1\xi_1 + \ldots + a_h\xi_h \neq \vec{0}$ since $a_j \neq 0$ for $j = 1, \ldots, h$ and at least for one $j \varepsilon \{1, \ldots, h\}$ we have $\xi_j \neq \vec{0}$. Therefore $EAE|N$ is nonsingular.

5.3 COROLLARY: Let A and E satisfy the hypotheses of 5.2. Then the proper values and proper vectors of $EAE|N$ are the same as the nonzero ones of A by restricting each proper value of A to its geometric multiplicity.

Theorem 4.6 states relationships between the arguments of sets of complex numbers when certain conditions are true. In order to apply the theorem to various sets of proper values, we use congruences of matrices.

5.4 THEOREM: Let A and B be nonsingular linear transformations on E_n. Suppose that A is congruent to B. If $\{ a_1, \ldots, a_n \}$ and $\{ b_1, \ldots, b_n \}$ are respectively the sets of proper values of A and B, then

$$\prod_{i=1}^{n} a_i / |a_i| = \prod_{i=1}^{n} b_i / |b_i| .$$

Proof. There exists a nonsingular linear transformation X on E_n such that $A = X^*BX$. Thus $\det A = \det(XX^*) \det B$. Therefore

$$\prod_{i=1}^{n} a_i = \det(XX^*) \prod_{i=1}^{n} b_i .$$

Since

$$\left| \prod_{i=1}^{n} a_i \right| = \det(XX^*) \left| \prod_{i=1}^{n} b_i \right|$$

we obtain

$$\prod_{i=1}^{n} a_i / |a_i| = \frac{\det (XX^*) \prod_{i=1}^{n} b_i}{\det (XX^*) \left| \prod_{i=1}^{n} b_i \right|} .$$

Thus

$$\prod_{i=1}^{n} a_i / |a_i| = \prod_{i=1}^{n} b_i / |b_i| .$$

5.5 COROLLARY: Let A and B satisfy the hypotheses of 5.4. Let $u_1, \ldots, u_n$ be proper values of the unitary part of B. Then

$$\prod_{i=1}^{n} a_i / |a_i| = \prod_{i=1}^{n} u_i .$$

5.6 THEOREM: Let A and B satisfy the hypotheses of 5.4. Then there exist determinations of arg a_i and arg b_i , $i = 1, \ldots, n$ such that

$$\max \arg b_i - \min \arg b_i \leq 2\pi$$

and
$$(\arg a_i) \prec (\arg b_i) .$$

Proof. Since $(a_i/|a_i|)$ and $(b_i/|b_i|)$ are n-tuples of complex numbers of absolute value one and

$$\prod_{i=1}^{n} a_i/|a_i| = \prod_{i=1}^{n} b_i/|b_i| ,$$

then by 4.6 there exist determinations of $\arg (a_i/|a_i|)$ and $\arg (b_i/|b_i|)$, $i = 1, \ldots, n$, such that

$$\max \arg (b_i/|b_i|) - \min \arg (b_i/|b_i|) \leq 2\pi$$

and

$$(\arg a_i/|a_i|) \prec (\arg b_i/|b_i|) .$$

But $\arg a_i = \arg (a_i/|a_i|)$ and $\arg b_i = \arg (b_i/|b_i|)$ for $i = 1, \ldots, n$. Thus the conclusion follows.

5.7 COROLLARY: Let A and B satisfy the hypotheses of 5.5. Then there exist determinations of arg a_i and arg u_i for $i = 1, \ldots, n$ such that
$$\max \arg u_i - \min \arg u_i \leq 2\pi$$
and
$$(\arg a_i) \prec (\arg u_i) .$$

If A is a nonsingular matrix with proper values $a_1, \ldots, a_n$, and $u_1, \ldots, u_n$ are the proper values of the unitary part of A, then by Theorems 3.1 and 4.6 there exist determinations of $\arg(a_i/|a_i|)$ and $\arg u_i$ and hence of $\arg a_i$ and $\arg u_i$, $i = 1, \ldots, n$ such that

$$\max \arg u_i - \min \arg u_i \leq 2\pi$$

and

$$(\arg a_i) \prec (\arg u_i) \quad .$$

We generalize this result to the case where $PAP|R(A)$ is nonsingular by establishing relationships between the arguments of the nonzero proper values of A and those of its subunitary part.

