COEFFICIENT BOUNDS FOR CERTAIN UNIVALENT FUNCTIONS

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TABLE OF CONTENTS

INTRODUCTION

In a paper published in 1916 dealing with the coefficients of power series, L. Bieberbach [3], in considering the class of functions

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n$$

with $|z|<1$, conjectured that the coefficients of the power series satisfy $|a_n| \leq n$. Bieberbach restricted this class of functions to univalent functions, that is one-to-one mappings, and proved that $|a_2| \leq 2$.

To this date, no one has proved or disproved the conjecture, but several special classes of functions have been shown to satisfy the conjecture. In 1921, Nevalinna [10] proved that the Bieberbach conjecture is true for functions that map the unit disk onto a star-shaped domain. Löwner [8] proved that $|a_3| \leq 3$ in 1923, but a proof for $n = 4$ did not follow until 1955 when Garabedian and Schiffer published an article in the Annals of Mathematics [4]. Rogosinski [12] proved that the conjecture holds for functions whose power series have real coefficients. In 1925, J. E. Littlewood [7] proved that $|a_n| < en$, and this result was refined by Bazilevic [1] in 1951 to be $|a_n| < \frac{en}{2} + 1.51$. W. K. Hayman [5] has recently demonstrated that the coefficients of a normalized univalent function satisfy the limit

$$\lim_{n\to\infty} \frac{a_n}{n} = \alpha < 1,$$

and thus, for $n > N$, $|a_n| \leq n$.

In 1955, M. O. Reade [11] proved that the Bieberbach conjecture is satisfied by a class of functions, termed close-to-convex functions, which had been defined by Kaplan [6]. This paper will discuss such functions and an analagous class, close-to-star functions.

CHAPTER 0

PRELIMINARIES

The following definitions will be used throughout the remainder of this paper.

0.1 Definition. A domain is an open, arc-wise connected set.

0.2 Definition. Let $f(z)$ be a single-valued continuous function of z in a domain D. $f(z)$ is differentiable at the point z_0 of D if

$$\lim_{h \to 0} \frac{1}{h}\left[f(z_0+h)-f(z_0)\right]$$

exists as a finite number and is independent of how the complex increment h tends to zero. The limit, when it exists, will be denoted by $f'(z_0)$ and called the derivative of $f(z)$ at z_0.

0.3 Definition. $f(z)$ is holomorphic at $z=z_0$ if and only if it is differentiable at every point in some neighborhood of z_0. $f(z)$ is holomorphic in a domain D if and only if $f(z)$ is holomorphic at every point of D.

0.4 Definition. A disk of radius r and center at z_0 is a domain defined by the equation $|z-z_0|<r$ and denoted by $K_r(z_0)$.

0.5 Definition. A circle of radius r with center at z_0 is the boundary of $K_r(z_0)$ defined by the equation $|z-z_0|=r$ and denoted by $C_r(z_0)$.

0.6 Definition. C is a continuous arc if and only if C is the set of all ordered pairs (x,y) such that $x=f(t)$, and $y=g(t)$, where $a \leq t \leq b$, and $f(t)$ and $g(t)$ are continuous.

0.7 Definition. C is a contour if and only if C is a continuous arc, $f(t)$ and $g(t)$ have sectionally continuous first derivatives, and $f(t)$ and $g(t)$ are not zero for the same t.

0.8 Definition. C is a simple arc if and only if

(1) C is a continuous arc, and

(2) if $t_1 \neq t_2$, then $\left[f(t_1),g(t_1)\right] \neq \left[f(t_2),g(t_2)\right]$.

0.9 Definition. Let z_0 be a point in the plane. $z=z_0+\lambda s$, where $0 \leq s < \infty$ and λ is a complex constant, is a ray from z_0 with direction λ.

0.10 Definition. Let $z=f(t), a \leq t \leq b$, represent a contour C. Let $z_0=f(t_0)$ be a fixed point on C and consider $z_h=f(z_0+h), h>0$. Let $\arg(z_0-z_h)=\theta_h$, and

$$e^{i\theta_h} = \frac{z_0-z_h}{|z_0-z_h|} .$$

If

$$\lim_{h\to 0} e^{i\theta_h} = e^{\alpha i} = \frac{f'(z_0)}{|f'(z_0)|},$$

exists, then $z=z_0+e^{i\alpha} s$, $0 \leq s < \infty$, is the tangent ray of contour C at z_0.

0.11 Definition. A function $f(z)$ defined in a domain D is conformal at z_0 in D if and only if for every pair of simple contours intersecting at z_0 with angle of intersection θ, the image curves intersect at $f(z_0)$ with angle θ. $f(z)$ is conformal in D if and only if $f(z)$ is conformal at each z in D.

0.12 Definition. $f(z)$ is normalized if and only if $f(0)=0$ and $f'(0)=1$.

Throughout this paper, all mappings will be conformal and holomorphic.

CHAPTER 1

CONVEX FUNCTIONS AND STAR-LIKE FUNCTIONS

Before examining the coefficient bounds for certain types of functions, it is necessary to develop some properties and relationships of convex and star-like mappings.

