INTRODUCTION TO SYMMETRY AND ITS TECHNIQUES

A THESIS

IN MATHEMATICS

by

Hamilton C. Dupont

Approved

Texas Technological College

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To

My Wife,

Lois Dupont

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CHAPTER I

INTRODUCTION

Symmetrical forms surround us in our daily lives. One cannot look about him without discovering that some form of symmetry is present. Indeed, symmetry is so commonplace that little thought is given it. Even among writers of elementary and advanced geometries there is little interest in detailed discussion of the subject, with usually little more than a few paragraphs about it. Writers in such fields as crystallography and art have found it necessary to use symmetry as a tool, but have developed the subject little beyond their own needs. Even writers in the same field do not agree on uniform ways of developing the subject, but usually use their own devices and symbols.[1] Therefore, it is the aim of this thesis to provide readily accessible information about the aspects of symmetry in general and specifically about its mechanics. It is intended not to cover the enormous field of possibilities in the subject, but to develop a thorough introduction to the field of symmetry.

Interest in symmetry should result as a natural consequence of the symmetry found in nature. Symmetry can be seen in the living as well as the inanimate world about

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[1]Wheeler P. Davey, A Study of Crystal Structure and Its Applications, (New York,) 1934, p. 222.

us. The narcissus, clover, star fish, and dragon fly[1] represent outstanding examples of symmetry in plant and animal life. Each of these examples represents a particular kind of symmetry which will be explained later. In the domain of the nonliving, the snow crystals provide an excellent example of symmetry. Nature is not content with displaying countless examples of symmetry in our presence, but also provides endless examples in the internal structure of life forms as well as nonliving objects. However, the symmetry of nature is not always perfect. Consider the oak leaf. There can be little doubt that symmetry is present about a central axis in the direction of the stem. This symmetry will probably not be perfect, for if the leaf is folded along the axis there will not be perfect matching. This can be considered as a result of imperfect development of the leaf. With ideal conditions the leaf should develop with complete symmetry.[2]

Perhaps man's earliest attempt to adapt the symmetry displayed in nature to his personal needs found expression in the field of art. Crude but partly symmetric drawings give testimony to this fact. As civilization progressed it was only natural to expect symmetry to become an

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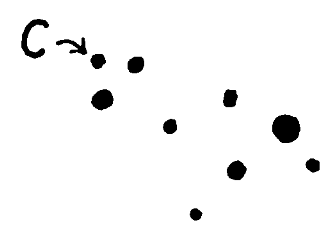
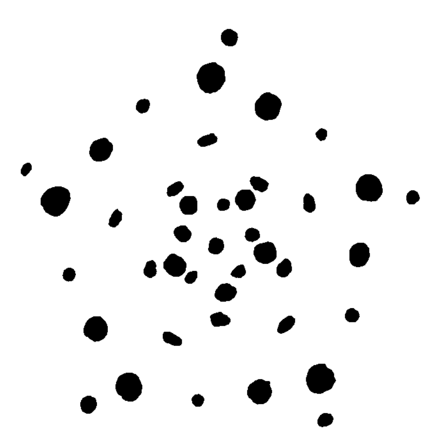
[1]E. L. Palmer, Fieldbook of Natural History, (New York,) 1949, pp. 141, 228, 338.

[2]F. O. Bower, Botany of the Living Plant, (London,) 1947, pp. 199-200.

integral part of art and architecture. The very nature of symmetrical designs and objects is reason enough for man's interest in the subject. In symmetry we find a visual impression of balance.[1] It is this sense of balance, repetition, regularity, and systematic order that gives symmetry its significance. Consider the following complex system of dots placed at random upon the paper.



This configuration of dots lacks symmetry and therefore has little meaning or interest. However, if we rotate the configuration about one of the dots, for instance C, and leave an impression of the configuration after every rotation of 72° we have the following figure.



This figure has meaning and interest, for it now exhibits

[1]Deman W. Ross, A Theory of Pure Design, (New York,) 1933, pp. 22-25.

symmetry. This system of meaningless dots has been transformed by repetition into one which is symmetrical. Approximately the same visual results can be achieved by use of repeated reflections in mirrors. When mirrors are placed in consecutive order and adjacent to each other, the object and the repeated images produce a rotational effect provided the dihedrel angle remains a constant divisor of 360°. In 1817 Sir David Brewster employed this principle in the kaleidoscope.[1] The small bits of colored glass in the kaleidoscope when brought together by chance into some meaningless and haphazard arrangement will, after regular repetition of reflection, produce a surprisingly beautiful pattern. The invention of Sir David is an admirable illustration of symmetry produced with mirrors.

Advancement in the studies of the physical sciences showed a need for some investigation of symmetrical forms. Most of the sciences employ the use of symmetry to some extent. In the field of crystallography we find an outstanding need for identifying crystal forms. The structure of the crystal provides an answer. Nature produces the rectilinear solids in different symmetric forms. Accordingly, the authorities in the subject have devised methods for identifying the crystals through symmetry groups. However, most authorities use a symmetry study unique to their

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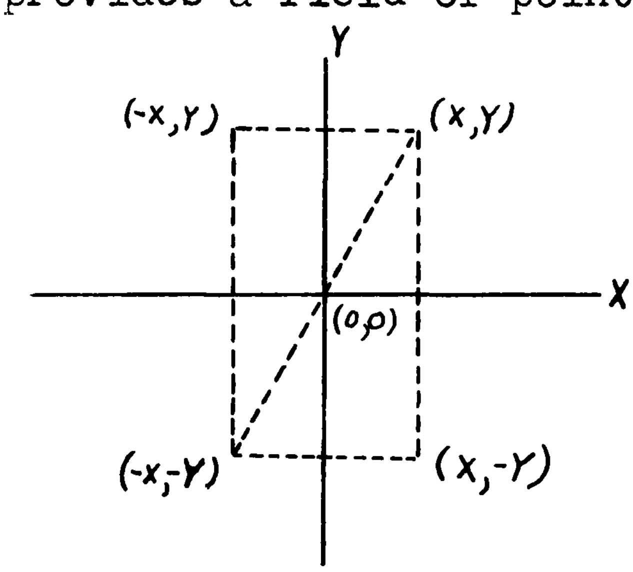
[1]"Kaleidoscope," Encyclopaedia Britannica, Vol. XIII, 1950 ed.

work.[1] Bragg has alluded to this nonuniformity of symmetry development by his collection into a table of the space-group nomenclature of several writers.[2] In biology the manifest need for recognizing symmetry is found in the many life forms. The environment in which life is to exist has an important influence on the type of symmetry inherent in the living forms.[3] In moving animals there is a symmetry plane dividing the body into two similar parts. This is referred to as bilateral symmetry.[4]

When Descartes[5] developed his idea of co-ordinate geometry, he provided additional material for investigation in symmetry in the field of mathematics. His system of co-ordinate axes provides a field of points together with



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[1]Davey, op. cit., p. 222.

[2]Sir Lawrence Bragg and W. H. Bragg, The Crystalline State, (London,) I, 1949, 341-346.

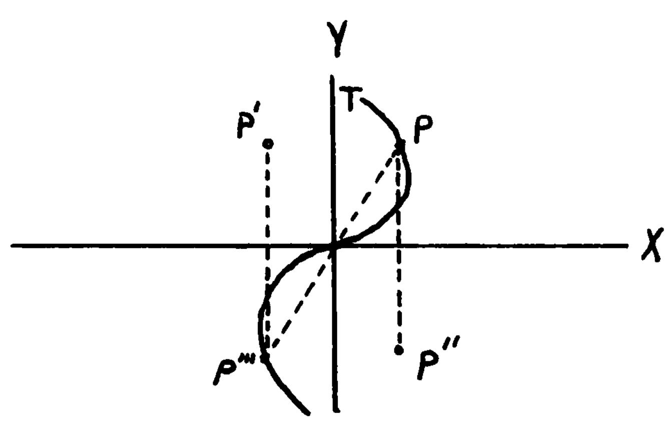
[3]"Symmetry," Van Nostrand's Scientific Encyclopedia (2d ed.; New York,) 1947.

[4]Ibid.

[5]Florian Cajori, A History of Mathematics, (London,) 1926, pp. 173-180.

a set[1] of axes which made possible certain symmetrical arrangements.

The points (x, y) and (-x, y) are symmetrical with respect to the y-axis.[2] Points (x, -y) and (-x, -y) are similarly placed about the y-axis as an axis of symmetry. The x-axis serves as an axis of symmetry for the points (x, y) and (x, -y) and also for the points (-x, y) and (-x, -y). The origin (0, 0) acts as a center of symmetry for the points (x, y) and (-x, -y). It is also a center of symmetry for the points (-x, y) and (x, -y). If on the plane of the axes there is a plane curve, this curve may be symmetric with respect to either axis or the origin. Consider any point P on a plane curve. There are three other points, P', P", and P"' which are symmetrical counterparts of P with respect to each axis and the origin respectively.



If the curve is denoted by T and the arbitrary point on the curve by P, the three symmetric counterparts of P will be P', P", and P"'. In this case only P"' lies on the curve T.

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[1]First consideration will be in only one plane.

[2]See the definitions of fundamental symmetry concepts in Chapter II.

Let P have the coordinates (x, y) so that P"' will have the coordinates (-x, -y). The coordinates of both P and P"' satisfy the equation of T. Therefore, if x and y in the equation of T are replaced by -x and -y the equation will remain unchanged. However, the points P and P" are symmetrical with respect to the origin, and consequently the curve T is symmetrical with the origin. Using this method, a test for the symmetry of a plane curve can be devised. A plane curve is symmetric with respect to the origin if, when we replace x and y in the equation of the curve by -x, -y, the equation is unaltered. If the equation is unaltered when y is replaced by -y, the equation is symmetric about the x-axis. When x may be replaced by -x without altering the equation, the curve is symmetric about the y-axis. The addition of a third axis normal to both the x and y axis, with similar reasoning, produces the tests for the symmetry of a space curve. The tests for the space curve are analogous to the plane curve tests for symmetry.[1]

There is general use of the word symmetry to represent similarity which is proportional whether the elements are mathematical or not. A symmetric determinant[2] represents a case of this kind in mathematics.

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[1] John M. H. Olmsted, Solid Analytic Geometry, (New York,) 1947, p. 97.

[2] H. W. Turnbull, The Theory of Determinants, Matrices and Invariants, (London,) 1929, p. 104.

$$\Delta = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 5 \\ 3 & 4 & 3 & 1 \\ 4 & 5 & 1 & 4 \end{vmatrix}$$

Each two elements of the symmetric determinant Δ, which are symmetrically situated with respect to the principal diagonal, are equal.

Jay Hambridge has reconstructed a type of symmetry different from the one of popular conception.[1] It is called dynamic symmetry. It is not a symmetry based upon axes but upon the relationship of areas in design to the area of the whole of an object. This symmetry which is commonly exhibited in nature was extensively used by the early Greeks. The subject is well treated by Hambridge and is demonstrated in the design of the Greek vase.

Symmetry is definitely a part of our world both in the visible and invisible spheres. Construction of a visible environment free from any type of symmetry would provide an interesting task for a student of architecture. In fact, symmetry is so universal that a case of nonsymmetry may often be considered as a special case of symmetry.

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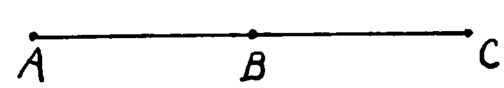
[1] Jay Hambridge, Dynamic Symmetry, (New York,) 1920.

CHAPTER II

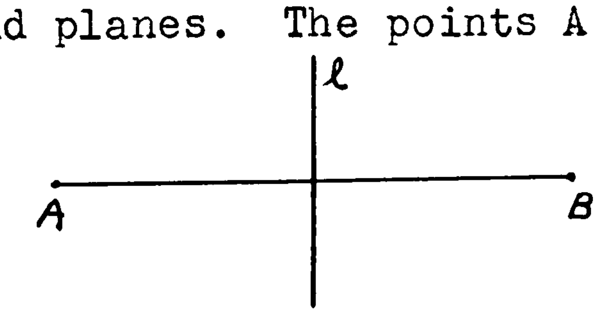
DEFINITIONS AND ELEMENTARY CONCEPTS

It is easy to understand symmetry from the standpoint of every day environment. However, as in the case of so many aspects of our physical universe, it is possible to examine the subject with the aid of mathematics. Such an examination will require the use of definitions and symbols subject to mathematical treatment. It will be necessary to introduce new definitions and symbols in the development of the mechanics of symmetry, but there are many basic definitions in common use which warrant previous study.

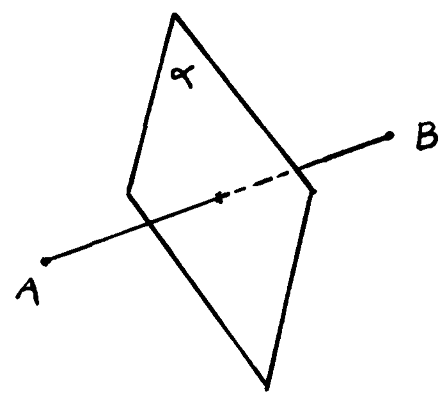
The primary element of symmetrical figures will be the point. A configuration may then be considered as a infinite number of points arranged in predetermined relationships. If we consider three isolated collinear points, consideration of a symmetrical case is possible. Two



points A and C are symmetric with respect to a third point B if B is the mid-point of the segment AC joining the points A and C. Points may also be symmetrical with respect to lines and planes. The points A and B are

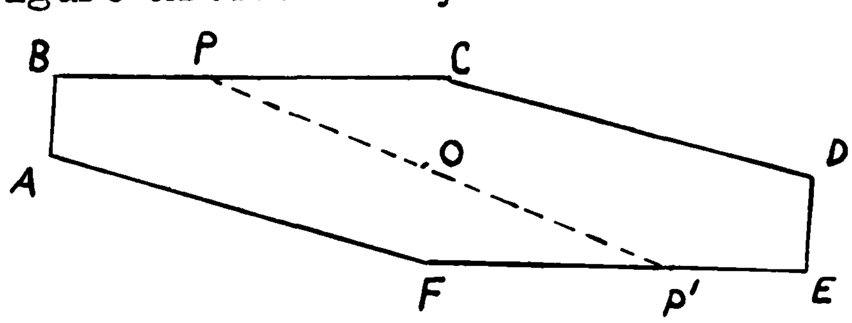


symmetrical with respect to the line l if this line bisects at right angles the segment joining A and B. The line l is referred to as the axis of symmetry. In a similar way a plane can serve to indicate symmetry between points. Two points A and B are symmetric with respect to a plane α if the plane is the perpendicular bisector of the segment



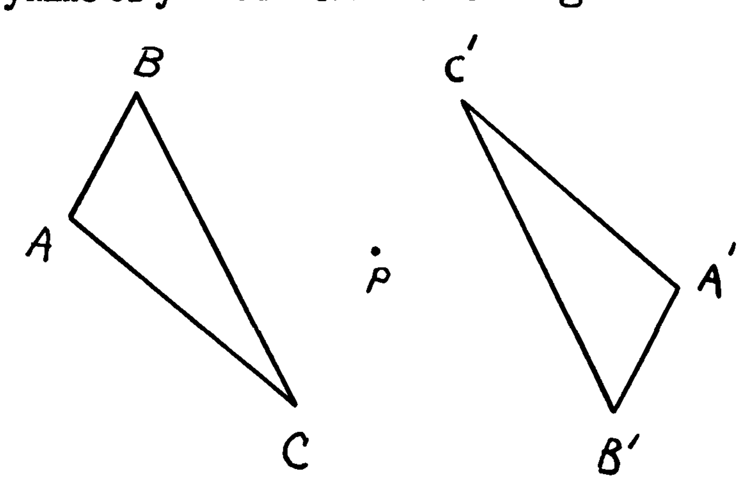
joining A and B. In this case the plane is called a plane of symmetry.

A geometric figure can be defined as a point, a line, a surface, a solid or a combination of these taken in any way. The figure may have its elements so related that it can be considered as symmetrical. If a point of the figure exists such that any line drawn through the point cuts the figure in two points which are symmetrical with respect to the given point, the figure can be considered as symmetrical about the given point. Therefore, if the rectilinear figure ABCDEF is symmetrical about O, any point



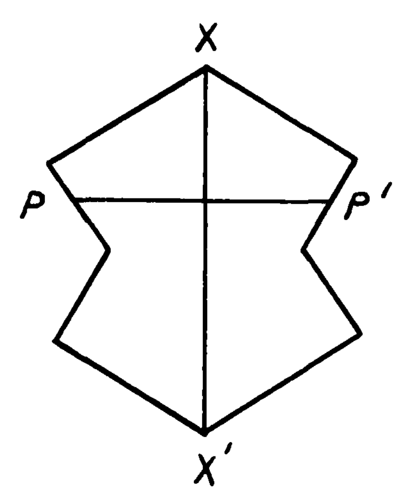
P on the figure is symmetric with another point P' on the figure found by drawing a line through P and O and intersecting the figure at P'. The point O is referred to as the center of symmetry for the figure. This symmetry can be defined in another way. Suppose that the figure is free to rotate. Select the point O as the center of rotation and rotate the figure through an angle of 180° in any given plane. This will cause the points A and D, B and E, and C and F to change places. The new position coincides with the old and the appearance of the figure remains unaltered. Thus, a figure is symmetric about a point if its appearance remains unchanged after a rotation of 180° about the point.