5.8 THEOREM: Let A be a linear transformation on E_n and S be the subunitary part of A. If $PAP|R(A)$ is nonsingular and $\{a_1, \ldots, a_k\}$ and $\{s_1, \ldots, s_k\}$ are sets of nonzero proper values of A and S respectively, then there exist determinations of $\arg a_i$ and $\arg s_i$, for $i = 1, \ldots, k$ such that

$$\max \arg s_i - \min \arg s_i \leq 2\pi$$

and

$$(\arg a_i) \prec (\arg s_i) \quad .$$

Proof. Let $A = \sqrt{AA^*}\, S$ be the subpolar decomposition of A. It follows that $PAP = P\sqrt{AA^*}\, SP = (P\sqrt{AA^*}P)(PSP)$

$$= \left(\sqrt{P\sqrt{AA^*}\,P}\right)\left(\sqrt{P\sqrt{AA^*}\,P}\right)(PSP).$$

Now let $B = \sqrt{P\sqrt{AA^*}\,P}\;(PSP)\sqrt{P\sqrt{AA^*}\,P}$.

We note that PBP = B and B has the same nonzero proper values as A which are the same as those of PAP. We also note that B is congruent to PSP.

Since PAP is nonsingular on R(A), it follows that PSP is also nonsingular on R(A) and we may write

$$PSP|R(A) = \sqrt{PSPS^*P}\; V$$

where V is the unitary part of PSP defined on R(A).

If $v_1, \ldots, v_k$ are the proper values of V, then it follows that

$$\prod_{i=1}^{k} s_i/|s_i| = \prod_{i=1}^{k} v_i \; .$$

But since B|R(A) is congruent to PSP|R(A) by 5.5 we have that

$$\prod_{i=1}^{k} a_i/|a_i| = \prod_{i=1}^{k} v_i \; .$$

Therefore,

$$\prod_{i=1}^{k} a_i/|a_i| = \prod_{i=1}^{k} s_i/|s_i|$$

and hence $(a_1/|a_1|, \ldots, a_k/|a_k|)$ and $(s_1/|s_1|, \ldots, s_k/|s_k|)$ satisfy the hypothesis of 5.6.

Thus there exists a determination of $\arg a_i$ and $\arg s_i$ for $i = 1, \ldots, k$ such that

$$\max \arg s_i - \min \arg s_i \leq 2\pi$$

and

$$(\arg a_i) \prec (\arg s_i) ,$$

and the theorem is proved.

If A is nonsingular and the arguments of the proper values of its unitary part meet certain restrictions, then conclusions are drawn in Theorem 4.4 about the sums of arguments of the proper values of A and those of its unitary part.

By requiring that $\pi > \arg(PSP|R(A)\ \xi, \xi) > 0$ for nonzero $\xi \in R(A)$, we establish an equality between the sums of arguments of nonzero proper values of A and S, where S is the subunitary part of A.

5.9 THEOREM: Suppose that A is a linear transformation on E_n and S is the subunitary part of A. If $PAP|R(A)$ is nonsingular and $\pi > \arg(PSP|R(A)\ \xi,\xi) > 0$ for all nonzero $\xi \in R(A)$, then

$$\sum_{i=1}^{k} \arg a_i = \sum_{i=1}^{k} \arg s_i$$

where $\{ a_1, \ldots, a_k \}$ and $\{ s_1, \ldots, s_k \}$ are sets of nonzero proper values of A and S respectively.