1.1 Definition. *A domain D is convex if and only if for each pair of points z_1 and z_2 in D, the set of all points z such that $z=z_1+\alpha(z_2-z_1)$, where $0\le\alpha\le 1$, are in D.*

Geometrically, D is convex if and only if any two points in D may be joined by a straight line lying completely in D.

1.2 Definition. *A function f(z) is convex if and only if f(z) maps $K_1(0)$ onto a convex domain.*

Let $z=re^{i\theta}$ with fixed $r<1$ describe $C_r(0)$ as θ increases continuously from 0 to 2π. Since f(z) is conformal, $f(re^{i\theta})$ will describe the image of $C_r(0)$ as arg $f(re^{i\theta})$ increases continuously. Furthermore, it has been shown [9] that if f(z) is convex, then $f(re^{i\theta})$ bounds a convex domain. Under these conditions, consider the following theorem.

1.3 Theorem. *f(z) is convex if and only if for* $|z|=r<1$,

$$\frac{\partial}{\partial\theta}\left[\arg izf'(z)\right] \geq 0$$

<u>when the partial derivative exists</u>.

Proof. If $f(z)$ is convex, the included angle between any two tangent rays to $f(re^{i\theta})$ is less than or equal to π. Thus, for fixed $z_1=re^{i\theta_1}$ and any point $z_2=re^{i\theta_2}$, where $\theta_2 > \theta_1$, the included angle between the tangent rays to $f(re^{i\theta_1})$ and $f(re^{i\theta_2})$ is

$$\pi + \arg iz_1f'(z_1) - \arg iz_2f'(z_2) .$$

Since $f(z)$ is convex,

$$\pi + \arg iz_1f'(z_1) - \arg iz_2f'(z_2) \leq \pi,$$

and

$$\arg iz_2f'(z_2) - \arg iz_1f'(z_1) \geq 0;$$

hence,

$$\lim_{\theta_2\to\theta_1} \frac{\arg iz_2f'(z_2) - \arg iz_1f'(z_1)}{\theta_2 - \theta_1} \geq 0.$$

Therefore, if the limit exists, for $C_r(0)$

$$\frac{\partial}{\partial\theta} \arg izf'(z) \geq 0.$$

Assume $f(z)$ is not convex. Then for some fixed $r < 1$ and $0 \leq \theta < 2\pi$,

$$\pi + \arg iz_1f'(z_1) - \arg iz_2f'(z_2) > \pi ,$$

where $z_1=re^{i\theta_1}$ and $z_2=re^{i\theta_2}$. Thus,

$$\arg iz_2f'(z_2) - \arg iz_1f'(z_1) < 0,$$

and

$$\lim_{\theta_2\to\theta_1} \frac{\arg iz_2f'(z_2) - \arg iz_1f'(z_1)}{\theta_2 - \theta_1} \leq 0.$$

Therefore, if the limit exists, for $C_r(0)$

$$\frac{\partial}{\partial\theta} \arg\left[izf'(z)\right] \leq 0.$$

If

$$\frac{\partial}{\partial\theta} \arg\left[izf'(z)\right] = 0,$$

then the arguments of the tangent rays at the points in a neighborhood of $f(z_1)$ are equal. Thus, the boundary at the point $f(z_1)$ is a straight line segment. This is contrary to the assumption; hence

$$\frac{\partial}{\partial\theta} \arg izf'(z) < 0.$$

But this is a contradiction, and $f(z)$ is convex.

If the limit in the preceding argument does not exist, $\arg\left[izf'(z)\right]$ is not continuous and $f(re^{i\theta})$ is not a smooth curve. Thus, at some point z_n, if the limit does not exist, $f(z_n)$ is a corner of a curvilinear polygon. If there are a finite number of points z_n for which $f(z_n)$ is a corner, the interior angle at each point may be measured directly. If there are infinitely many points z_n for which $f(z_n)$ is a corner, then there must be a limit point of these corners since the set of z_n's is a bounded infinite set. Since $f(z)$ is convex, each interior angle α_n at the point $f(z_n)$ must be less than or equal to π. From a theorem of plane geometry, the sum of the exterior angles $\pi - \alpha_n$ of a closed polygon is 2π. Hence,

$$\sum_{n=1}^{\infty} (\pi - \alpha_n) = 2\pi.$$

The infinite series converges; thus,

$$\lim_{n\to\infty} (\pi - \alpha_n) = 0,$$

and for every positive ϵ, there is an N such that for $n > N$, $|\pi - \alpha_n| < \epsilon$. Since f(z) is convex, $\pi - \alpha_n > 0$, and $0 < \pi - \alpha_n < \epsilon$. But this implies that $\pi - \epsilon < \alpha_n < \pi$; hence, $\alpha_n \leq \pi$ for all n.

1.4 Definition. A domain D is star-like with respect to the origin if and only if for any point z_1 in D, all the points αz_1, where $0 \leq \alpha \leq 1$, are in D.

In contrast to a convex domain, a star-like domain is one in which every point in the domain lies on a ray from the origin, and the line segment lies in the domain. Certainly every convex domain is star-like with respect to every point in the domain, but not conversely.