Two figures may be symmetrical about a point. The mathematical description of this symmetry is best described by a definition similar to the latter definition given for the symmetry of a single figure about a point. Consider the two triangles ABC and A'B'C'. The point P is the point about which the triangles are symmetrical. Therefore, P is the center of symmetry for the two figures.

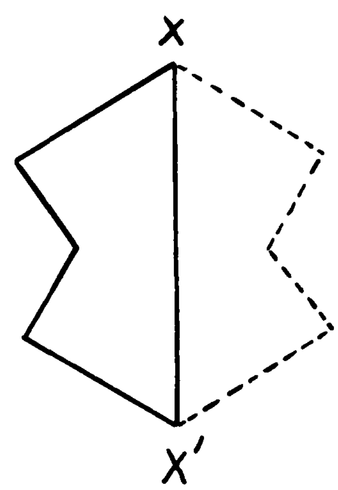


If either triangle is revolved through an angle of 180° about the center of symmetry it will coincide with the other triangle. In general, two figures may be defined as symmetrical with respect to a point if, after one is revolved 180° about the point as a pivot, it coincides with the other figure.

A single figure may be symmetrical with respect to a line instead of a point, but symmetry with respect to both line and point is possible in some figures. Symmetry with respect to a line for a single figure exists when the line will bisect any line perpendicular to it and terminated by the boundary of the figure. The line about which the symmetry is exhibited is called the axis of symmetry. The following figure has symmetry with respect to the axis XX'. The consequence of this symmetry is that any segment



PP' perpendicular to XX' will also be bisected by XX'. There is another definition which embodies a principle important in the analysis of symmetry that should be given for a single figure symmetric about an axis. If either half of the above figure is folded over about the axis, it can be made to coincide with the other half. For instance, if we consider only the left half of the figure and the result of the fold as a dashed line appearance, we have the following figure. Therefore, any figure in which one half



may be folded about a line to coincide with the other half is defined as symmetrical about this line or axis.

The method of producing symmetry for single figures about an axis using the fold technique is employed by psychologists to construct their Rorschach test.[1] The test consists of ten standard ink blots each formed by folding a sheet of paper containing a blot of wet ink about a line through the blot. When the paper is opened the resulting blot is a figure symmetrical about the fold line as an axis.

The description of the symmetry of two figures about an axis corresponds to the description of the fold procedure involved in the symmetry of a single figure about an axis. Once again it is possible to produce superposition

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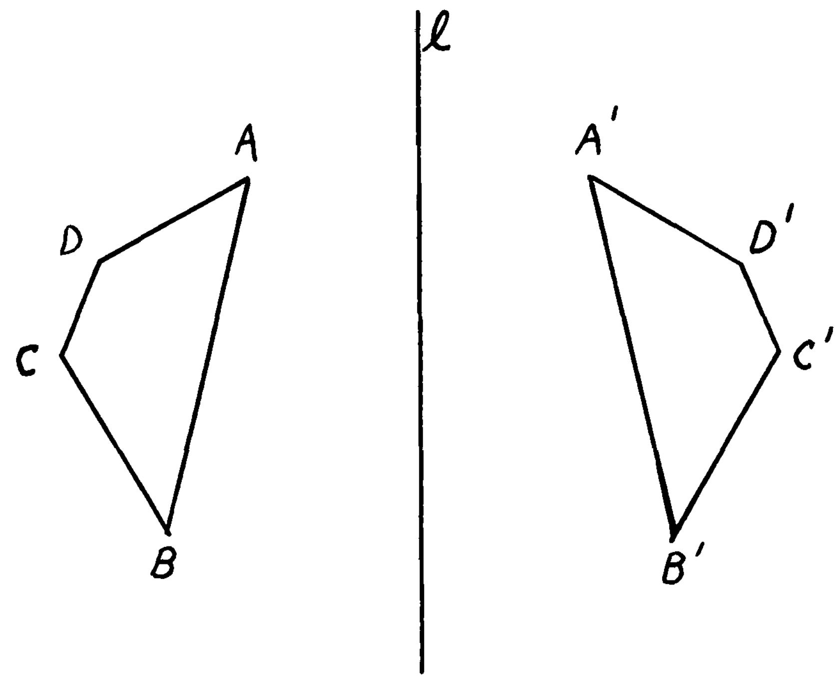
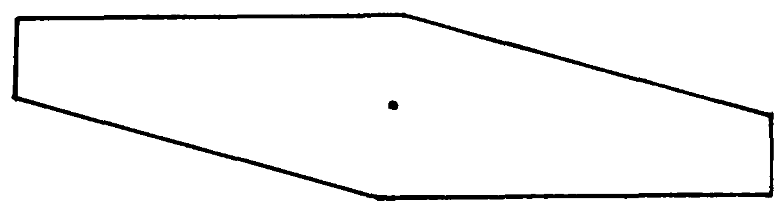
[1]Norman L. Munn, Psychology, (New York,) 1946, p. 466.

by folding about the axis. The line ℓ is the axis of symmetry for the two figures ABCD and A'B'C'D'. If this page is folded along the line ℓ the corresponding parts of the

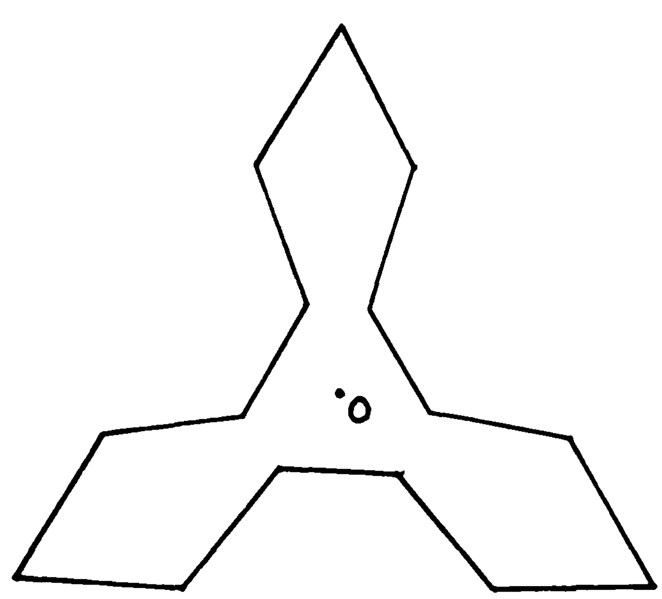


two quadrilaterals will be superimposed upon each other. By employment of this principle two figures are said to be symmetrical about an axis if one can be made to coincide with the other by a revolution of 180° around the axis. Of course, a simpler definition can be given in this case. Two figures are symmetric with respect to an axis if every point on one has a corresponding symmetric point on the other.

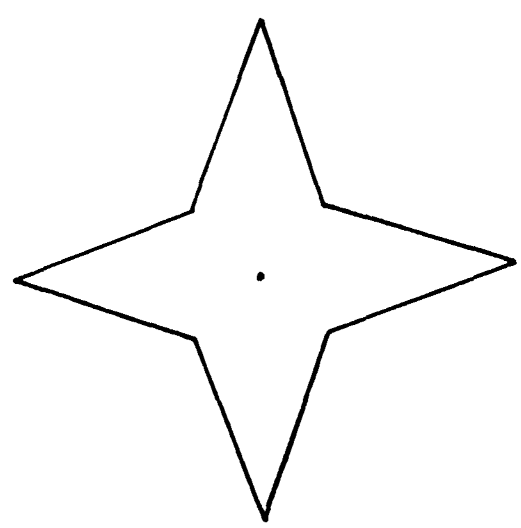
The idea of folding to cause a figure to coincide is important and is often referred to as <u>fold symmetry</u>. Reference to fold symmetry is often made concerning single figures symmetric about a point. However, use of the word "fold" is a matter of convenience since rotation is the cause of superposition. The previous definition of a single figure symmetrical about a point required rotation of 180° about the point. This may now be classified as a special case. A rotation of some angle α about the center is called a fold. However, the angle must be an integral divisor of 360°. Two folds of 180° will return a figure to the identical position from which it started. Symmetry of a figure about a point due to a rotation of 180° is called <u>twofold symmetry</u>. If we examine a twofold figure again, we find



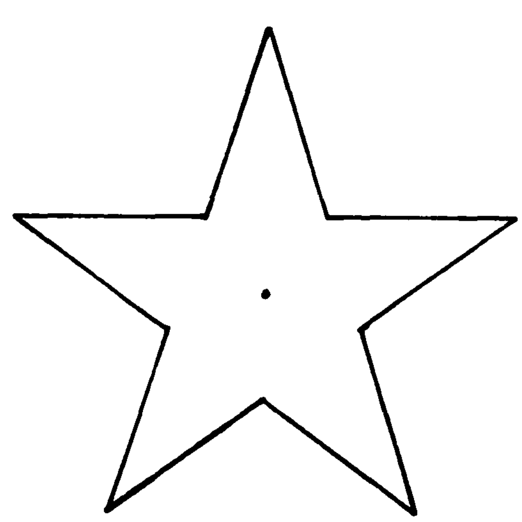
that the angle of rotation is equal to 360° divided by the number of the fold. It is also obvious that there are two parts of the figure. Each part may be made to coincide with the other. If a figure coincides with its original impression after a rotation of 120° about a point the figure is one of <u>threefold symmetry</u>. Consider the following threefold figure. This figure, after a rotation of 120°,



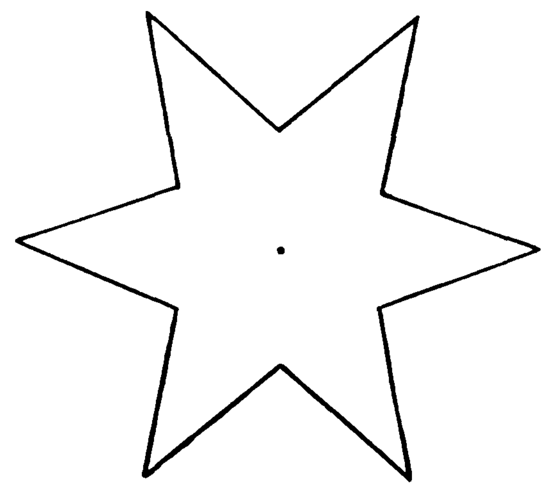
will coincide with itself. The position has the same appearance as the original which means that after another rotation of 120° it will once again coincide with the first position. Thus, after every consecutive rotation of 120° about O, the figure will be superimposed upon its original position. The definition can be extended to include figures which coincide after every rotation of 90°, 72°, 60°, etc. They are called figures with four, five, and sixfold symmetry. If any figure after a rotation of 360/n degrees coincides with its initial appearance, it may be considered as having n-fold symmetry.



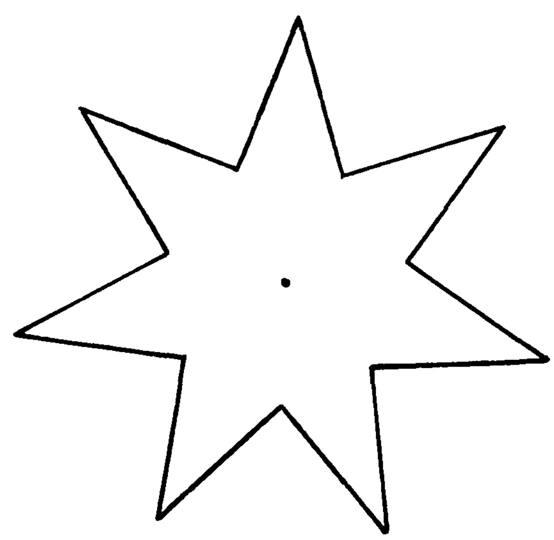
4-fold symmetry



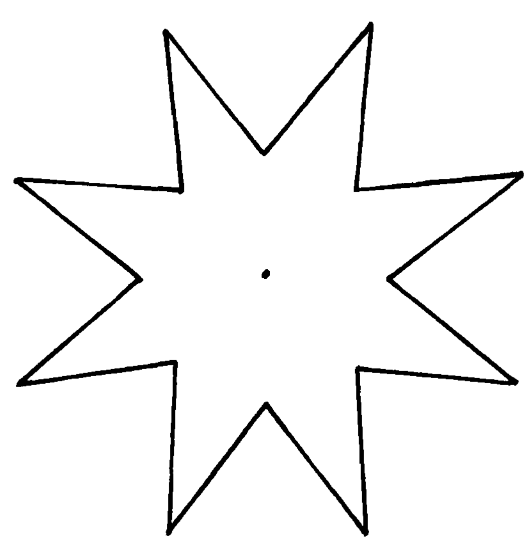
5-fold symmetry



6-fold symmetry

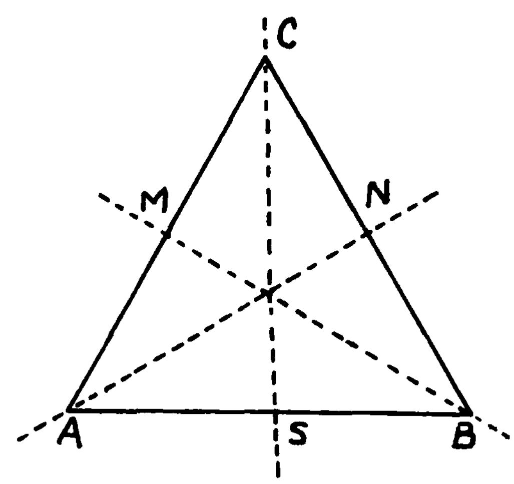


7-fold symmetry

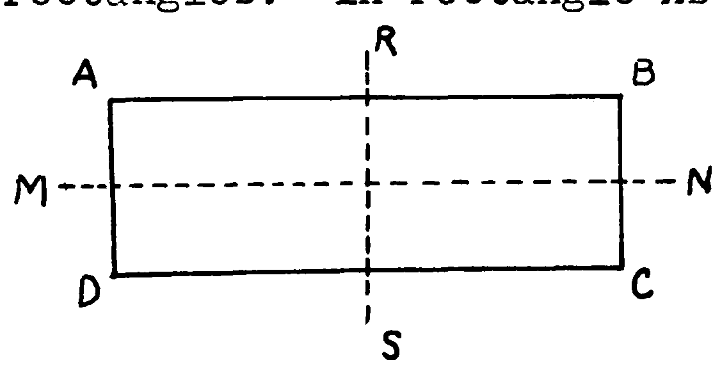


8-fold symmetry

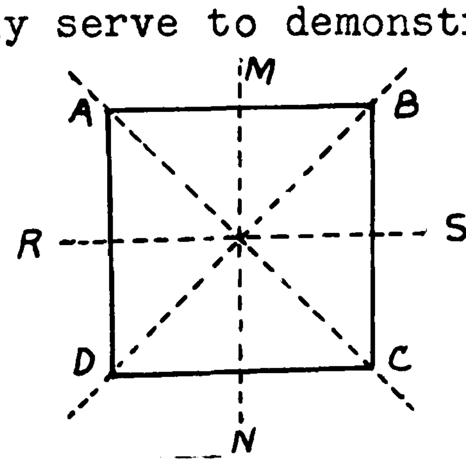
The fact that an object or figure may be symmetric with respect to a line has been discussed. The line is called an axis of symmetry. A figure symmetrical about one axis may or may not be symmetrical about other axes. This fact is easily seen when one considers the perpendicular bisectors of an equilateral triangle. The three bisectors



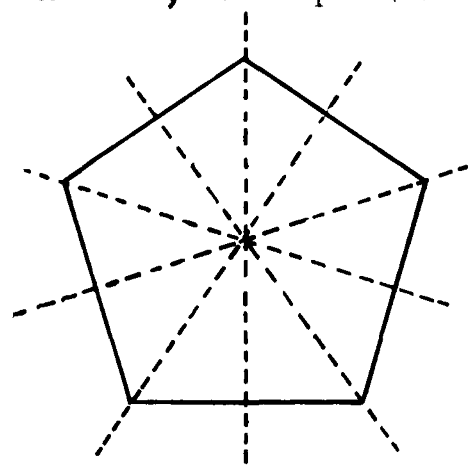
AN, BM, and CS represent three different axes of symmetry for triangle ABC. If one side of the triangle is folded over along any of the three axes it will fit exactly over the other half. A case of two axes of symmetry is found in the study of rectangles. In rectangle ABCD the lines MN