Proof. By 4.3, $PSP|R(A)$ is congruent to a unitary transformation U on $R(A)$. Thus there exists a nonsingular linear transformation X on $R(A)$ such that $(PSP|R(A)) = X^*UX$. It is clear that $PAP = (PKP)(PSP)$. Now let $B = (\sqrt{PKP}\,(PSP)\sqrt{PKP})|R(A)$. Then $B|R(A)$ is congruent to $PSP|R(A)$ and $B|R(A)$ has the same proper values as $PAP|R(A)$. Now B is congruent to $PSP|R(A)$ which is congruent to U and hence B is congruent to U. Therefore there exists a nonsingular linear transformation Y on $R(A)$ such that $B = Y^*UY$.

Since $\pi > \arg(U\xi,\xi) > 0$ for all nonzero ξ in $R(A)$, it follows that the proper values $u_1, \ldots, u_k$ of U satisfy the inequality $\pi > \arg u_1 \geq \ldots \geq \arg u_k > 0$.

Note that $C = (XX^*)U$ has the same proper values as $PSP|R(A)$ and U is the unitary part of C.

By 4.4 we have

$$\sum_{i=1}^{k} \arg s_i = \sum_{i=1}^{k} \arg u_i \quad .$$

Similarly the transformation $D = YY^*U$ has the same nonzero proper values as B which has the same nonzero proper values as A, and U is the unitary part of D.

Thus

$$\sum_{i=1}^{k} \arg a_i = \sum_{i=1}^{k} \arg u_i \quad .$$

Therefore

$$\sum_{i=1}^{k} \arg a_i = \sum_{i=1}^{k} \arg s_i \quad .$$

We again generalize results which have been proved for a nonsingular matrix A and its unitary part by considering the case where A is singular and establishing relationships between proper values of its subunitary part and those of A restricted to a subspace.

5.10 THEOREM: Let A be a linear transformation on E_n and A = KS be the subpolar decomposition of A. Let $s_1, \ldots, s_p$ be distinct nonzero proper values of S, the subunitary part of A. Let $N_i = \{ \eta \mid S\eta = s_i\eta, \ \eta \neq \vec{0} \}$, and $N = [N_1] \oplus \ldots \oplus [N_p]$. If E is the projection on N and $d_1, \ldots, d_h$ are distinct proper values of $EAE|N$, then

$$\prod_{\lambda=1}^{h} \frac{d_\lambda^{m_\lambda}}{|d_\lambda^{m_\lambda}|} = \prod_{\lambda=1}^{p} \frac{s_\lambda^{n_\lambda}}{|s_\lambda^{n_\lambda}|}$$

where m_i and n_i are the respective geometric multiplicities of the proper values.

Proof. Note that EAE = EKSE = (EKE)(ESE). Since $K = \sqrt{AA^*}$, $K^2 = AA^*$ and since R(S) = R(A) and N(S) = N(A) it follows that $R(S) \cap N(A^*) = 0$ and $N(S) \cap R(A^*) = 0$. Thus $K^2|N$ is nonsingular and hence $K|N$ is nonsingular.

Now for $\xi \in N$, $\xi \neq \vec{0}$, we have

$(EKE\xi,\xi) = (K\xi,\xi) = (\sqrt{K}\,\xi,\sqrt{K}\,\xi) = ||\sqrt{K}\,\xi||^2$. Since $K\xi \neq \vec{0}$ for $\xi \neq \vec{0}$, it follows that $\sqrt{K}\,\xi \neq \vec{0}$ and hence $(EKE\xi,\xi) \neq 0$. Consequently $EKE\xi \neq \vec{0}$ for $\xi \in N$, $\xi \neq \vec{0}$. Hence $EKE|N$ is nonsingular. But $ESE|N$ is also nonsingular by 5.2. Thus we have that $EAE|N$ is nonsingular.

Therefore $\det(EAE|N) \neq 0$ and $\det(EAE|N)$ equals the product of $\det(EKE|N)$ and $\det(ESE|N)$.