1.5 Definition. A function f(z) is a star-like function or star map if and only if f(z) maps $K_1(0)$ onto a domain which is star-like with respect to the origin.

Let $z = re^{i\theta}$ with fixed $r < 1$ trace $C_r(0)$ as θ increases continuously from 0 to 2π. Since f(z) is conformal, $f(re^{i\theta})$ must describe the image of $C_r(0)$ as $\arg f(re^{i\theta})$ increases continuously. It has been shown [9] that if f(z) is star-like, then $f(re^{i\theta})$ bounds a star-like domain. Now consider the following theorem.

1.6 Theorem. f(z) is a star map if and only if,

$$\frac{\partial}{\partial \theta} \arg f(z) \geq 0.$$

Proof. For $\theta_1 < \theta_2$, let $\arg f(re^{i\theta_1}) = \phi_1$ and $\arg f(re^{i\theta_2}) = \phi_2$. If $\phi_2 - \phi_1 < 0$, there are two points on $f(re^{i\theta})$ that lie on the same ray from the origin, since ϕ is continuous. Thus, there is one point on $f(re^{i\theta})$ which cannot be joined to the origin by a straight line lying completely in the domain, and $f(z)$ is not star-like. Therefore, $\phi_2 - \phi_1 \geq 0$.

Thus,

$$\frac{\arg f(re^{i\theta_2}) - \arg f(re^{i\theta_1})}{\theta_2 - \theta_1} \geq 0,$$

and

$$\lim_{\theta_2 \to \theta_1} \frac{\arg f(re^{i\theta_2}) - \arg f(re^{i\theta_1})}{\theta_2 - \theta_1} \geq 0.$$

Hence, if the limit exists, for $C_r(0)$

$$\frac{\partial}{\partial \theta} \arg f(re^{i\theta}) \geq 0.$$

Now, if $f(re^{i\theta}) = u(r,\theta)+iv(r,\theta)$,

$$\frac{\partial}{\partial \theta} \arg f(re^{i\theta}) = \frac{\partial}{\partial \theta} \arctan \frac{v}{u}$$

$$= \frac{1}{1+\frac{v^2}{u^2}} \frac{\partial}{\partial \theta} \left(\frac{v}{u}\right)$$

$$= \frac{u\frac{\partial v}{\partial \theta} - v\frac{\partial u}{\partial \theta}}{u^2 + v^2} .$$

Since $f(z)$ is holomorphic; $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, and $\frac{\partial v}{\partial y}$ exist.

Let $x = r\cos\theta$ and $y = r\sin\theta$; therefore,

$$\frac{\partial u}{\partial\theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial\theta}$$

and

$$\frac{\partial v}{\partial\theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial\theta}$$

exist. Also, $u^2+v^2=r^2\neq 0$;

therefore,

$$\frac{\partial}{\partial\theta}\arg f(re^{i\theta}) \text{ exists.}$$

Assume $f(z)$ is not star-like. Then for some $\phi = \arg f(re^{i\theta})$ the ray from the origin intersects the contour $f(re^{i\theta})$ in at least two distinct points. Therefore, for some $\theta_1 < \theta_2$,

$$\phi_1 = \arg f(re^{i\theta_1}) = \arg f(re^{i\theta_2}) = \phi_2.$$

Thus, by Rolle's theorem, there must be a ϕ_3 such that for $\phi_1 < \phi_3 < \phi_2$,

$$\frac{\partial}{\partial\theta}\arg f(re^{i\theta_3}) = 0.$$

This implies that $\arg f(re^{i\theta})$ has a maximum, minimum, or point of inflection between $\arg f(re^{i\theta_1})$ and $\arg f(re^{i\theta_2})$. If $\phi_3 = \arg f(re^{i\theta_3})$ is a maximum, then ϕ decreases from ϕ_3 to ϕ_2. If ϕ_3 is a minimum, then ϕ decreases from ϕ_1 to ϕ_3. If ϕ_3 is a point of inflection, then decreases from either ϕ_1 to ϕ_3 or ϕ_3 to ϕ_2. Thus ϕ is not a monotonic increasing function of θ , and

$$\frac{\partial}{\partial\theta}\arg f(re^{i\theta}) \not> 0.$$

But this is a contradiction, and $f(z)$ is star-like.

1.7 Theorem. A univalent function $f(z)$ maps $K_r(0)$ onto a star-shaped domain if and only if, for $K_1(0)$,

$$\operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] \geq 0.$$

Proof. From theorem 1.6, since $f(z)$ is star-like,

$$\frac{\partial}{\partial\theta} \arg f(z) \geq 0.$$

Now

$$\arg f(z) = \operatorname{Im}\left[\log f(z)\right],$$

and

$$\begin{aligned}
\frac{\partial}{\partial\theta} \arg f(z) &= \frac{\partial}{\partial\theta}\left[\operatorname{Im} \log f(z)\right] \\
&= \operatorname{Im}\left[\frac{\partial}{\partial\theta} \arg f(z)\right] \\
&= \operatorname{Im}\left[\frac{izf'(z)}{f(z)}\right] \\
&= \operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right];
\end{aligned}$$

therefore,

$$\operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] \geq 0.$$

All the statements are necessary and sufficient and the theorem is proved.