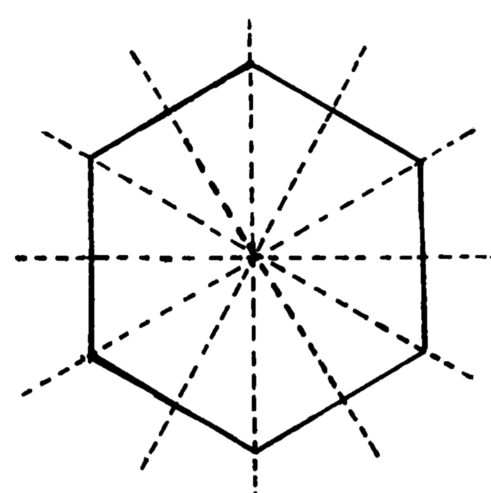
and RS are the two axes of symmetry. As previously shown the equilateral triangle is a case of three axes of symmetry. The square may serve to demonstrate the presence



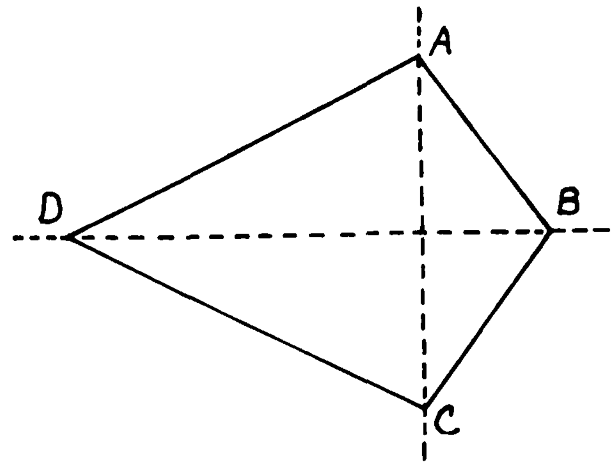
of four axes of symmetry. Lines MN, RS, and the diagonals AC and BD represent the four axes of symmetry for the square ABCD. The five perpendicular bisectors of the sides of a regular pentagon prove to be a case of five axes of symmetry. In a like manner, the perpendicular bisectors



of the sides and the diagonals form six axes of symmetry for the hexagon. Extension of this procedure will produce

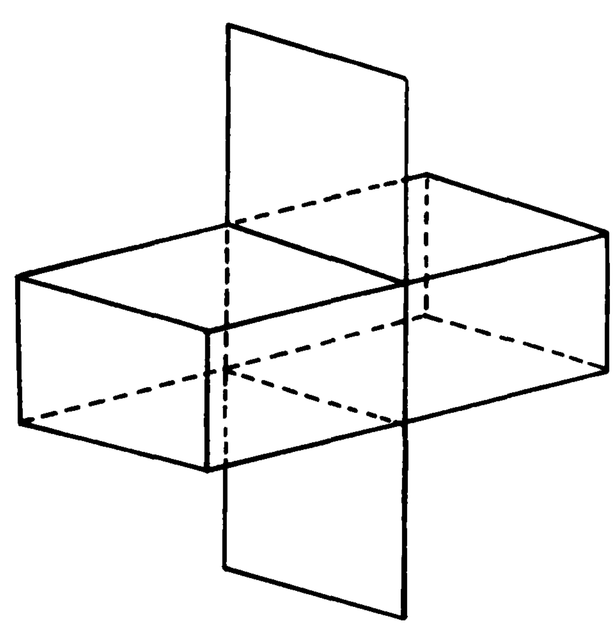


regular polygons with any desired number of axes of symmetry. The circle represents the limiting case since it possesses an infinite number of axes. Even though many figures possess several axes of symmetry, not every line of the figure will always be an axis. The quadrilateral ABCD is an example of a polygon which has two diagonals of which only one is an axis of symmetry. AB is not an axis of symmetry since the figure could not be folded over



on this axis and made to coincide. However, BD is an axis of symmetry and a folding procedure would cause superposition.

Solid figures are symmetrical about points and lines in much the same way as plane figures. In addition, they may be symmetrical with respect to a plane. Such a plane is called a plane of symmetry. A solid will be symmetrical about a plane of symmetry if the plane divides the solid into two parts, each part being the duplicate of the other. An example is a rectangular parallelepiped which is cut by a plane perpendicular to and bisecting four of the sides. The parallelepiped is symmetrical with respect to the plane.



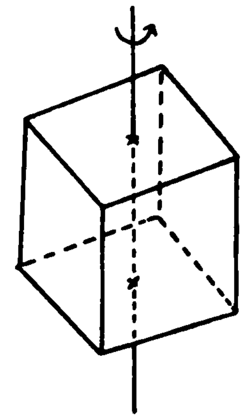
Solids may have one, two, or any number of symmetry planes. Often a solid will have an infinite number of symmetry planes as is the case of a sphere. Any plane which intersects the sphere in a great circle represents a plane of symmetry. One may cite other cases. For instance, there are an infinite number of symmetry planes passing through the apex of a cone and perpendicular to its base. Most solids, nevertheless, contain only a finite number of symmetry planes. A rectangular parallelepiped has only three such planes. In the previous figure, two other planes may be drawn as perpendicular bisectors of the remaining two sets of parallel sides.

CHAPTER III

CHARACTERISTIC SYMMETRIC ACTIONS AND SIMPLIFICATIONS

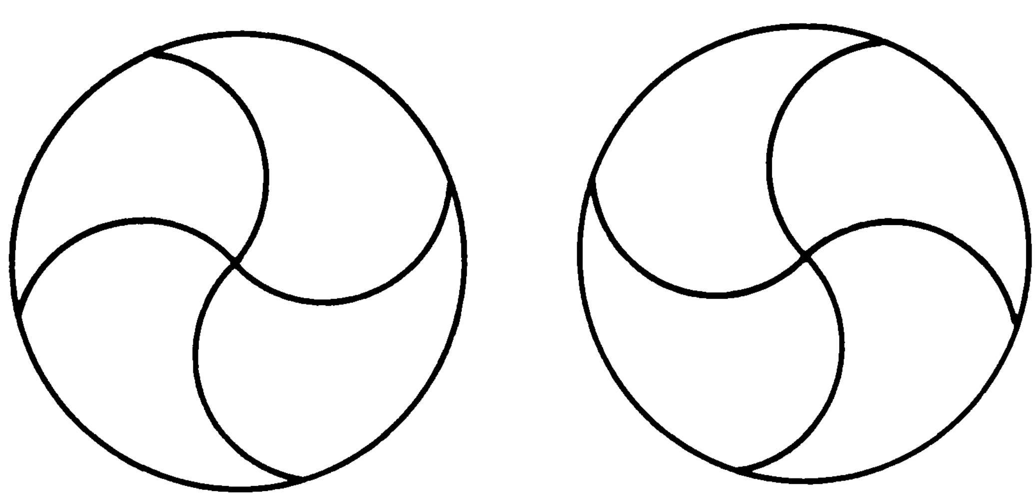
In chapters I and II the general aspects of symmetry and many basic mathematical definitions were discussed. Now that sufficient background has been presented, we may begin a discussion of the mechanics of symmetry.

It is now apparent that symmetry initiates a change in symmetric figures or objects, but there is a definite restriction about this change. This change will be considered as one which preserves the original relations and distances of the different elements to each other in a symmetric figure or object. Such a change can be produced by a rotation. The rotation is the kind used to define symmetry in the previous chapters. It is the rotation about a center of symmetry producing the plane figure patterns of the many n-fold symmetries. The folding over along an axis to make two parts of a figure or two figures coincide is an example of rotation to produce symmetry. The rotation technique has an important place in the symmetry of solids. In a solid the rotation is usually about some axis. Rotation of the solid about the axis will cause it to take a position which will coincide with an initial position. Several cases of coincidence may occur as would be the situation for a cube rotated about an axis through the center and perpendicular to two opposite faces. Every rotation through 90° about the axis will cause the cube to

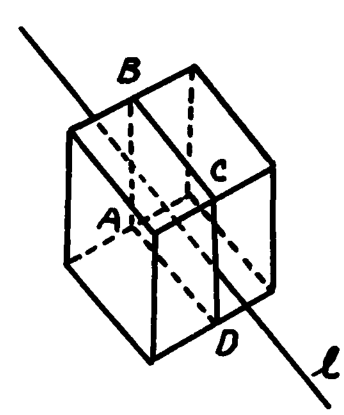


coincide with its original position. The symmetry of the solid in this case is analogous to the symmetry of the plane figure produced by a plane section through the cube normal to the axis. It has fourfold symmetry produced by a rotation. Rotation is therefore one fundamental way of producing or examining symmetry. The importance of the rotation effect justifies referring to it as <u>characteristic rotational symmetric action</u>.

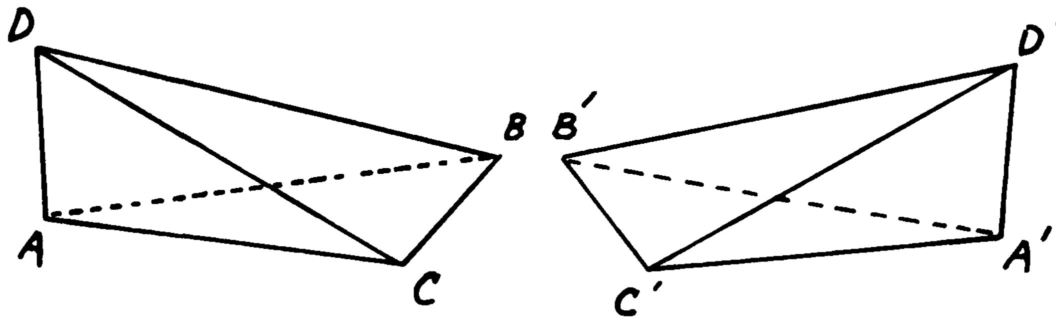
Symmetry in a figure or figures may also be caused by reflection. There are cases when rotation alone will not cause two symmetrical figures to coincide. Consider the two plane figures.



There is no possible way to rotate or translate in the plane so that the two figures will coincide. However, the figures are symmetrical for there is a definite preservation of the relations and distances of the different elements to each other. One of the figures may be thought of as a mirror image of the other. Consequently, symmetry may be brought about by a reflection. For three dimensional figures there are cases when reflection is the only way to cause superposition. Two examples are right and left hand gloves and screw threads. Sometimes an image of an object may be obtained by either a rotational or reflectional process. In such cases the reflected image can be thought of as being essentially the same as the object. The human body is an example of this case. If the image of the human body is shifted parallel to itself until it is in a position as far in front of the mirror as previously behind it, a rotation of 180° about a vertical axis will cause coincidence. Another case is the cube with a plane of symmetry taken as a reflecting plane. Reflection of one half of the cube in the



plane ABCD produces an image which is superposed on the other half of the cube. The same results could have been achieved by rotation of the cube through 180° about the axis ℓ. If, on the other hand, we consider an irregular tetrahedron there is only a reflective approach to the symmetrical counterpart. The image A'B'C'D' can be obtained



only by reflection in a mirror plane perpendicular to and bisecting the segment BB'. Unique symmetrical relations may be produced by reflections. Therefore, it may be referred to as <u>characteristic reflective symmetric action</u>.

The creation of a symmetrical situation may be caused by rotational or reflective characteristic symmetric action. A combination of both kinds of characteristic symmetric action may also be present. Therefore, we may classify symmetry according to the symmetric actions present. The characteristic symmetric actions are then fundamental to the mechanics of symmetry.

Many situations occur where rotation and reflection make up the characteristic symmetric action. The geometric concept of inversion provides a technique for combining the two symmetric actions into one operation. To define inversion[1] we consider two points P, P' collinear with a given

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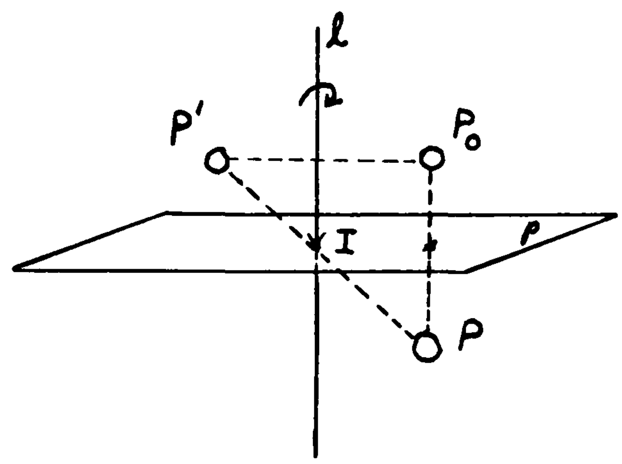
[1]Nathan Altshiller-Court, College Geometry, (New York,) 1925, p. 201.

point O. The two points are said to be inverse points with respect to O if they are located such that the product OP•OP' is equal to a given constant K. Point O is the <u>center of inversion</u> and K is the <u>constant of inversion</u>. If the elements of two figures are inverse points, then one is an inversion of the other. An inversion may take the place of a rotation followed by a reflection. The truth of this is evident when we consider the following figure.[1]



Rotation of point P' through 180° about the axis ℓ places it in the position P_o. A reflection of P_o in a plane p perpendicular to the axis ℓ produces the image P. Now this image P can also be obtained by an inversion of P' with respect to I and a constant of inversion equal to minus one. Therefore, we may use inversion as a substitute for rotation followed by a reflection. However, all inversions involved in the mechanics of symmetry must have a constant of inversion equal to minus one.

Using an inversion will often simplify the mechanics

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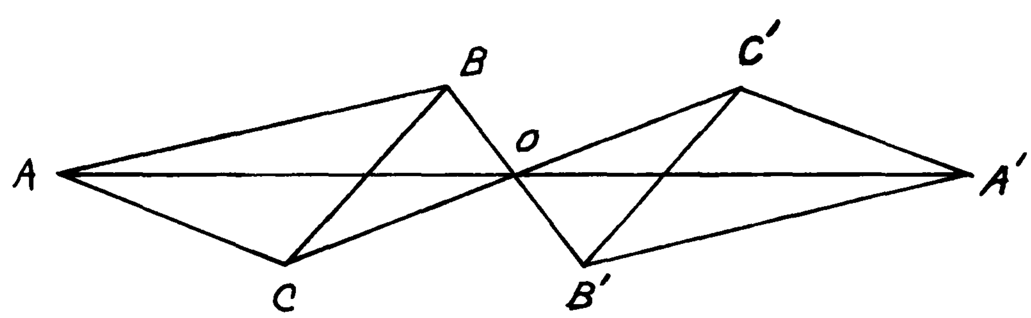
[1]Sir Lawrence Bragg and W. H. Bragg, <u>The Crystalline State</u>, (London,) I, 1949, 68-69.

of symmetry. Other methods of simplification can be devised. An image after reflection may not be in the proper position. In this case it may be displaced by a simple motion of translation. It may be thought of as the image of an object reflected in a glide-mirror.[1] The glide-mirror is an imaginary mirror which after reflection automatically shifts the image to a new position by a simple motion of translation. Inversion also involves translation. Consider the following plane figures where inversion is involved. Because of the inversion BO = OB' and CO = OC'. Angles BOC and B'OC' are vertical angles which means that tri-



angles BOC and B'OC' are congruent. Therefore, angle BCO = angle OC'B'. Since the alternate interior angles formed by the transversal CC' are equal, the lines CB and C'B' are parallel. In a similar way AB, A'B' and AC, A'C' are parallel. Now if the triangle A'B'C', produced by an inversion of triangle ABC with respect to O, is translated along AA' until A lies on C'B' and A' lies on CB we have the following figure. The translation will not alter the sides

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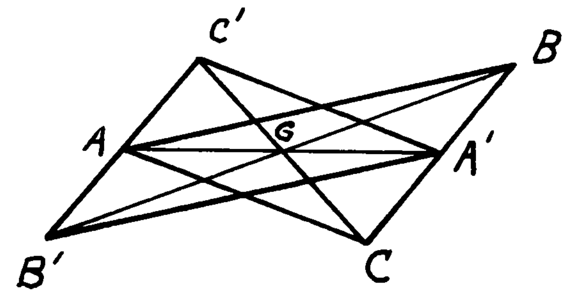
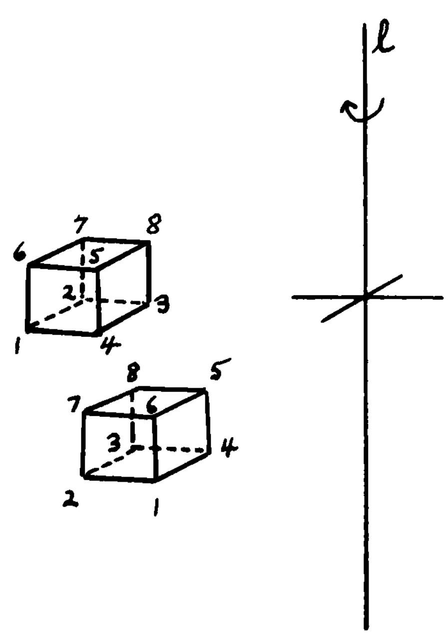
[1]Wheeler P. Davey, A Study of Crystal Structure and Its Applications, (New York,) 1934, p. 213.



from their parallel positions. This results in the three pairs of corresponding sides of the triangles meeting in three ideal points. Desargues' theorem[1] verifies that the lines BB', CC', and AA' are concurrent at G. Hence, the triangle A'B'C' is still produced by an inversion. The translation shifted the center of inversion from O to G. The revolution of a cube about an external axis produces a figure that is different only by translation from a figure produced by rotation about an axis through the geometrical center of the cube. A rotation of the cube 90° about a



vertical axis through the geometric center leaves the cube in a position such that a translation will cause the cube to

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[1]William C. Graustein, Introduction to Higher Geometry, (New York,) 1948, p. 23.

coincide, in accordance with the numbered corners, with the position of the cube after a revolution of 90° about axis ℓ. Translation is also involved in the helical motion of certain kinds of symmetry. In this case there is rotation combined with translation. It is best described by using the idea of a screw axis.[1] The spiral staircase is a good example of the screw axis. Many very interesting examples are provided by nature in leaf arrangement.[2] For instance, corn is two ranked, sedge three ranked, mint four ranked and poplar five ranked. The presence of translation in many cases of symmetry makes it possible to simplify the mechanics of symmetry by omission of the motion of translation. A figure will remain fixed in space during characteristic symmetric actions. The mechanics of symmetry will be unaffected by the addition of a translation if in the final results a translation is necessary.