So
$$\prod_{\lambda=1}^{h} d_\lambda^{m_\lambda} = \det(EKE|N)\prod_{\lambda=1}^{p} s_\lambda^{n_\lambda} .$$

Hence
$$\prod_{\lambda=1}^{h} \frac{d_\lambda^{m_\lambda}}{|d_\lambda^{m_\lambda}|} = \frac{\det(EKE|N)}{\det(EKE|N)}\prod_{\lambda=1}^{p} \frac{s_\lambda^{n_\lambda}}{|s_\lambda^{n_\lambda}|}$$

and thus
$$\prod_{\lambda=1}^{h} \frac{d_\lambda^{m_\lambda}}{|d_\lambda^{m_\lambda}|} = \prod_{\lambda=1}^{p} \frac{s_\lambda^{n_\lambda}}{|s_\lambda^{n_\lambda}|} .$$

5.11 <u>COROLLARY</u>: Consider the hypothesis of 5.10. Then there exist determinations or arg d_i and arg s_i such that $\max \arg s_i - \min \arg s_i \leq 2\pi$ and $(\arg d_i) \prec (\arg s_i)$.

In the introduction it was pointed out that Horn and Steinberg place conditions on two sets of numbers which allow them to conclude that one set is the set of proper values of a matrix and the other set coincides with the proper values of the unitary part of the matrix. The theorem is 3.2. In [5] Halmos and McLaughlin construct a subunitary matrix with a prescribed set of proper values.

We combine the two ideas and give conditions in the next theorem which are sufficient to establish the existence of a matrix with predetermined nonzero proper values $a_1, \ldots, a_k$ whose subunitary part has predetermined nonzero proper values $s_1, \ldots, s_k$.

5.12 THEOREM: Let $a_1, \ldots, a_k$ be nonzero complex numbers and $s_1, \ldots, s_k$ be nonzero complex numbers with $|s_i| \leq 1$, for $i = 1, \ldots, k$ such that

$$\prod_{i=1}^{k} (s_i/|s_i|) = \prod_{i=1}^{k} (a_i/|a_i|) ,$$

and exactly one of the following holds:

(a) There is a line through 0 containing the $(s_i/|s_i|)$'s and $(a_1/|a_1|, \ldots, a_k/|a_k|)$ is a rearrangement of $(s_1/|s_1|, \ldots, s_k/|s_k|)$.

(b) There is no line through 0 containing all of the $(s_i/|s_i|)$' s , but there is a closed half plane H with 0 on its boundary containing all of the $(s_i/|s_i|)$' s , and, if we choose a branch of the argument function which is continuous in $H - \{0\}$, then $(\arg a_i) \prec (\arg s_i)$.

(c) There is no closed half plane with 0 on its boundary which contains all of the $(s_i/|s_i|)$' s.

Then there exists a matrix A whose nonzero proper values are $a_1, \ldots, a_k$ and $s_1, \ldots, s_k$ are nonzero proper values of S, the subunitary part of A.

<u>Proof.</u> By 3.2 and 4.1 there exists a k by k matrix B with proper values $a_1, \ldots, a_k$ which is congruent to T where

$$T = \begin{pmatrix} s_1/|s_1| & & 0 \\ & \ddots & \\ 0 & & s_k/|s_k| \end{pmatrix} .$$

Thus there exists a nonsingular k by k matrix X such that $B = X^*TX$. Let

$$T_1 = \begin{pmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_k \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} \sqrt{|s_1|} & & 0 \\ & \ddots & \\ 0 & & \sqrt{|s_k|} \end{pmatrix},$$

then $T_1 = KTK$.

Thus $B = X^*K^{-1}T_1K^{-1}X$. One observes that the matrix $C = (K^{-1}X)(X^*K^{-1})\ T_1$ has the same proper values as B, and thus the proper values of C are $a_1, \ldots, a_k$. It is clear that $(K^{-1}X)(X^*K^{-1})$ is a positive definite matrix.

Now consider the 2k by 2k matrix S such that

$$S = \begin{pmatrix} s_1 & & 0 & t_1 & & 0 \\ & \ddots & & & \ddots & \\ 0 & & s_k & 0 & & t_k \\ 0 & & 0 & 0 & & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & & 0 & 0 & & 0 \end{pmatrix}$$

where $\begin{pmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_k \end{pmatrix}$ and $\begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_k \end{pmatrix}$

are in diagonal form and $|s_i|^2 + |t_i|^2 = 1$ for $i = 1, \ldots, k$.