1.8 Theorem. $f(z)$ is convex for $|z|<1$ if and only if $zf'(z)$ is star-like for $|z|<1$.

Proof. By theorem 1.3, $f(z)$ is convex if and only if

$$\frac{\partial}{\partial\theta} \arg\left[izf'(z)\right] \geq 0.$$

Now

$$\frac{\partial}{\partial\theta} \arg izf'(z) = \frac{\partial}{\partial\theta}\left\{\arg i + \arg\left[zf'(z)\right]\right\}$$

$$= \frac{\partial}{\partial\theta}\left\{\frac{\pi}{2} + \arg\left[zf'(z)\right]\right\}$$

$$= \frac{\partial}{\partial\theta} \arg\left[zf'(z)\right] .$$

Consequently,

$$\frac{\partial}{\partial\theta} \arg zf'(z) = \frac{\partial}{\partial\theta} \arg\left[izf'(z)\right] \geq 0. .$$

From theorem 1.6,

$$\frac{\partial}{\partial\theta} \arg F(z) \geq 0$$

if and only if $F(z)$ is a star map. It follows that $F(z) = zf'(z)$ is a star map. The argument is reversible, and the converse follows.

1.9 Theorem. *f(z) is convex for $|z| < 1$ if and only if*

$$1 + \mathrm{Re}\left[\frac{zf''(z)}{f'(z)}\right] \geq 0.$$

Proof. By theorem 1.8, $F(z) = zf'(z)$ is a star-map. Thus, for $K_1(0)$,

$$\mathrm{Re}\left[\frac{zF'(z)}{F(z)}\right] \geq 0.$$

Now

$$\frac{zF'(z)}{F(z)} = \frac{z\ f'(z) + f''(z)}{zf'(z)}$$

$$= 1 + \frac{zf''(z)}{f'(z)} .$$

Therefore,

$$1 + \operatorname{Re}\left[\frac{zf''(z)}{zf'(z)}\right] = \operatorname{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right]$$

$$= \operatorname{Re}\left[\frac{zF'(z)}{F(z)}\right]$$

$$\geq 0.$$

Again, all statements are both necessary and sufficient, so the converse follows immediately.

CHAPTER II

COEFFICIENT BOUNDS AND CLOSE-TO-CONVEX FUNCTIONS

In this chapter, the coefficient bounds for convex and star-like functions will be developed. A new class of functions, which satisfies the Bieberbach conjecture, will be defined and some of its properties investigated.

2.1 <u>Theorem</u>. *If* (i) $g(z)$ *is holomorphic for* $K_1(0)$,

(ii) $g(0) = 1$,

(iii) $\operatorname{Re} g(z) > 0$ for $K_1(0)$,

and (iv) $$g(z) = \sum_{n=0}^{\infty} c_n z^n,$$

then $|c_n| \leq 2$ *for all* n.

Proof. If $F(z) = U(z)+iV(z)$ is holomorphic in $K_1(0)$, then $F(z)$ is continuous for $C_R(0)$, and

$$F(z) = \sum_{n=0}^{\infty} c_n z^n,$$

where

$$c_n = \frac{1}{n!} F^{(n)}(0).$$

By Cauchy's integral representation,

$$F^{(n)}(0) = \frac{n!}{2\pi i} \int_C \frac{F(t)}{t^{n+1}}\, dt.$$

If C is $C_R(0)$, where R is fixed and $t=Re^{i\phi}$, then

$$c_n = \frac{1}{2\pi i}\int_0^{2\pi} \frac{F(Re^{i\phi})}{(Re^{i\phi})^{n+1}} \frac{dt}{d\phi}\, d\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{F(Re^{i\phi})}{R^n}\, e^{-in\phi}\, d\phi\ .$$

Thus,

$$c_n R^n = \frac{1}{2\pi}\int_0^{2\pi} F(Re^{i\phi})e^{-in\phi}\, d\phi\ .$$

Since $F(Re^{i\phi})$ is holomorphic,

$$\Phi = \frac{1}{2\pi}\int_0^{2\pi} F(Re^{i\phi})e^{in\phi}\, d\phi = 0.$$

Hence,

$$c_n R^n + \overline{\Phi} = \frac{1}{2\pi}\int_0^{2\pi} \left[F(Re^{i\phi}) + \bar{F}(Re^{i\phi})\right] e^{-in\phi}\, d\phi\ .$$