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[1]Davey, op. cit., p. 212.

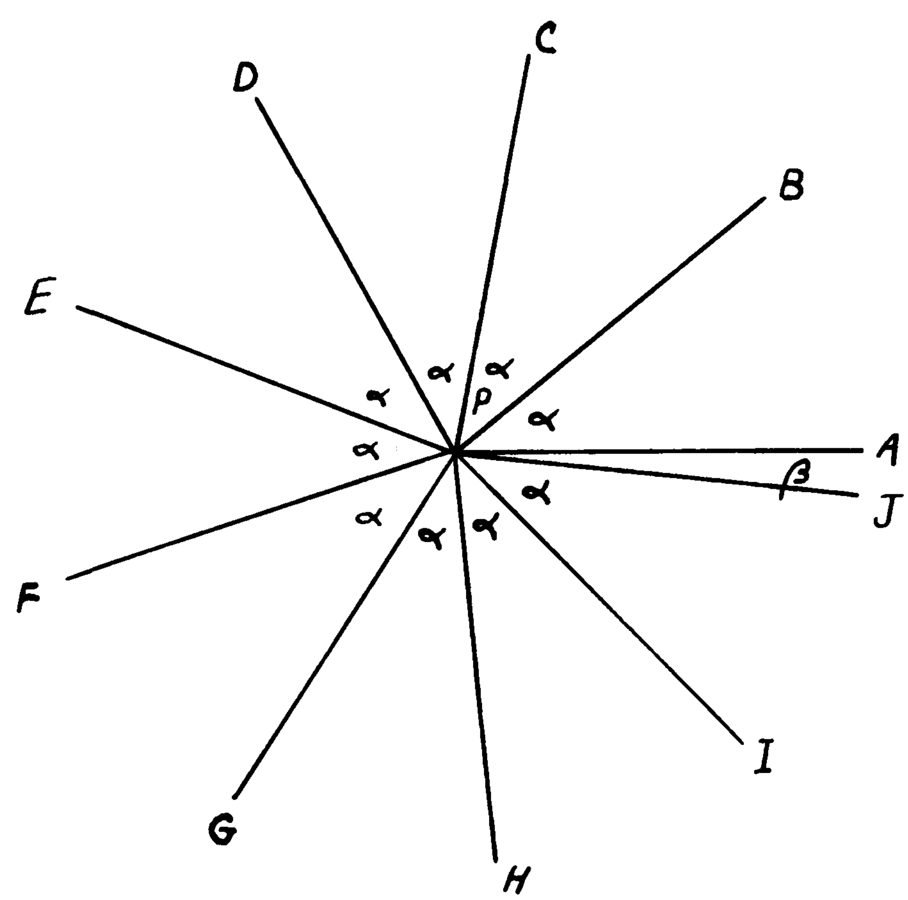
[2]Mary Stuart MacDougall, Biology: The Science of Life, (1943,) p. 135.

CHAPTER IV

THE FIRST-CLASS SYMMETRY AXIS

Rotational characteristic symmetric action is the most common symmetry development. There is also less complication in the mechanics of this kind of symmetry. Consideration of a figure static in space will be in accordance with the technique of chapter III. We may then consider the figure as rotated about an axis through its geometric center. The axis is an important factor in this case since it defines the location and direction of the figure. Such an axis can be referred to as a <u>rotation axis</u>. Likewise, the rotatory motion is a major factor for symmetry about the rotation axis. It is defined by some angle α about the axis. In order to distinguish this axis from the type involving a reflection, a name and symbol are necessary. We shall call the axis a <u>first-class symmetry axis</u>. Because rotation is the most important idea involved, we shall use a capital R to designate the axis.

The angle of rotation can be designated by α. It is defined as the smallest angle through which a figure must be rotated about the axis R in order to coincide with itself. The multiples 2α, 3α, 4α, ... $n\alpha$ will likewise cause the figure to coincide with itself, provided the angle α is a divisor of 360°. A proof of this is as follows. Consider the plane section of a first-class axis which is perpendicular to the axis R at P. Take as the initial position some line PA of the figure. After each rotation through α the figure should be in a position to coincide with itself. If this were not so, then the line



would take a position PJ with an angle β less than α between PJ and PA. However, this is impossible since α was defined to be the smallest angle which will cause the figure to coincide with itself. Thus, the angle β is zero and $\alpha = \frac{2\pi}{n}$, where n = 1, 2, 3, 4, ... ∞ and determines the period of the axis.

The value of n in $\alpha = \frac{2\pi}{n}$, for any particular case, can be defined as the period number of a first-class symmetry axis. The period number can have values between 1 and ∞. The value of ∞ is established when α = zero. For the first-class symmetry axis the period number determines what has been called twofold, threefold, fourfold, etc. kinds of symmetry. That is, a period number of 2 represents twofold symmetry and a period number of 3 represents threefold symmetry. Considering the period number for any particular case, a better description of a first-class symmetry axis can be given. Suppose that the period number is 2. Then the symmetry axis can be called a binary symmetry axis of the first class. If the period number is 3, we have a ternary symmetry axis of the first class. Increasing period numbers will produce quaternary, quinary, senary, etc. symmetry axes of the first class.

There is, of course, the possibility that a symmetric figure will have more than one rotation axis passing through its geometric center. It will be shown in the chapter on reflecting planes that the two axes are equivalent to a third axis through their intersection. Because this third axis is equivalent to the other two axes, it may be considered alone as the first-class symmetry axis of the symmetric figure in question.

CHAPTER V

THE SECOND-CLASS SYMMETRY AXIS

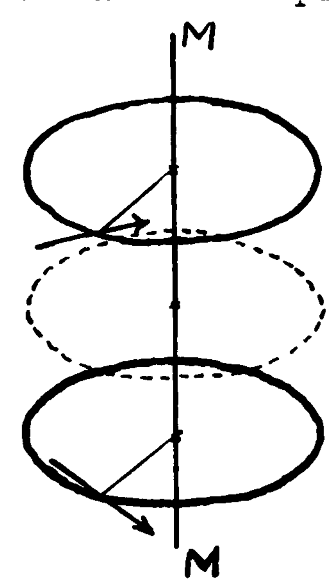
Reflective characteristic symmetric action is the second important method of symmetry development. It is less common than rotation symmetry, but more complicated. However, the mechanics of this symmetry provide many cases where simplification is possible.

A study of this symmetry can be achieved in a helpful way by considering the symmetrical figure involved as fixed in space about an axis. The situation is much the same as in the case of a first-class symmetry axis. This axis will differ from the first-class symmetry axis in that after every rotation there is a reflection in a mirror plane perpendicular to the axis. The reflection after rotation is an absolute necessity and rotation without reflection in this case will be meaningless. Because reflection is the most important factor involved, the axis can be called a mirror axis. However, we shall follow the scheme of chapter IV and call the axis a second-class symmetry axis. The mirror axis can best be designated by the capital M.

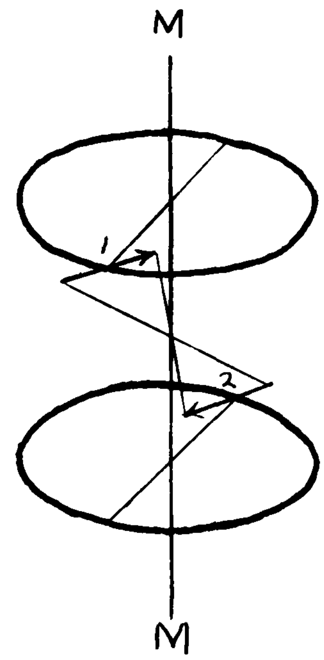
As in the case of a first-class symmetry axis, the period of the second-class symmetry axis will be defined by its angle of rotation α. Also in this case we have $\alpha = \frac{2\pi}{n}$, where n = 1, 2, 3, 4, ... ∞. The number n is the period number. Likewise, the period of a second-class axis can be used as a basis for referring to an axis as a binary, ternary, quaternary, quinary, etc. axis.

Many symmetric figures of the second-class symmetry axis can be produced by other devices which will often simplify the procedure. Sometimes a second-class symmetry axis can be replaced by a first-class symmetry axis and some other simple operation. Usually the additional operation is a simple reflection or inversion with respect to the geometric center.

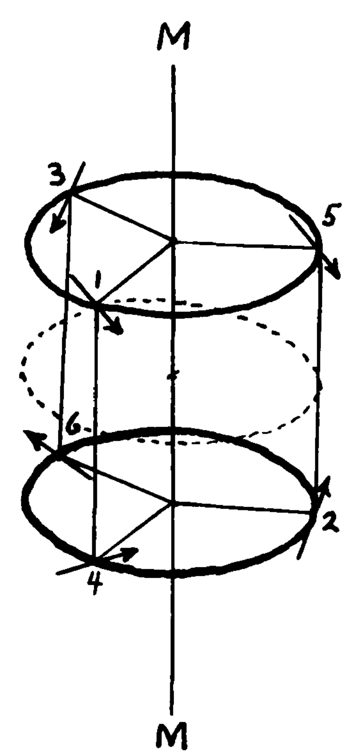
The different periods exhibit different possibilities for simplification. When the period number is n = 1, the second-class symmetry axis is equivalent to a first-class axis with a reflection in a real plane. The label M on the following figure can be replaced by R if a reflec-



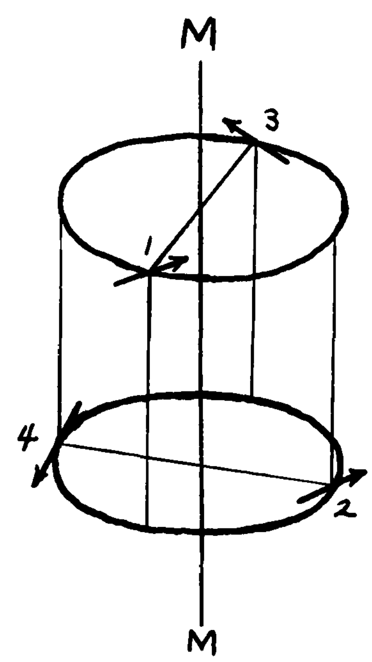
tion plane (dotted) is added. When n = 2 the second-class symmetry axis can be replaced by a simple inversion. Consider the second-class symmetry axis when n = 2. The arrow 1 is rotated 180° about the axis and reflected into arrow 2. The second-class operation can be replaced by a



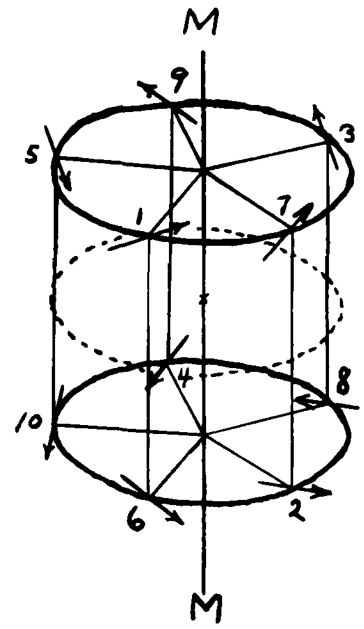
simple inversion. Consider next a ternary axis of the second class. The arrow 1 is rotated to the location of



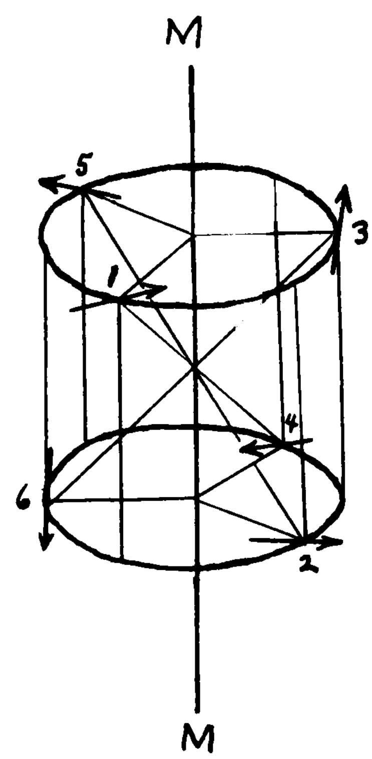
arrow 5 and reflected into arrow 2. Arrow 2 is rotated to the location of arrow 6 and reflected into arrow 3. A rotation of arrow 3 carries it to the position of arrow 1 where it is reflected into arrow 4. Arrow 4 is carried by rotation into the location of arrow 2 and reflected into arrow 5. Arrow 5 is carried to the location of arrow 3 by a rotation and is then reflected into arrow 6. A rotation carries arrow 6 to the location of arrow 4 where it is reflected into arrow 1. Inspection shows that the second-class operations can be replaced by rotations about a first-class axis and a reflecting plane perpendicular to it. When n = 4 the mirror axis can be represented by the following figure. Here a rotation of 90° followed by a reflection carries arrow 1 into arrow 2. Likewise, consecutive rota-



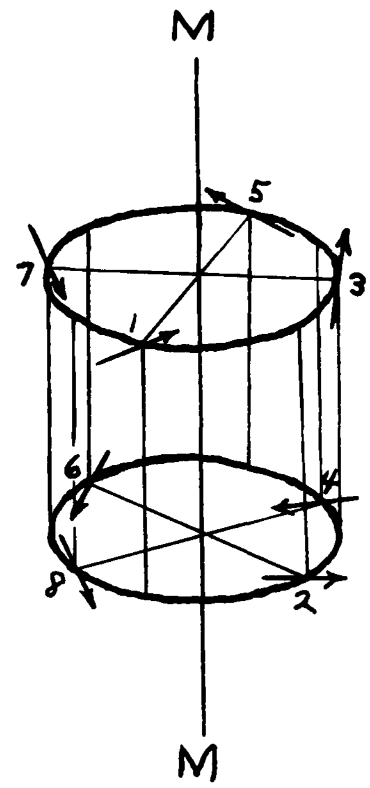
tions of 90° followed by reflections carries arrow 2 into arrow 3, arrow 3 into arrow 4, and arrow 4 back to arrow 1. Unlike the cases for n = 1, 2, 3, there is no substitute for the second-class operation when n = 4. Therefore, the symmetry can be described only by using a second-class symmetry axis. For n = 5 we have the following descriptive figure. It is obvious that the case of n = 5 is similar to



the case of n = 3 and can be replaced by a first-class axis and a reflection plane perpendicular to the axis. The second-class symmetry axis with a period number n = 6 can be replaced by a ternary first-class symmetry axis combined with an inversion. Illustration of this is evident in the representative figure.



The period number n = 7 provides a case similar to the case for n = 3 and n = 5. The second-class symmetry axis can be replaced by a first-class axis and a reflection plane. Examination of the figure for n = 8 will show that it is similar to the case for n = 4.