It is clear that S is subunitary.

We shall define the 2k by 2k matrix H by

$$H = \begin{pmatrix} (K^{-1}XX^*K^{-1}) & (0) \\ (0) & (0) \end{pmatrix} .$$

Then $A = HS$ has nonzero proper values $a_1, \ldots, a_k$ and $s_1, \ldots, s_k$ are nonzero proper values of S, the subunitary part of A.

CHAPTER VI

SOME PROPERTIES OF SUBUNITARY MATRICES

In this chapter some additional properties of subunitary matrices are established. We begin by stating a theorem which is proved in [4].

6.1 <u>THEOREM</u>: Let S be a subunitary transformation on E_n. Let a be a proper value of S and $S\xi = a\xi$, $\xi \neq \vec{0}$. Then

(1) $a = 0$ if and only if $\xi \in N(S)$,

(2) $|a| = 1$ if and only if $\xi \in [N(S)]^{\perp}$

(3) $0 < |a| < 1$ if and only if $\xi \notin N(S) \cup [N(S)]^{\perp}$.

The proper vectors of subunitary transformations need not be orthogonal. The next theorem states some conditions which do imply orthogonality.

6.2 <u>THEOREM</u>: Let s_1 and s_2 be two distinct proper values of a subunitary transformation S on E_n. Suppose that $|s_1| = 1$ and that ξ_1 and ξ_2 are proper vectors of S corresponding to s_1 and s_2 respectively. Then $\{ \xi_1, \xi_2 \}$ is an orthogonal set.

<u>Proof</u>. If $s_2 = 0$, then by 6.1 the conclusion is obvious. Since $|s_1| = 1$, we have $\xi_1 \in R(S^*)$. Let $\xi_2 = \eta_2 + \zeta_2$ such that $\eta_2 \in R(S^*)$ and $\zeta_2 \in N(S)$.

Then

$$(\xi_1,\xi_2) = (\xi_1,\eta_2 + \zeta_2) = (\xi_1,\eta_2)$$
$$= (S\xi_1, S\eta_2)$$
$$= (S\xi_1, S\xi_2)$$
$$= s_1\overline{s}_2(\xi_1,\xi_2) \quad .$$

Now if $|s_2| = 1$, then $\overline{s}_2 = 1/s_2 \neq \overline{s}_1$. Thus $s_1\overline{s}_2 \neq 1$ which implies that $(\xi_1\ \xi_2) = 0$. Hence the set $\{ \xi_1, \xi_2 \}$ is orthogonal and the theorem is proved.

In establishing minimax principles for unitary [6] and normal [3] matrices, orthonormal bases of proper vectors were used. By the preceding theorem we observe that certain sets of proper vectors of a subunitary transformation may be orthogonal. Thus very limited minimax principles for a subunitary transformation restricted to a subspace of these special proper vectors could be given.

6.3 THEOREM: Let A be a linear transformation on E_n with a polar form $A = U\sqrt{A^*A}$. Let P and Q be respectively the projections on R(A) and R(A*). If α is a proper vector of U corresponding to the proper value u, then $|| P\alpha || = || Q\alpha ||$.

<u>Proof</u>. If A is nonsingular, then the theorem is trivial.

Suppose that A is singular. The subpolar form of A is $A = S\sqrt{A^*A} = (PU)\sqrt{A^*A}$. Since $SS^* = P$ and $S^*S = Q$, one observes that

$$\begin{aligned} ||\, Q\alpha \,||^2 = (Q\alpha, Q\alpha) = (Q\alpha, \alpha) &= (S^*S\alpha, \alpha) \\ &= (S\alpha, S\alpha) \\ &= (PU\alpha, PU\alpha) \\ &= u\bar{u}\,(P\alpha, P\alpha) \\ &= ||\, P\alpha \,||^2 . \end{aligned}$$

Consequently $||\, Q\alpha \,|| = ||\, P\alpha \,||$.