But $\overline{\Phi} = 0$; thus,

$$c_n R^n = \frac{1}{2\pi}\int_0^{2\pi} \left[F(Re^{i\phi}) + \bar{F}(Re^{i\phi})\right] e^{-in\phi}\, d\phi$$

$$= \frac{1}{\pi}\int_0^{2\pi} U(Re^{i\phi})e^{-in\phi}\, d\phi\ .$$

For $n=0$,

$$\frac{1}{\pi}\int_0^{2\pi} U(Re^{i\phi})d\phi = \frac{1}{2\pi}\int_0^{2\pi} \left[F(Re^{i\phi}) + \bar{F}(Re^{i\phi})\right] d\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (c_0 + \bar{c}_0)d\phi$$

$$= c_0 + \bar{c}_0.$$

Let $g(z) = U(re^{i\phi}) + iV(Re^{i\phi})$;

$$\text{Re}\left[g(z)\right] = U(Re^{i\phi}) > 0.$$

Thus,

$$|c_n|R^n \leq \frac{1}{\pi}\int_0^{2\pi} U(Re^{i\phi})d\phi = c_\circ+\bar{c}_\circ,$$

and, as R approaches unity,

$$|c_n| \leq 2 \text{ since } g(0) = c_\circ = 1.$$

This inequality is sharp, since

$$g(z) = \frac{1+z}{1-z} = 1+2\sum_{n=1}^{\infty} z^n$$

satisfies the conditions of theorem 2.1.

2.2 <u>Theorem</u>. <u>If $\phi(z)$ is a normalized univalent convex function on $K_1(0)$ and has a power series expansion</u>

$$\phi(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

then $|b_n| \leq 1$.

Proof. From theorem 1.9, since $\phi(z)$ is convex,

$$\text{Re}\left[1 + \frac{z\phi''(z)}{\phi'(z)}\right] > 0.$$

Let

$$g(z) = 1 + \frac{z\phi''(z)}{\phi'(z)} = 1 + \sum_{n=1}^{\infty} c_n z^n.$$

Then

$$\frac{z\phi''(z)}{\phi'(z)} = \sum_{n=1}^{\infty} c_n z^n .$$

Since

$$\phi'(z) = \sum_{n=1}^{\infty} nb_n z^{n-1},$$

and

$$z\phi''(z) = \sum_{n=1}^{\infty} n(n-1)b_n z^{n-1} ,$$

then

$$\frac{\sum_{n=1}^{\infty} n(n-1)b_n z^{n-1}}{\sum_{n=1}^{\infty} nb_n z^{n-1}} = \sum_{n=1}^{\infty} c_n z^n .$$

Thus,

$$\sum_{n=1}^{\infty} n(n-1)b_n z^{n-1} = \left[\sum_{n=1}^{\infty} c_n z^n\right]\left[\sum_{n=1}^{\infty} nb_n z^{n-1}\right];$$

consequently,

$$\sum_{n=1}^{\infty}\left[n(n+1)b_{n+1} - \sum_{j=0}^{n-1} (j+1)c_{n-j}b_{j+1}\right]z^n = 0.$$

Since $z^n \not\equiv 0$, the coefficients of z^n must be identically zero; therefore,

$$n(n+1)b_{n+1} = \sum_{j=0}^{n-1} (j+1)c_{n-j}b_{j+1},$$

and

$$n(n+1)\left|b_{n+1}\right| \leq \sum_{j=0}^{n-1} (j+1)\left|c_{n-j}\right|\left|b_{j+1}\right| .$$

From theorem 2.1, $|c_n| \leq 2$ for all n; hence,

$$n(n+1)|b_{n+1}| \leq 2\sum_{j=0}^{n-1} (j+1)|b_{j+1}|.$$

If n=1, $|b_2| \leq |b_1|$.

Assume that $|b_{k-1}| \leq |b_1|$. For n=k-1,

$$k(k-1)|b_k| \leq 2\sum_{j=0}^{k-2} (j+1)|b_{j+1}|$$

$$\leq 2|b_1|\sum_{j=0}^{k-2} (j+1)$$

$$= k(k-1)|b_1|,$$

and $|b_k| \leq |b_1|$.

Therefore, by induction, $|b_n| \leq |b_1|$ for all integral n.

But $b_1=1$ and $|b_n| \leq 1$.

This inequality is also sharp since

$$f(z) = \frac{z}{1-z} = \sum_{n=1}^{\infty} z^n$$

is convex.

2.3 Theorem. *If $\phi(z)$ and g(z) are univalent mappings in $K_1(0)$, then $f(z)=g[\phi(z)]$ is univalent in $K_1(0)$.*

Proof. Since g(z) and $\phi(z)$ are holomorphic, then $g[\phi(z)]$ is holomorphic. For any two distinct points, z_1 and z_2, in the domain of definition, $\phi(z_1) \neq \phi(z_2)$. Let $\phi(z_1) = \alpha_1$, and $\phi(z_2) = \alpha_2$.

Then

$$f(z_1) = g\left[\phi(z_1)\right] = g(\alpha_1),$$

and

$$f(z_2) = g\left[\phi(z_2)\right] = g(\alpha_2).$$

But $g(\alpha_1) \neq g(\alpha_2)$, since $g(z)$ is univalent. Therefore, $f(z_1) \neq f(z_2)$, and $f(z)$ is univalent.