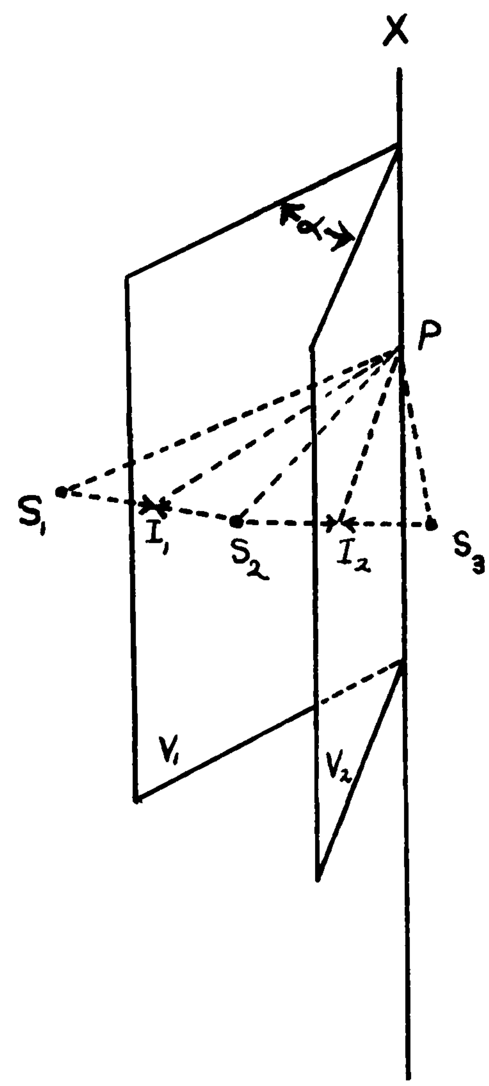
Continued examination of the second-class symmetry axis for additional period numbers will show the same pattern of behavior as demonstrated by the numbers 1 through 8. We can generalize this behavior. When the period number n of a second-class symmetry axis is an odd number, the axis can be replaced by a first-class symmetry axis and a reflecting plane perpendicular to the first-class axis. Two cases arise when the number n is an even number. If n is divisible by 4, the second-class axis cannot be replaced by any substitute method. The remaining case is that of n not divisible by 4. This means that $\frac{n}{2}$ is an odd number since $\frac{n}{4} = \frac{n}{2 \cdot 2}$ and the quotient for $\frac{n}{2}$ cannot be divided again by 2. Thus, the second-class axis in this case can be replaced by a first-class axis with period number $\frac{n}{2}$ and the addition of an inversion.

CHAPTER VI

REFLECTION PLANES

In the discussion of the mechanics of symmetry only one reflecting plane was considered. Actually this is sufficient for our study of symmetry. The cases of more than one plane can be reduced to the study of a symmetric figure about either a first or second-class symmetry axis.

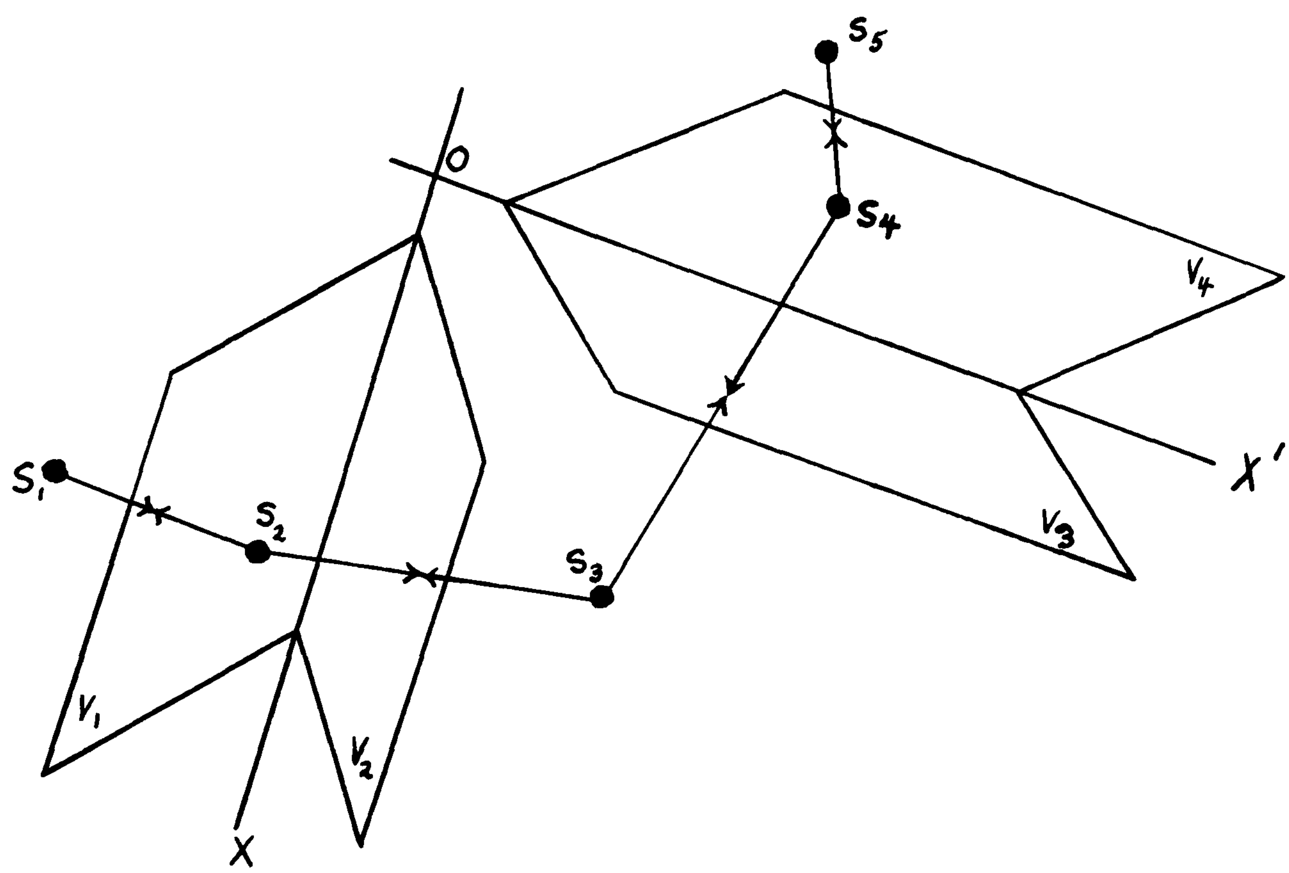
The first case to be considered is that of two reflecting planes intersecting in a line X. The angle between the planes is α and the planes are labeled V_1 and V_2.



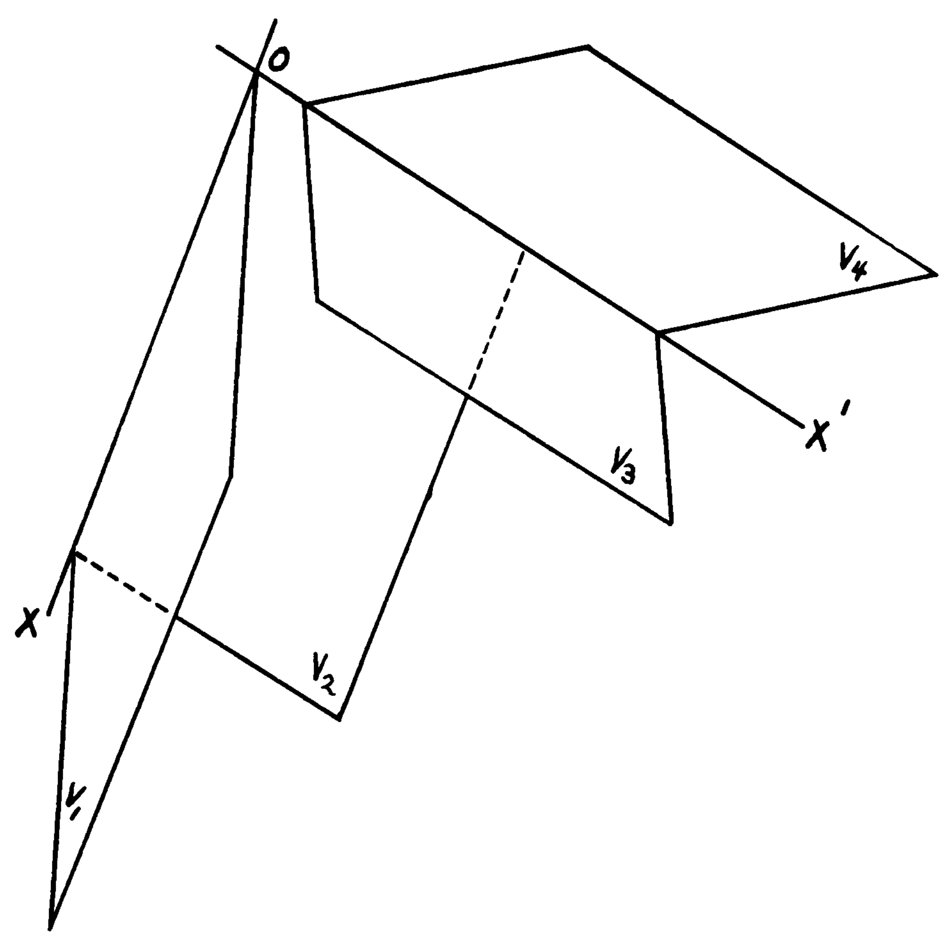
To examine the combined action of the two reflection planes an element S_1 of a symmetric figure will be considered.

The reflection of S_1 in the reflection plane V_1 produces the image S_2. However, S_2 is the object for reflection in the reflection plane V_2 and this reflection produces the image S_3. Because the image S_3 is the result of a reflection of a reflection, S_3 will be the same as S_1 except for position. It is clear that this change in position can be achieved by a rotation about the axis X. Angles S_1PI_1 and S_2PI_1 are equal because of the nature of reflection. In the same way angles S_2PI_2 and S_3PI_2 are equal. Since, from the figure, the sum of angles S_2PI_1 and S_2PI_2 is α, the sum of angles S_1PI_1 and S_3PI_2 is equal to α. Therefore, the rotation to carry S_1 into S_3 must be through $\alpha + \alpha$ or 2α. Specifically, this means that the combined reflection in two reflection planes separated by an angle α can be replaced by a rotation of 2α about the line of intersection as an axis. This gives a situation which can be described as a first-class symmetry axis. Another phenomenon of the two intersecting reflecting planes is that of rigid rotation. If the two planes are kept in a rigid relation to each other so that the angle α between them remains constant, they may be rotated about their intersection X without affecting the result of the reflection in any way. The total of angles S_1PI_1 and S_3PI_2 remain equal to α even though their relative sizes change. Thus, the positions of S_1 and S_3 are unaffected by the rigid rotation of planes V_1 and V_2. This kind of operation will play an important part in simplification of the next case.

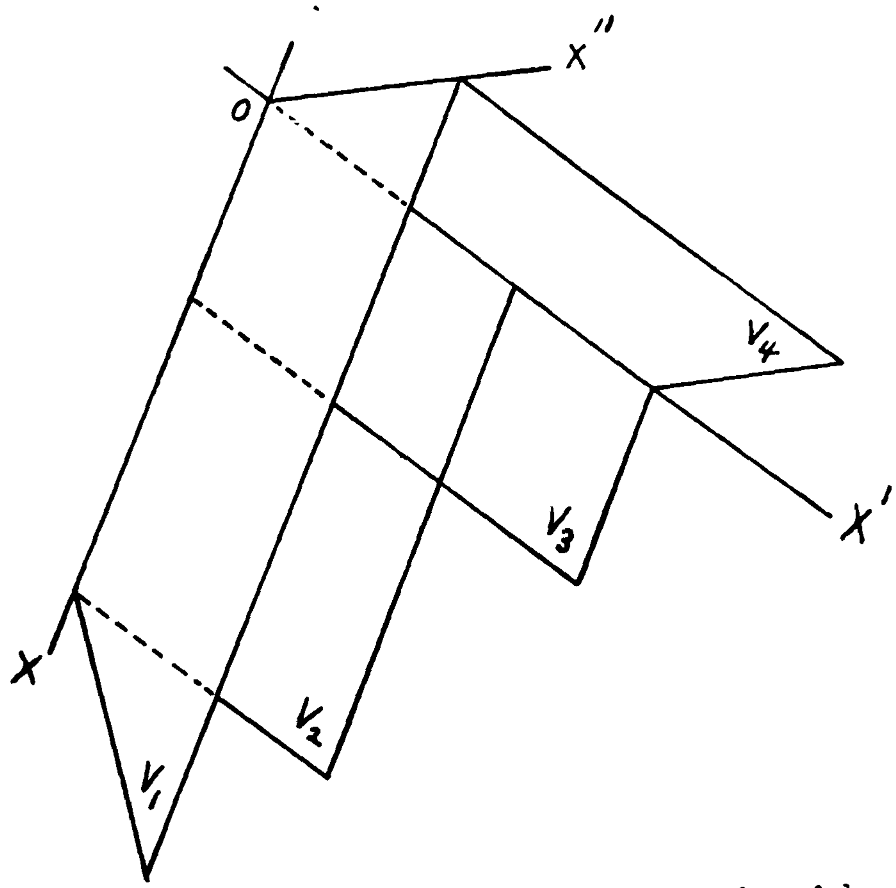
The second case for consideration is the combined reflection in four planes say V_1, V_2, V_3 and V_4 which pass through a common point O. The line of intersection of V_1 and V_2 is OX and the intersection of planes V_3 and V_4 is line OX'. The four planes represent a double case of two intersecting reflecting planes with one set operating on the results of the other set. The element S_1 of a sym-



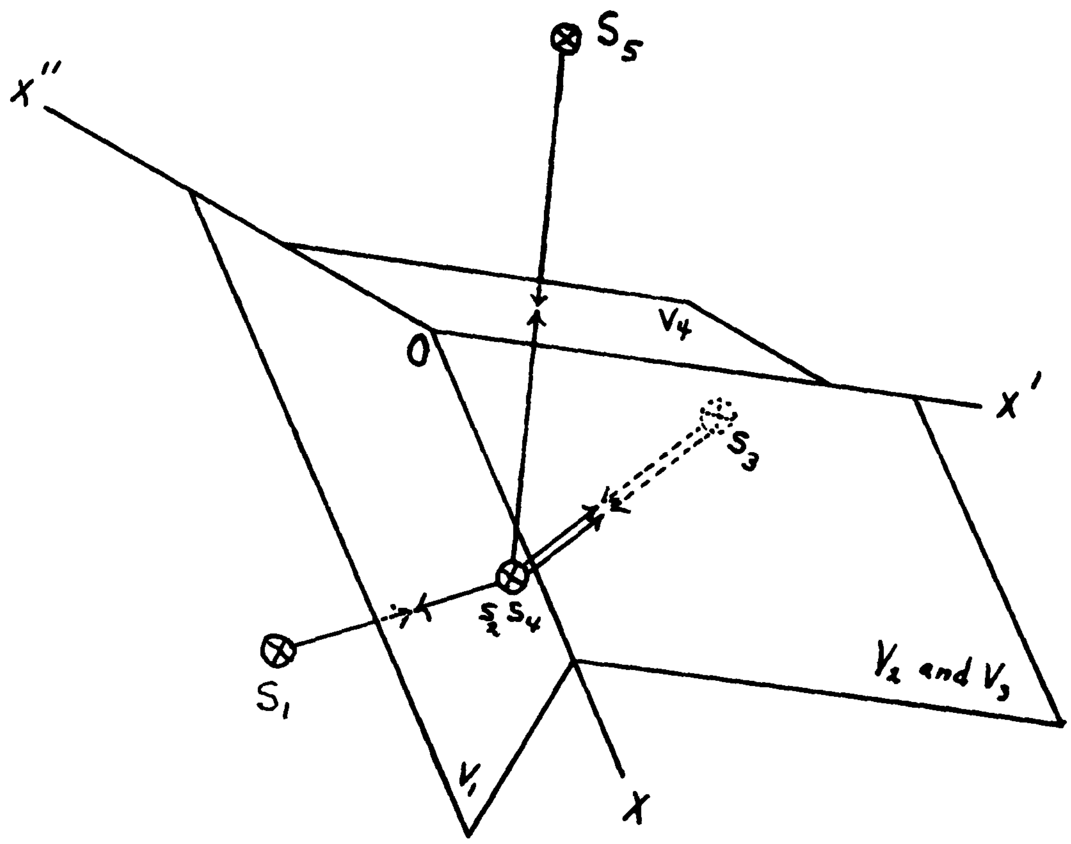
metric figure when reflected in V_1 produces the image S_2. Reflection of S_2 in V_2 produces S_3 which is the image of S_2 and therefore S_3 is the same as S_1 except for position. But S_3 is now reflected in V_3 and the resulting image S_4 is reflected in plane V_4 to give S_5. The number of reflections are even, which makes the final result S_5 the same as S_1 except for position. This reflective action in four planes can be reduced to a reflective action in two planes without affecting the results. To prove this we recall that a set of intersecting reflecting planes can be rotated together without affecting the results if the angle between the planes is kept constant. Rotate the planes V_1 and V_2 rigidly about OX until OX' lies in the plane V_2. Since the angle between plane V_1 and plane V_2 remains con-



stant the results of the reflections remains unchanged. The next step is to rotate the planes V_3 and V_4 about the axis OX' until the line OX lies in the plane V_3. As in the rotation of planes V_1 and V_2, the rotation of planes V_3 and V_4 must be performed so as to keep the angle between them constant. Therefore, this rotation does not affect the results of the reflections. The following figure



illustrates that the planes V_2 and V_3 now coincide with the plane of OX and OX'. The outcome of the reflections after the total rotations about OX and OX' remains unchanged but the reflections in planes V_2 and V_3 now neutralize each other in the way illustrated by the following figure.