6.4 <u>COROLLARY</u>: If A satisfies the hypothesis of 6.3, the $||\, S\alpha \,|| = ||\, S^*\alpha \,||$.

<u>Proof</u>. In the process of proving 6.3, we actually proved that $||\, Q\alpha \,|| = ||\, S\alpha \,||$ and $||\, P\alpha \,|| = ||\, S^*\alpha \,||$. Thus we have that $||\, S\alpha \,|| = ||\, S^*\alpha \,||$.

Several theorems in this paper have required that the arguments of the proper values of a linear transformation be between zero and pi. For a unitary transformation, we show that the above restriction is equivalent to the condition $\pi > \arg (U\xi, \xi) > 0$ for all nonzero ξ.

6.5 <u>THEOREM</u>: Let U be a unitary transformation on E_n with proper values $u_1, \dots, u_n$. Let ξ be a nonzero vector in E_n. Then $\Pi > \arg(U\xi,\xi) > 0$ if and only if $\Pi > \arg u_i > 0$ for $i = 1, \dots, n$.

<u>Proof</u>. Let $\{\xi_1, \dots, \xi_n\}$ be the orthonormal set of proper vectors of U such that $U\xi_i = u_i \xi_i$ for $i = 1, \dots, n$. If $\Pi > \arg(U\xi,\xi) > 0$ for all nonzero $\xi \in E_n$, then we have that $\Pi > \arg(U\xi_i, \xi_i) = \arg u_i \|\xi_i\|^2 = \arg u_i > 0$ for $i = 1, \dots, n$.

Conversely if $\Pi > \arg u_i > 0$ for $i = 1, \dots, n$, it is clear that for any nonzero vector $\xi \in E_n$ we have $\xi = (\xi,\xi_1)\xi_1 + \dots + (\xi,\xi_n)\xi_n$ where at least one of the coefficients is different from zero.

Thus $(U\xi,\xi) = u_1 |(\xi,\xi_1)|^2 + \dots + u_n|(\xi,\xi_n)|^2$.

This implies that $\arg(U\xi,\xi) = \arg(u_1|(\xi,\xi_1)|^2 + \dots + u_n|(\xi,\xi_n)|^2)$.

Since

$$\begin{aligned} \Pi > \max \arg u_i &= \max \arg u_i|(\xi,\xi_i)|^2 \\ &\geq \arg(u_1|(\xi,\xi_1)|^2 + \dots + u_n|(\xi,\xi_n)|^2) . \\ &\geq \min \arg u_i|(\xi,\xi_i)|^2 \\ &= \min \arg u_i > 0 , \end{aligned}$$

we have that $\Pi > \arg(U\xi,\xi) > 0$.

6.6 <u>COROLLARY</u>: Let S be a subunitary transformation on E_n. Then $S = PU$ for some unitary transformation on E_n with P the projection on R(S). Suppose that $u_1, \ldots, u_n$, the proper values of U, satisfy $\pi > \arg u_1 \geq \ldots \geq \arg u_n > 0$ and $s_1, \ldots, s_k$ are nonzero proper values of S. Then $\pi > \arg s_i > 0$ for $i = 1, \ldots, k$.

<u>Proof</u>. Observe that for $|\xi_i| = 1$ and $S\xi_i = s_i\xi_i$ we have

$$s_i = (S\xi_i, \xi_i) = (PU\xi_i, \xi_i) = (U\xi_i, \xi_i) .$$

The proof then follows by 6.5 .

A converse to Theorem 6.6 is desirable; however, at the present time we draw no conclusions about the arguments of $u_1, \ldots, u_n$ if we place restrictions on the arguments of $s_1, \ldots, s_k$.

If S is a subunitary transformation on E_n, then there exists a subunitary transformation T such that $S + T$ is unitary. We call T a supplement of S.

Additional theorems are being pursued by using techniques of extending the subunitary part of a linear transformation and studying the proper values of S , T , and $S + T$.

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