2.4 Theorem. *If $g(z)$ is holomorphic in a convex domain D and $\text{Re}\left[g'(z)\right] > 0$ in D, then $g(z)$ is univalent in D.*

Proof. Let z_1 and z_2 be two distinct points in D. Since D is convex, $z_2 = z_1 + re^{i\theta}$.

$$g(z_2)-g(z_1) = \int_{z_1}^{z_2} g'(z)dz$$

$$= e^{i\theta}\int_0^r g'(z+te^{i\theta})dt,$$

where the path of integration is along the straight line $te^{i\theta}$. If $g(z_2) = g(z_1)$, then

$$\int_0^r g'(z_1+te^{i\theta})dt = 0,$$

and

$$\int_0^r \text{Re}\left[g'(z_1+te^{i\theta})\right]dt + i\int_0^r \text{Im}\left[g'(z_1+te^{i\theta})\right]dt = 0;$$

hence,

$$\int_0^r \mathrm{Re}\left[g'(z_1+te^{i\theta})\right]dt = 0.$$

But $\mathrm{Re}\left[g'(z)\right] > 0$; thus, $r=0$.

Therefore, $z_1=z_2$, and $g(z)$ is univalent.

2.5 <u>Definition</u>. <u>If</u> $f(z)$ <u>is</u> <u>holomorphic</u> <u>for</u> $|z|<1$ <u>and</u> $f'(z)\neq 0$ <u>for</u> $|z|<1$, <u>then</u> $f(z)$ <u>is</u> <u>close-to-convex</u> <u>if</u> <u>and</u> <u>only</u> <u>if</u> <u>there</u> <u>exists</u> <u>a</u> <u>univalent</u> <u>convex</u> <u>function</u> $\phi(z)$ <u>such</u> <u>that</u> <u>for</u> $|z|<1$,

$$\mathrm{Re}\left[\frac{f'(z)}{\phi'(z)}\right] > 0.$$

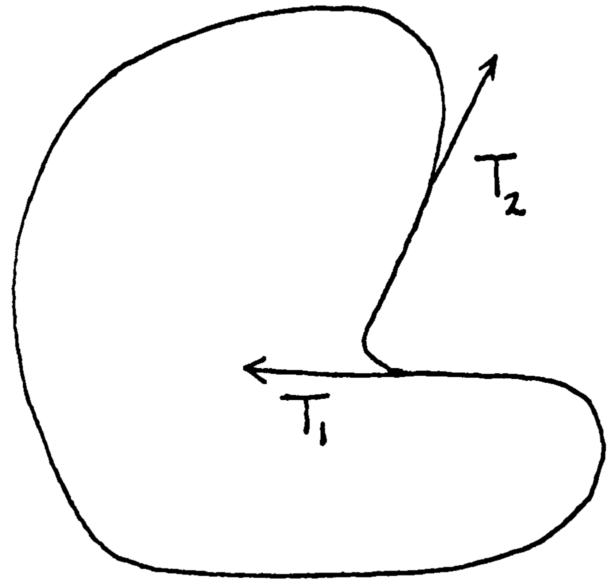
From theorem 1.3, convex functions are characterized by the fact that the argument of the tangent ray to the image of $C_r(0)$, with fixed $r<1$, is a monotonic increasing function of θ as θ increases continuously from 0 to 2π. For close-to-convex functions, the argument of the tangent ray to the image of $C_r(0)$, with fixed $r<1$, can decrease as θ increases continuously from 0 to 2π, but the amount of decrease must be less than π. In other words,

$$\frac{\partial}{\partial\theta} \arg izf'(z) > -\pi$$

if $f(z)$ is close-to-convex, and

$$\frac{\partial}{\partial\theta} \arg izf'(z) \geq 0$$

if $f(z)$ is convex. Thus, a change in direction of the tangent ray, like the situation illustrated below, is possible for close-to-convex functions.



arg T_2 - arg T_1 is slightly greater than $-\pi$.

2.6 Theorem. A close-to-convex function $f(z)$ is univalent.

Proof. Since $f(z)$ is close-to-convex, there exists a univalent convex function $\phi(z)$ such that

$$\operatorname{Re}\left[\frac{f'(z)}{\phi'(z)}\right] > 0.$$

Let

$$g'[\phi(z)] = \frac{f'(z)}{\phi'(z)}.$$

Then

$$g'[\phi(z)]\phi'(z) = f'(z),$$

and $f(z)=g[\phi(z)]$. $f(z)$ is univalent if $g[\phi(z)]$ and $\phi(z)$ are univalent, by theorem 2.3. From definition 2.5, $\phi(z)$ is univalent. $g[\phi(z)]$ is holomorphic in the domain of $\phi(z)$, which is a convex domain since $\phi(z)$ is convex. Furthermore,

$$\operatorname{Re}\{g'[\phi(z)]\} = \operatorname{Re}\left[\frac{f'(z)}{\phi'(z)}\right] > 0.$$

Thus, by theorem 2.4, $g[\phi(z)]$ is univalent; therefore, f(z) is univalent.

The next two theorems show that the class of close-to-convex functions contains as subclasses convex functions and star-like functions.

2.7 Theorem. Convex functions are close-to-convex functions.

Proof. Let f(z) be any univalent convex function. Choose $\phi(z)=f(z)$, and $\phi'(z)=f'(z)$. Thus,

$$\operatorname{Re}\left[\frac{f'(z)}{\phi'(z)}\right] = \operatorname{Re}\left[\frac{f'(z)}{f'(z)}\right] = 1 > 0;$$

hence, f(z) is close-to-convex.