The combined reflections in four planes can now be considered as a reflection in two planes V_1 and V_4 which intersect in the line OX". Now that reflections in only two planes are established, the reflections can be replaced by a rotation about OX" equal to twice the angle between planes V_1 and V_4.

We can now establish the proposition of chapter IV: that rotations about two intersecting axes are equivalent to a rotation about a third axis through their point of intersection. In the consideration of four reflecting planes the reflections in planes V_1 and V_2 can be replaced by a rotation about OX as an axis. Likewise, the reflections in planes V_3 and V_4 can be replaced by a rotation about OX' as an axis. However, it has been established that the combined reflections in the four planes are equivalent to a rotation about OX" as an axis. Thus, the rotations about OX and OX', taken together, may be replaced by a rotation about OX" and the proposition is established. This theorem is Euler's rotation theorem.[1]

The third case of reflecting planes to consider is that of combined reflection in three planes. If the planes have a common line of intersection, two of the planes may be rigidly rotated until one of the planes coincides with the nonrotated plane. This causes two

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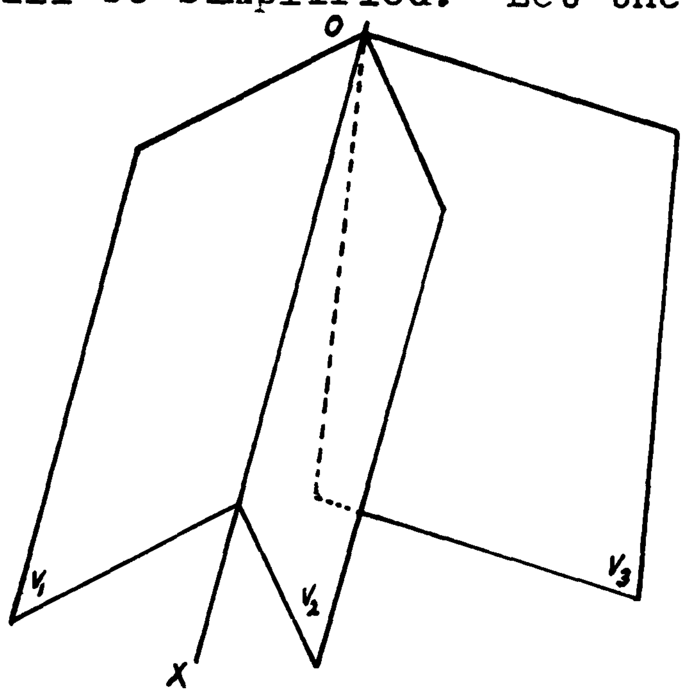
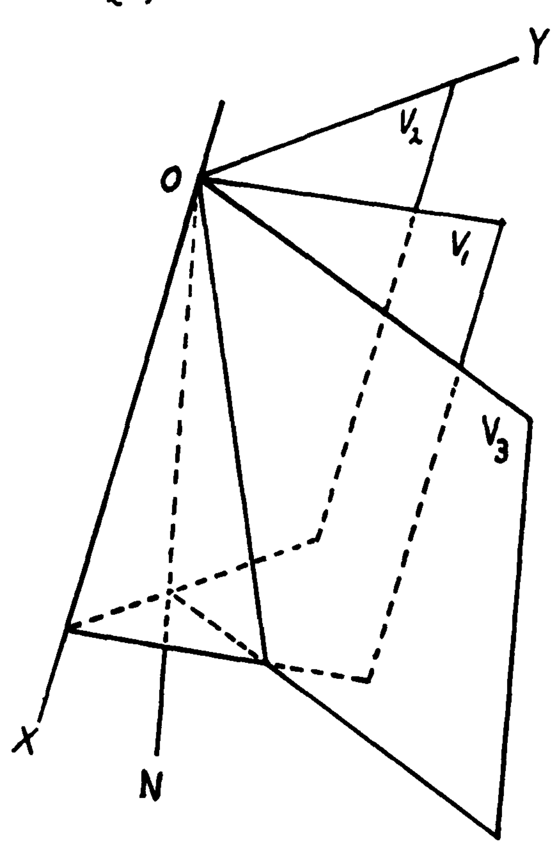
[1] H. S. M. Coxeter, Regular Polytopes, (London,) 1948, p. 37.

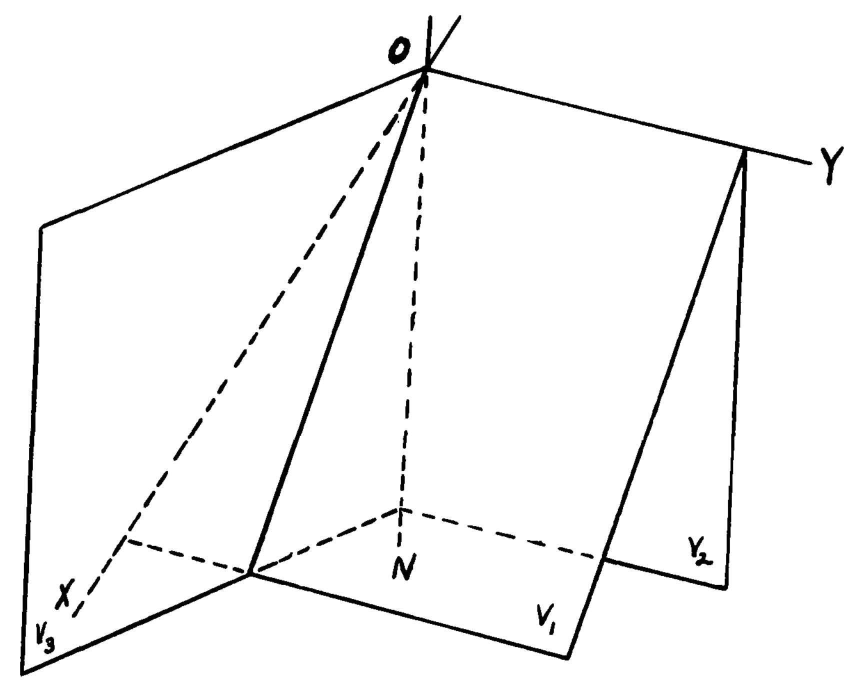
reflections to neutralize each other and the case becomes one of reflections in two reflecting planes. If the three reflecting planes do not have a common intersection line, the case may still be simplified. Let the planes be V_1, V_2,



and V_3. Rotate the planes V_1 and V_2 about the line OX, keeping the angle between them constant, until the plane V_2 is perpendicular to the nonrotated reflection plane V_3. The line OY in plane V_2, of the following figure, will now



be perpendicular to the line ON which is the intersection of planes V_3 and V_2. This operation does not affect the original reflections. Now rotate planes V_2 and V_3, keeping the 90° angle between them constant, about the line ON until plane V_3 is perpendicular to plane V_1. This operation will



also cause no change in the reflections, but the planes V_1 and V_2 are now both perpendicular to the plane V_3. As a consequence of this arrangement the reflections in the three planes can be replaced by a rotation about the intersection of planes V_1 and V_2 followed by a reflection in a plane V_3 perpendicular to the intersection of V_1 and V_2. The intersection of V_1 and V_2 is OY, which is perpendicular to plane V_3. Therefore, the combined reflections in three planes, not having a common line of intersection, are equivalent to a rotation about an axis combined with a reflection in a plane perpendicular to the axis. This situation amounts to the setup for a second-class symmetry axis, and a symmetric figure under consideration would be changed into its mirror image.

We can now generalize the results of reflections in more than one reflection plane. Generally, if the number of reflection planes is even, the reflections can be reduced to the case of two planes and hence to a rotation. It is symmetry of a first-class symmetry axis. If the number of reflecting planes is odd, the reflections can be reduced to the case of three reflection planes and hence to a rotation followed by a reflection in a plane perpendicular to the axis of rotation. It is symmetry of a second-class symmetry axis.

CHAPTER VII

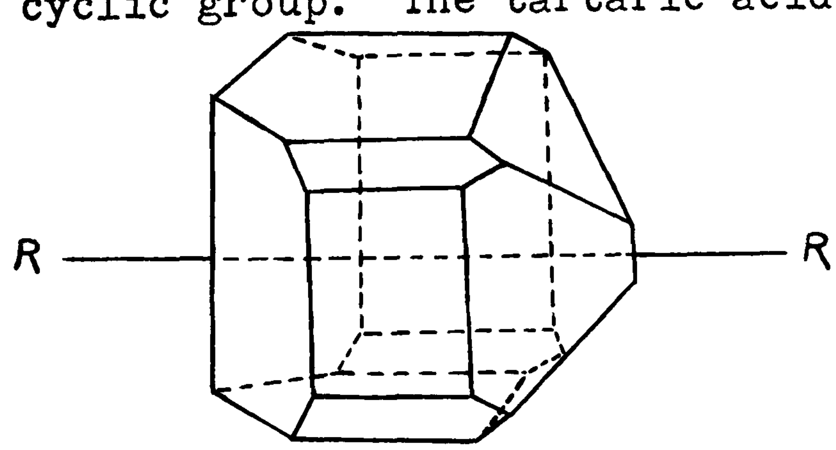
CLASSIFICATION OF SYMMETRIC FIGURES

The material in the previous chapters describes the mechanics of symmetry in terms of the fundamental aspects of rotation and reflection. Now the problem is to demonstrate methods for describing and classifying symmetric objects together with examples of this classification. Prerequisite to the problem is a definition concerning the positions of coincidence for symmetric objects. If a cube is rotated 90° about the perpendicular bisector of two opposite faces, it will occupy the same place in space as before. However, the cube is not the same as before. The corners of the cube are interchanged. This kind of coincidence occurs after every rotation of 90°. A total of 360° results after the fourth rotation and the cube is in identically the original position. Thus, we may define a symmetric action which carries an object to its original position as an identity-motion. Symmetric motions which cause an object to occupy the same place in space but not to coincide with the original position are defined as non-identity-motions.

First consideration of symmetric objects will be those of the first-class type. The most common of this type are symmetrical objects having one axis with a period of $\frac{2\pi}{n}$. Rotation is the impetus for this symmetry. The symmetry is entirely of a cyclic nature. For this reason we shall refer to objects belonging to this group as members of the cyclic group. A member of the group can be classified by the symbol C_n where C stands for the cyclic group and n is the period number which identifies the period of rotation for the object.

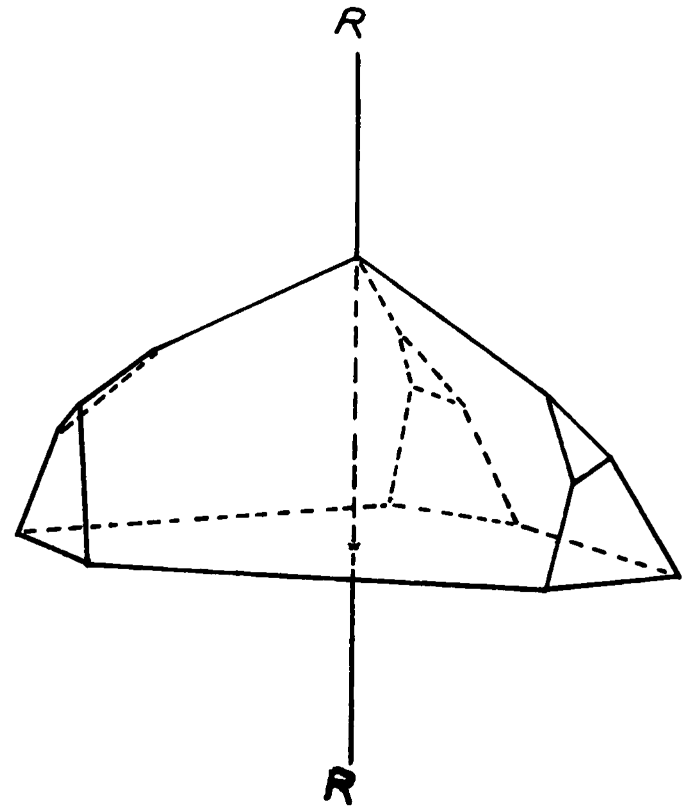
Examples of the cyclic group are very common. Perhaps the most numerous are the flowers of many plants. Illustrations by Rendole[1] provide ample evidence of the cyclic nature in the flowers of plants. The flower of the Eriocaulon septangulare is an example of C_2 and the flower of Rhubarb is C_3. The male flower of the Black Mulberry is symmetry C_4, and the floral diagram of Tibouchina shows it to be symmetry C_5. Plant fruit and seed also often exhibit symmetry of the cyclic type. The fruit of Illicium anisatum has symmetry C_{13}. Swietenia mahagoni, a mahogany tree of the West Indies, provides an example of seed with symmetry C_3.

In the lifeless world, many crystals can be classified in the cyclic group. The tartaric acid crystal is an

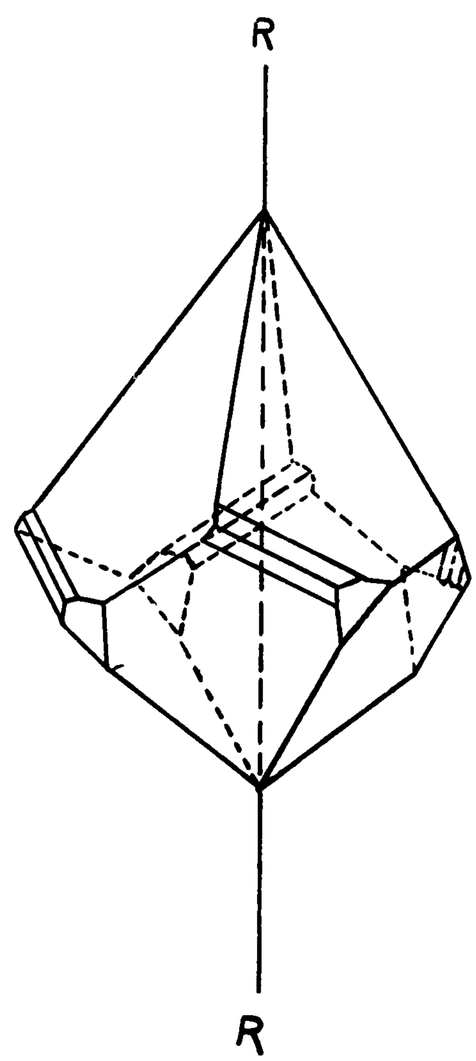


[1]Alfred Barton Rendole, The Classification of Flowering Plants, 2 Vols. (Cambridge), 1925.

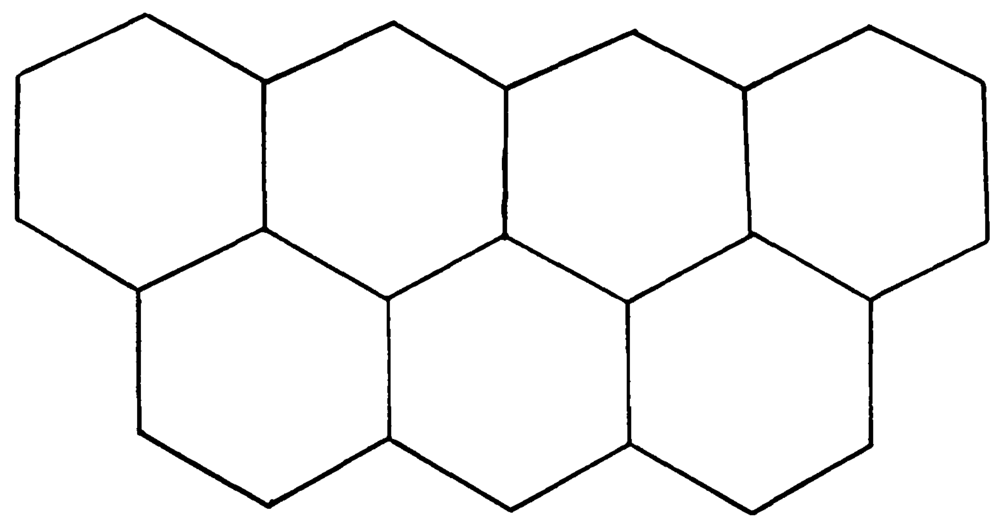
example of C_2. Crystals of sodium metaperiodate trihydrate, $NaIO_4 \cdot 3H_2O$, have symmetry C_3, and symmetry C_4 is the classi-



fication of wulfenite, $PbMoO_4$.

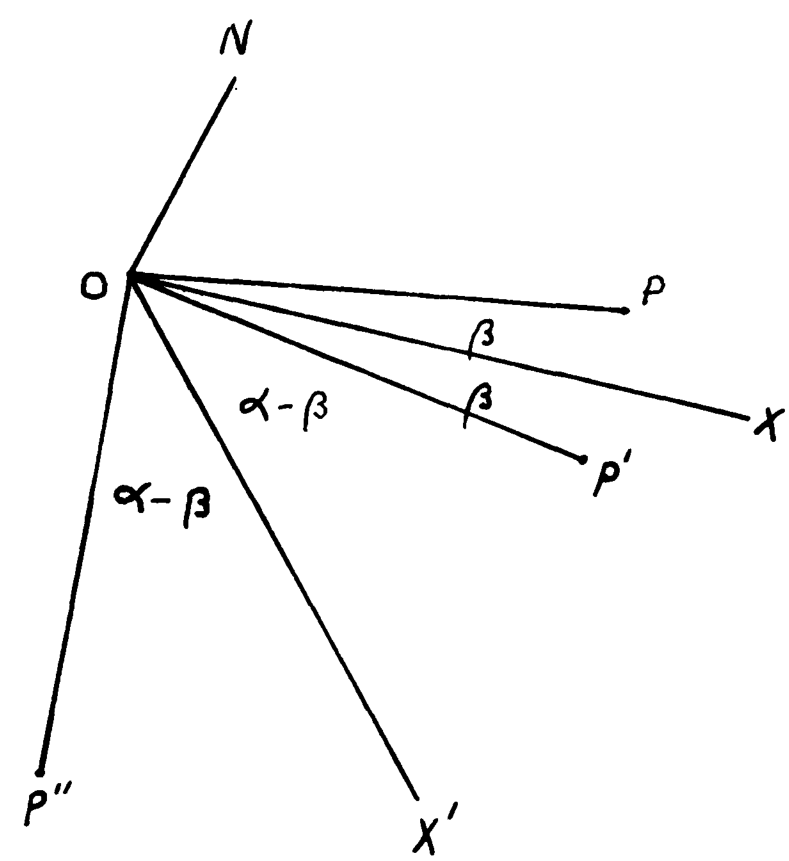


Geometric designs used as a form of art to ornament flooring, wallpaper, ceilings, etc. are usually of the cyclic group. The following design can be said to consist

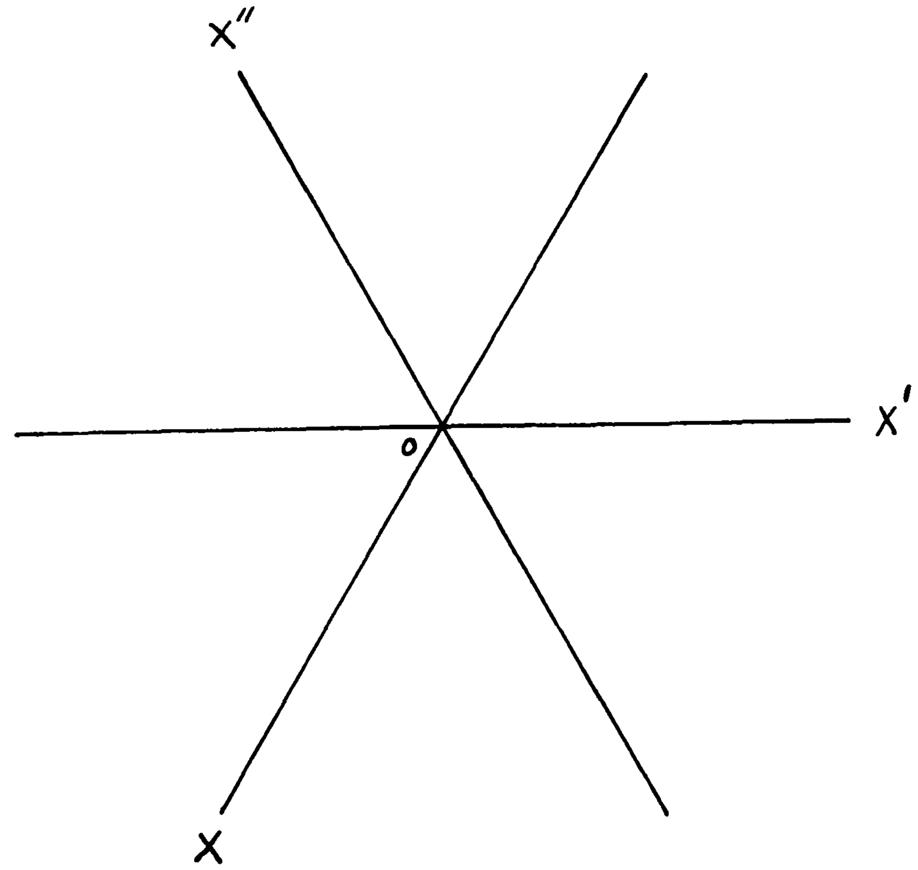


of figures having symmetry C_6.

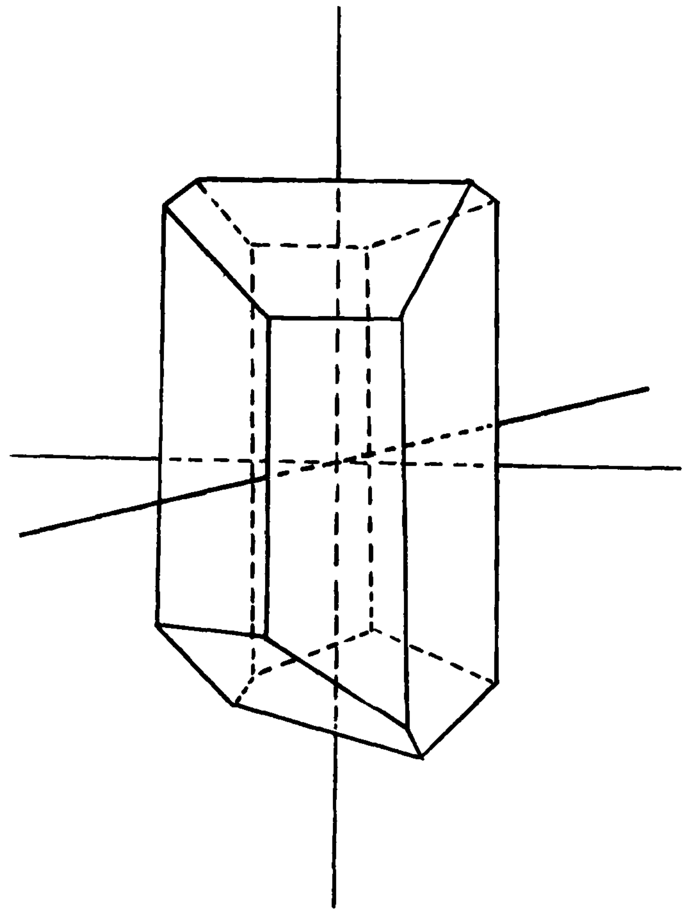
The cyclic cases are symmetries with one axis. Cases of more than one axis also exist. Let us consider the case of a symmetric figure having two intersecting binary axes. Previous conclusions lead us to believe that a third



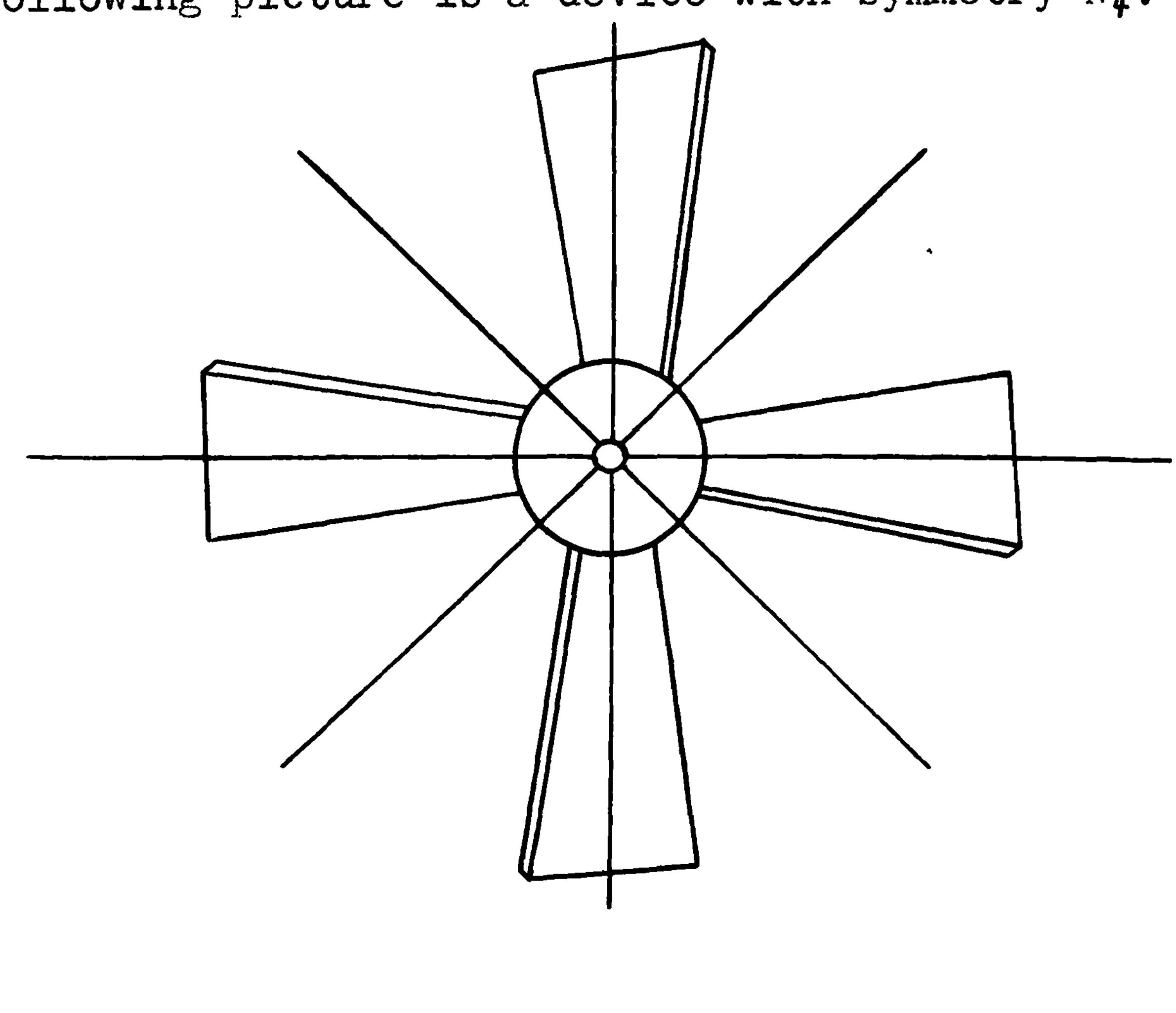
axis is present in the figure which will describe symmetry. Suppose that the two binary axes are OX and OX'. Let the angle between the axes be α and ON be a perpendicular to the plane of the axes at O. Because both X and X' are binary axes, a rotation about OX of 180° will invert ON. However, the rotation about OX' of 180° will also invert ON. This amounts to an identity-motion, and ON remains in the same place. Thus, ON is an axis of symmetry about which rotation will occur. What is its period? To find the period of ON, an element P of the figure will be considered. Rotation about OX carries P into P', and P' is carried into P" by the rotation about OX'. Let the angle POX equal β. Then angle XOP' is also equal to β. Angle XOX', the angle between the binary axes, is α. This means that angle P'OX' is $\alpha - \beta$, and angle X'OP" must also be equal to $\alpha - \beta$. Using these values we find that the sum of angles POX and X'OP" is $(\alpha - \beta) + \beta$ or α. Therefore, angle POP" is $\alpha + \alpha$ or 2α, and n determines the period of ON where $2\alpha = \frac{2\pi}{n}$. The number n is the number quantity of binary axes involved and the angle between any two of them is $\alpha = \frac{\pi}{n}$. Since rotation about ON may be other than two fold, the value of n can be greater than 2. In general, we can refer to this kind of symmetry as a group which is characterized by two or more binary axes in a plane normal to an axis through their intersection. A good symbol for this group is N_n since there is always an axis normal to the binary axis. Thus, n is the number of binary axes and determines the period of the n-fold axis. If the ends of the binary axes are thought of as poles, then a sense of polarity prevails according to whether n is even or odd. If n is even, then it is obvious that either end of a binary axis will coincide with either end of another binary axis. This is not true if n is odd. Consider the case of N_3 where n = 3 and ON is normal to the plane of the pa-



per. Only the poles of the axes marked with an X coincide after rotation about ON. A convenient reference to this kind of axes may be devised. If n is even, the axes may be called bipolar, and if n is odd, the axes may be called monopolar. For an example of symmetry of the group N_n we may look to the field of crystallography. Therefore, 1-brom, 2-hydroxy-naphthalene represents a crystal of symmetry N_2.



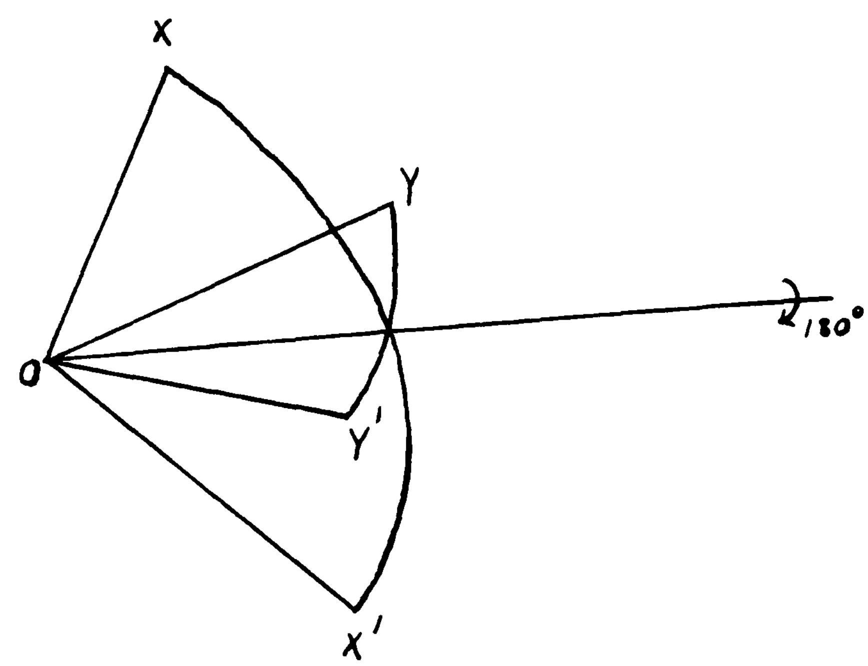
Many kinds of man made objects exhibit symmetry of the N_n group. For instance, certain kinds of propellers with flat blades can be classified in this group. The propeller in the following picture is a device with symmetry N_4.



The axis normal to the binary axes is the center of the axle on which the propeller turns.

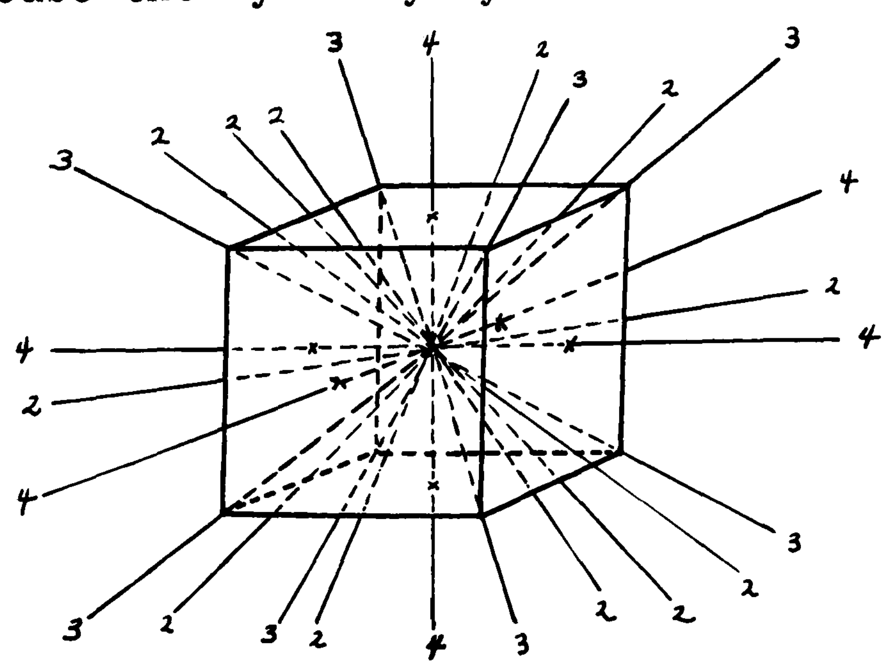
Our next symmetric group for consideration is the case of a symmetric figure with two axes having periods greater than 2. Let the periods of the axes be $\frac{2\pi}{n}$ and $\frac{2\pi}{m}$. The rotation of the axis having period $\frac{2\pi}{n}$ will produce n positions of the axis having period $\frac{2\pi}{m}$. Likewise, rotation of the axis having period $\frac{2\pi}{m}$ will produce m positions of the axis having period $\frac{2\pi}{n}$. This means that the symmetric figure has n axes with period $\frac{2\pi}{m}$ and m axes with period $\frac{2\pi}{n}$.