2.8 Theorem. Star-like functions are close-to-convex functions.

Proof. Let f(z) be any star-like function. Choose

$$\phi(z) = \int_0^z \frac{f(t)}{t}\,dt.$$

Since f(z) is star-like, f(0)=0, and

$$\lim_{t\to 0} \frac{f(t)}{t} = \lim_{t\to 0} f'(t) = f'(0)\neq 0.$$

Hence,

$$\phi(z) = \int_0^z \frac{f(t)}{t}\,dt$$

exists for $|z| < 1$, and

$$\phi'(z) = \frac{f(z)}{z}.$$

Thus, $z\,\phi'(z) = f(z)$. By theorem 1.8, $\phi(z)$ is a convex function. From theorem 1.7, since $f(z)$ is a star map,

$$\operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] > 0,$$

and

$$\operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] = \operatorname{Re}\left[\frac{f'(z)}{\frac{f(z)}{z}}\right] = \operatorname{Re}\left[\frac{f'(z)}{\phi'(z)}\right].$$

Therefore,

$$\operatorname{Re}\left[\frac{f'(z)}{\phi'(z)}\right] > 0,$$

and $f(z)$ is close-to-convex.

Now consider the following theorem which shows that close-to-convex functions satisfy the Bieberbach conjecture.

2.9 <u>Theorem</u>. If $f(z)$ <u>is a normalized univalent close-to-convex function, and, for</u> $|z| < 1$,

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

then $|a_n| \leq n$ for all n.

Proof. Since $f(z)$ is close-to-convex, there exists a univalent convex function, say

$$\phi(z) = z + \sum_{n=2}^{\infty} c_n z^n,$$

such that

$$\operatorname{Re}\left[\frac{f'(z)}{\phi'(z)}\right] > 0.$$

Let

$$g(z) = \frac{f'(z)}{\phi'(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n \ ;$$

thus,

$$(n+1)a_{n+1} = b_n + \sum_{k=1}^{n-1} (k+1)c_{k+1}b_{n-k} + (n+1)c_{n+1},$$

and consequently,

$$(n+1)\left|a_{n+1}\right| \leq \left|b_n\right| + \sum_{k=1}^{n-1} (k+1)\left|c_{k+1}\right| \left|b_{n-k}\right| + (n+1)\left|c_{n+1}\right|.$$

From theorem 2.1, $\left|b_n\right| \leq 2$, and from theorem 2.2, $\left|c_n\right| \leq 1$; therefore,

$$(n+1)\left|a_{n+1}\right| \leq 2 + 2\sum_{k=1}^{n-1}(k+1) + (n+1).$$

Hence,

$$\begin{aligned}(n+1)\left|a_{n+1}\right| &\leq 2 + 2\left[\frac{n(n+1)}{2} - 1\right] + (n+1).\\ &= (n+1)(n+1).\end{aligned}$$

Therefore, $\left|a_{n+1}\right| \leq n+1$, or $\left|a_n\right| \leq n$ for all n.

2.10 Corollary. Convex functions and star-like functions satisfy the Bieberbach conjecture.

Proof. The proof is immediate from theorems 2.7, 2.8, and 2.9.

There are other classes of functions whose coefficients are bounded in the same manner as close-to-convex functions but are not close-to-convex, since these classes are not univalent. The following class of functions is an example of functions which are not univalent and have bounded coefficients.

2.11 Definition. If a necessary and sufficient condition for f(z) to be real is that z is real, then f(z) is typically-real, and conversely.

An example of a typically real function is, for $|z| < 1$,

$$f(z) = z - \frac{1}{1-z} = 1 + \sum_{n=2}^{\infty} z^n .$$

For w=f(z), the inverse of f(z) is

$$z = \frac{(1+w) \pm \sqrt{(1+w)(3-w)}}{2}$$

which is not single-valued. Therefore, by theorem 2.6, f(z) is not univalent and not close-to-convex.

CHAPTER III

CLOSE-TO-STAR FUNCTIONS

As an interesting contrast to close-to-convex functions, Reade [11] has discussed close-to-star functions which are defined in a manner similar to that which is used to define close-to-convex functions. Although the class of close-to-star functions behaves in a manner analagous to that of close-to-convex functions and possesses similar properties, a function of this class may not have the same coefficient bounds as the functions previously discussed. However, the coefficients of the power series of close-to-star functions are bounded.

3.1 Definition. *Let* $f(z) = \sum_{n=1}^{\infty} a_n z^n$, $a_1 \neq 0$, *be holomorphic for* $|z| < 1$ *and let* $f(z) \neq 0$ *for* $z \neq 0$. $f(z)$ *is close-to-star if and only if there exists a univalent star map* $\psi(z) = \sum_{n=1}^{\infty} b_n z^n$ *such that, for* $|z| < 1$,

$$\mathrm{Re}\left[\frac{f(z)}{\psi(z)}\right] > 0.$$

3.2 Theorem. $f(z)$ *is close-to-star if and only if*

$$F(z) = \int_0^z \frac{f(t)}{t}\,dt$$

is close-to-convex.