The presence of two axes indicates there should be a third equivalent axis in the figure. Let the axes with periods $\frac{2\pi}{n}$ and $\frac{2\pi}{m}$ be OX and OY. If the figure is rotated

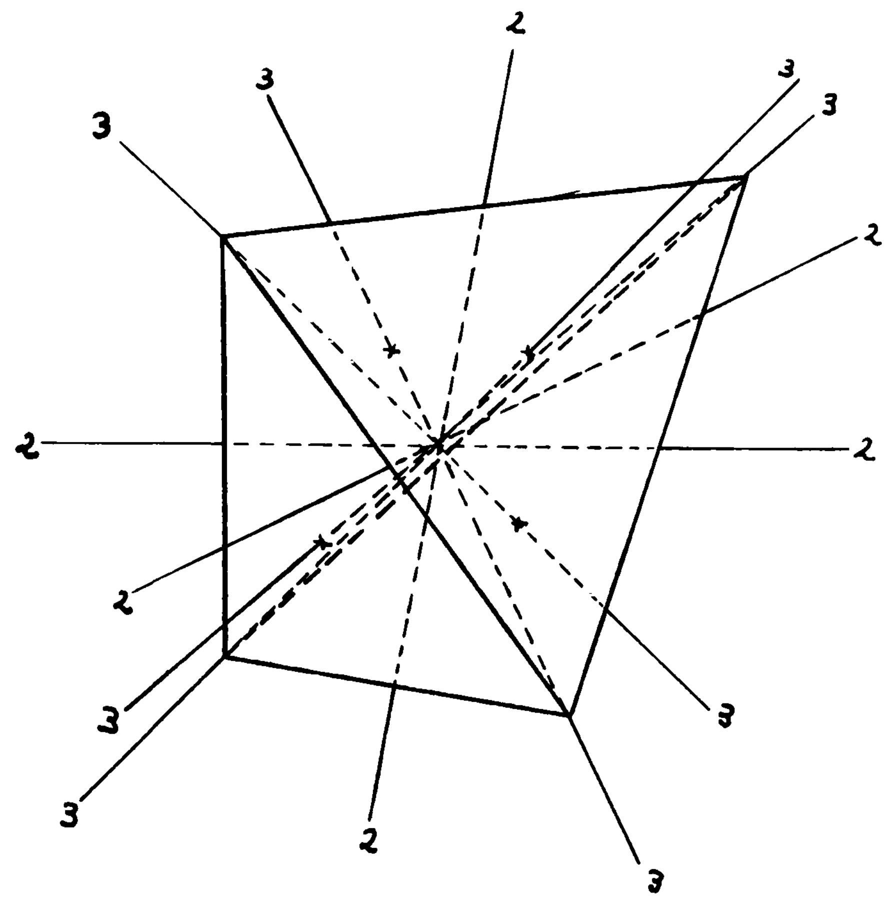


about OX, OY is carried into an equivalent axis OY' and OX remains unchanged. In a similar way, rotation about OY carries OX into an equivalent axis OX' leaving OY unchanged. Reference to the figure shows that the results of the two rotations taken together can be replaced by a binary axis. If the arcs XX' and YY' are considered parts of great circles of a sphere with center at O, the location of the binary axis is obvious. The binary axis is located midway between two axes having the same period.

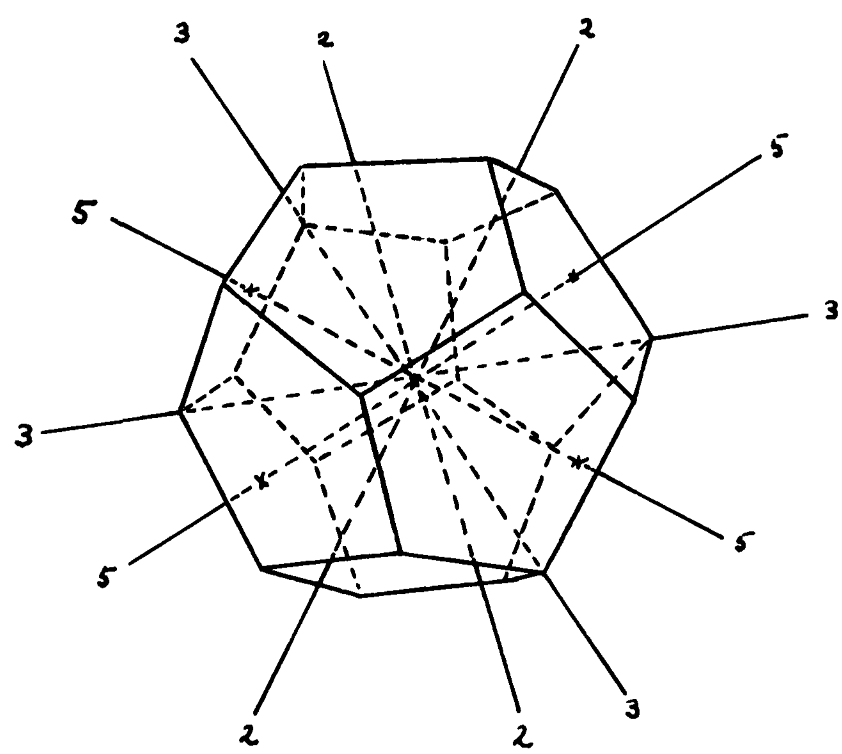
Consider, again, a sphere about O. All the axes for each set of periods intersect the sphere in regular and finite numbers. This distribution over the surface of the sphere establishes directional relations with inscribed regular polyhedrons. Thus, the symmetry of this group is limited because only five regular polyhedrons exist. They are the regular hexahedron or cube, tetrahedron, octahedron, dodecahedron, and icosahedron. However, the axes of these five solids are not all different. For instance, the regular octahedron has the same axial arrangement as the cube. Therefore, the symmetry under discussion can be compared with particular regular polyhedrons. With reference to the cube the symmetry symbol will be a capital K.



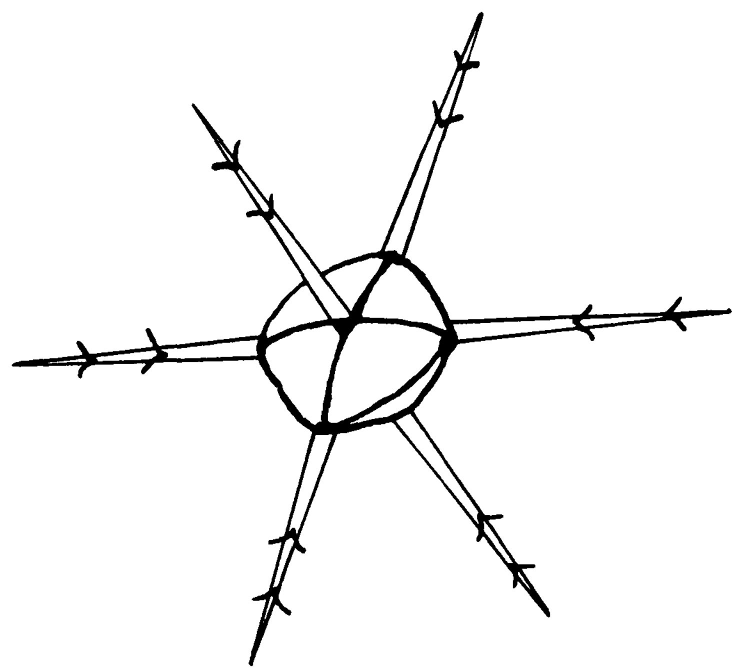
A good symbol for objects having the symmetry of the tetra-



hedron is the capital T. The symmetry of the regular dodecahedron can be designated by a capital P. To avoid too many lines the following figure displays only a few of the



fifteen binary axes, ten ternary axes, and six quinary axes present. Excellent examples of the K, T and P symmetries are found in life forms among the radiolaria.[1] In particular, the circoporus octahedrus is a beautiful example of symmetry K.



Crystalization provides us with an example of T symmetry in the form of a crystal of tetrahedrite.[2]

There is much similarity between symmetry descriptions of the first and the second-class symmetry objects. As already described, many second-class operations are equivalent to a first-class operation followed by some additional operation. Consider the cyclic rotations about a second-class symmetry axis. Because of the similarity, we can use the same symbol C_n of the cyclic group with one

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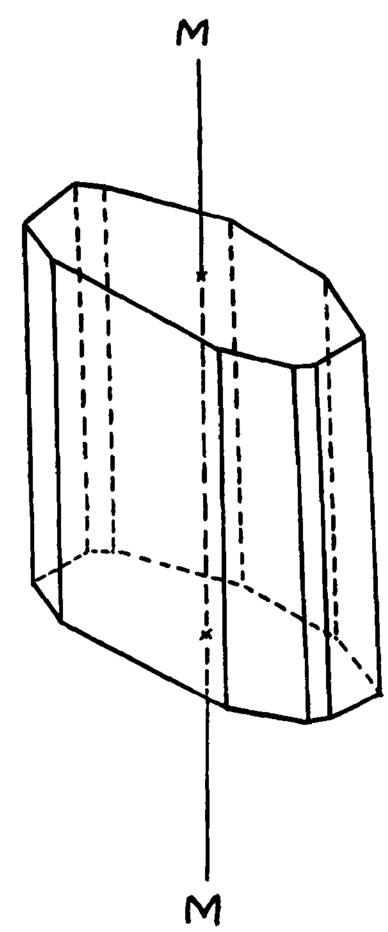
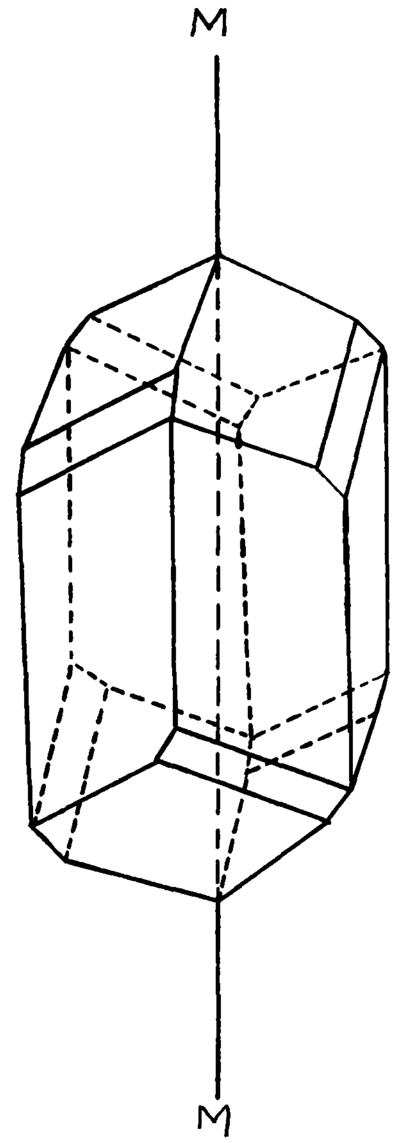
[1] Richard R. Kudo, Protozoology, (Springfield,) 1947, pp. 421-425.

[2] Sir Lawrence Bragg and W. H. Bragg, The Crystalline State, (London,) 1949, p. 71.

exception. Indication that the axis is a second-class axis is accomplished by placing a bar over the symbol C_n giving $\overline{C_n}$. The number n establishes the period of rotation. Since this is a second-class operation, after every rotation a reflection must occur. For the symmetry $\overline{C_1}$ the value n = 1 means an identity-motion. Actually this amounts to no rotation. The symmetry is simply a reflection in a reflecting plane. The symmetry $\overline{C_1}$ describes a great many living things. It is the symmetry of the oak leaf in the vegetable kingdom and of man in the animal kingdom. Only an inversion is necessary to describe symmetry $\overline{C_2}$. An example is the copper sulphate crystal, $CuSo_4 . 5H_2O$, illustrated in the following figure.

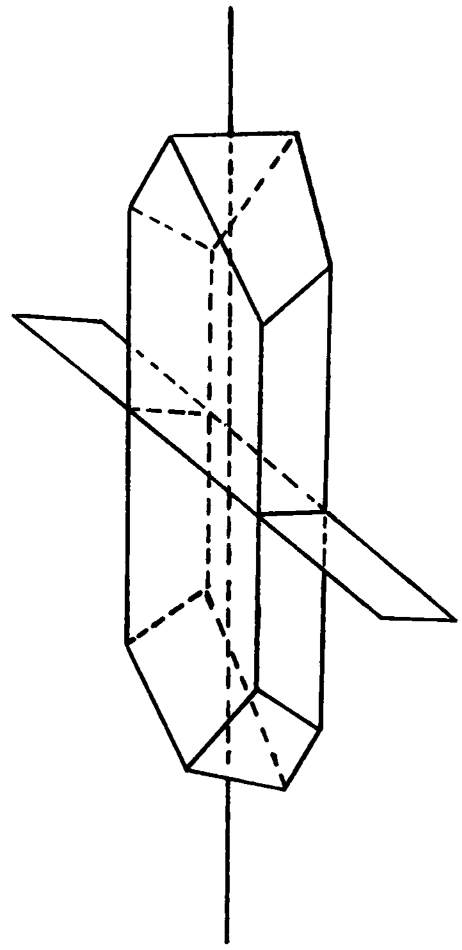


Another example of second-class symmetry is dioptase, CuH_2SiO_4. The crystal is an example of symmetry $\overline{C_6}$.



In accordance with chapter V the crystal is an object of symmetry C_3 combined with inversion.

In some symmetric forms a first-class axis is combined with reflection planes. A plane in this case may be perpendicular to the axis or the axis may lie in the plane. Figures falling into these two classes can be classified by the symbols VC_n and HC_n where VC_n refers to a vertical plane or a plane containing the axis and HC_n refers to a horizontal plane or a plane perpendicular to the axis. There are many crystals and living forms which serve as examples for the symmetries VC_n and HC_n. The symmetry of paraquinone is an example of HC_2.



Crystal forms for picric acid and metabromonitrobenzene are examples of symmetries VC_2 and VC_3 respectively.[1] Another example of symmetry HC_n is phloroglucinol diethyl ether which has symmetry HC_4.[2] Examples of symmetry with period numbers as high as 6 are crystals of apatite, $3Ca_3(PO_4)_2 \cdot CaF_2$, and hydrocinchonine sulphate hydrate,[3] $(C_{19}H_{24}ON_2)_2 \cdot H_2SO_4 \cdot 11H_2O$. Their respective symmetries are HC_6 and VC_6.

It will be worth-while to numerate a few examples of the symmetries VC_n and HC_n for life forms. Symmetry VC_n is abundant among plant life. The fruit of psittacanthus

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[1]For illustrations of crystals see C. W. Bunn, Chemical Crystallography, (Oxford,) 1945, pp. 42, 49.

[2]*Ibid.*, p. 51.

[3]*Ibid.*, p. 50.

claviceps[1] has symmetry VC_2. Three vertical planes are present in the fruit of _colchicum autumnale_[2] which, of course, has symmetry VC_3. Symmetry VC_4 is illustrated by the fruit of _rhamnus catharticus_,[3] and symmetry VC_5 is exampled by the fruit of _nigella damascena_.[4] An example of symmetry HC_5 is found in the sea. It is the well-known starfish.[5]

The world has countless symmetry forms in both the living and nonliving domains. Classification and description of all the possible cases would, very likely, fill several volumes. However, the technique used in such an undertaking would be only a continuation and added development of the introductory procedures described in this thesis. Classification of symmetrical forms in a limited field is sometimes undertaken using the mechanics of symmetry. A classification of this kind is no longer in the strict field of mathematics, though the applied use of the mathematics of symmetry is essential.

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[1]For illustrations of plant see A. B. Rendle, _The Classification of Flowering Plants_, (Cambridge,) II, 1925, 67.

[2]_Ibid._, I, 295.

[3]_Ibid._, II, 310.

[4]_Ibid._, p. 141.

[5]M. W. DeLaubenfels, _Pagent of Life Science_, (New York,) 1949, p. 249.

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