Proof. If $F(z)$ is close-to-convex, then there exists a normalized convex function $\phi(z)$ such that

$$\operatorname{Re}\left[\frac{f'(z)}{\phi'(z)}\right] > 0.$$

Since

$$F(z) = \int_0^z \frac{f(t)}{t}\,dt,$$

$$F'(z) = \frac{f(z)}{z},$$

and thus

$$\frac{F'(z)}{\phi'(z)} = \frac{f(z)}{z\,\phi'(z)} .$$

From theorem 1.8, since $\phi(z)$ is convex, $z\phi'(z)$ is star-like. Let $z\,\phi'(z) = \psi(z)$. Then

$$\operatorname{Re}\left[\frac{f(z)}{\psi(z)}\right] = \operatorname{Re}\left[\frac{F'(z)}{\phi'(z)}\right] > 0,$$

3.4 <u>Theorem</u>. <u>If</u> $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ <u>is close-to-star</u>, <u>then</u> $|a_n| \leq n^2$.

Proof. Let $F(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ be close-to-convex.

From 2.4, $|b_n| \leq n$. By theorem 3.3, $f(z) = zF'(z)$ is close-to-star. Now,

$$f(z) = z\sum_{n=1}^{\infty} nb_n z^{n-1} = \sum_{n=1}^{\infty} nb_n z^n .$$

Since

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n,$$

$$z+\sum_{n=2}^{\infty} a_n z^n = \sum_{n=1}^{\infty} nb_n z^n,$$

and $$\sum_{n=1}^{\infty} a_n z^n - \sum_{n=1}^{\infty} nb_n z^n = 0;$$

thus,

$$\sum_{n=1}^{\infty} (a_n - nb_n) z^n = 0.$$

But $z^n \not\equiv 0$, so $a_n - nb_n = 0$. Therefore,

$$|a_n| = n|b_n| \leq n^2.$$

As an example of close-to-star functions, there is a class of functions which was defined by M. S. Robertson [12] in 1939. These functions have since become known as Robertson functions, star-like in one direction.

3.5 Definition. Either

(A) 1. $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ is holomorphic for $|z|<1$,

and 2. There exists an $m=m(f)>0$ such that, for each r in the open interval $1-m<r<1$, $f(z)$ maps $C_r(0)$ onto a contour C_r which is cut by the real axis in exactly two points, *or* (B) 1. *$f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ is holomorphic for $|z|<1$,*

and 2. *$f(z)$ maps $C_1(0)$ onto a contour which is cut by the real axis in exactly two points,*

if and only if $f(z)$ is star-like in the direction of the real axis with respect to $C_1(0)$.

Note that if $f(z)$ is star-like in some other direction, then it can be reduced to the type in definition 3.5 by a rotation

$$e^{i\alpha} f(e^{-i\alpha} z)$$

where α is real.

The function

$$f(z) = \frac{z+iz^2}{(1-iz)^3} = \sum_{n=1}^{\infty} i^{n-1} n^2 z^n$$

is a well known Robertson function for which $a_n = n^2$. In order to show that this function is close-to-star, consider the function

$$F(z) = \frac{z}{(1-iz)^2}$$

where $|z|<1$.

$$\frac{zF'(z)}{F(z)} = \frac{z+iz^2}{(1-iz)^3} \cdot \frac{(1-iz)^2}{z}$$

$$= \frac{1+iz}{1-iz} .$$

Let $z=x+iy$ where x and y are real.

$$\frac{1+iz}{1-iz} = \frac{1+i(x+iy)}{1-i(x-iy)}$$

$$= \frac{1-(x^2+y^2)}{x^2+(1+y)^2} + i\,\frac{2x}{x^2+(1+y)^2} \,.$$

Thus,

$$\mathrm{Re}\left[\frac{zF'(z)}{F(z)}\right] \quad \frac{1-(x^2+y^2)}{x^2+(1+y)^2} \,.$$

Since x and y are real, $x^2+(1+y)^2 > 0$, and $x^2+y^2=|z|^2 < 1$. Therefore

$$\mathrm{Re}\left[\frac{zF'(z)}{F(z)}\right] > 0$$

and, by theorem 1.7, F(z) is star-like. For $|z|<1$,

$$\int_0^z \frac{f(t)}{t}\,dt = \int_0^z \frac{t+it^2}{t(1-it)^3}\,dt$$

$$= \int_0^z \left[\frac{2}{(1-it)^3} - \frac{1}{(1-it)^2}\right]dt$$

$$= \frac{i}{(1-iz)^2} - i - \frac{i}{(1-iz)} + i$$

$$= \frac{z}{(1-iz)^2}$$

$$= F(z).$$

Therefore, from theorem 3.2,

$$f(z) = \frac{z+iz^2}{(1-iz)^3}$$

is close-to-star.

This particular function shows that the bounds of theorem 3.4 are sharp and that close-to-star functions may not satisfy the boundedness problem of univalent functions.

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