

A NON-LINEAR TRANSFORMATION FOR
SEQUENCES AND INTEGRALS

by

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CHAPTER I

INTRODUCTION

The rate of convergence of convergent sequences is an interesting subject in pure mathematics, but the ability to increase the above rate is important to applied mathematics. A transformation, which in some situations increases the rate of convergence of given sequences, is discussed here as has been introduced and discussed by Lubkin (1) and Shanks (2). Both of these papers consider the transformation for the purpose of evaluating infinite series. The transformation discussed here is for the purpose of evaluating infinite sequences and improper integrals(3) instead of evaluating infinite series.

This thesis describes a method of obtaining an equivalent sequence or an equivalent improper integral from a given convergent sequence or a convergent improper integral, respectively. In some cases the new sequence and the new improper integral increase the rate of convergence and, moreover, the same method may usually be repeated indefinitely to obtain a new sequence or a new improper integral each equivalent to the original and each better than the predecessor in a convergence sense.

Techniques used to test the new sequence or integral for faster convergence are discussed in Chapter II and Chapter III.

CHAPTER II

A NON-LINEAR TRANSFORMATION FOR SEQUENCES

Before commencing a discussion of the sequences to sequences transformation some pertinent definitions are in order.

Definition 2.1. Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences with $a_n \rightarrow a$ and $b_n \rightarrow a$, then $\{b_n\}$ is said to converge no less rapidly than $\{a_n\}$ if and only if, there exists a constant $K > 0$ such that

$$|b_n - a| \leq K |a_n - a|.$$

Definition 2.2. Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences with $a_n \rightarrow a$ and $b_n \rightarrow a$, then $\{b_n\}$ is said to converge with the same order of rapidity as $\{a_n\}$ if and only if, there exist positive constants K_1 and K_2 such that

$$|b_n - a| \leq K_1 |a_n - a|$$

and

$$|a_n - a| \leq K_2 |b_n - a|.$$

Definition 2.3. Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences

with $a_n \rightarrow a$ and $b_n \rightarrow a$, then $\{b_n\}$ is said to converge more rapidly than $\{a_n\}$ if and only if, for every $\epsilon > 0$, there exists an integer N such that

$$|b_n - a| \leq \epsilon |a_n - a| \quad \text{for all } n > N.$$

The following lemmas are useful for investigating the relative rate of convergence of two sequences.

Lemma 2.1. Let

(1) $a_n \rightarrow a$ and $b_n \rightarrow a$ with $\Delta a_n > 0, \Delta b_n \geq 0$, where

$$\Delta a_n = a_{n+1} - a_n, \Delta b_n = b_{n+1} - b_n, \text{ and}$$

(2)

$$\frac{\Delta b_n}{\Delta a_n} < K \quad (\text{a positive constant}),$$

then $\{b_n\}$ converges no less rapidly than $\{a_n\}$.

Proof: Let m, n be two positive integers such that $n > m$.

Since

$$\frac{\Delta b_n}{\Delta a_n} < K,$$

we have

$$b_{n+1} - b_n < K(a_{n+1} - a_n) \quad \text{for all } n.$$

Consider

$$0 \leq b_n - b_{n-1} < K (a_n - a_{n-1})$$

$$0 \leq b_{n-1} - b_{n-2} < K (a_{n-1} - a_{n-2})$$

•
•
•

$$0 \leq b_{m+2} - b_{m+1} < K (a_{m+2} - a_{m+1})$$

$$0 \leq b_{m+1} - b_m < K (a_{m+1} - a_m),$$

this implies

$$0 \leq b_n - b_m < K (a_n - a_m)$$

or

$$|b_n - b_m| < K |a_n - a_m|,$$

then, as $n \rightarrow \infty$

$$|a - b_m| \leq K |a - a_m|.$$

Therefore, by Definition 2.1, we have $\{b_n\}$ converges no less rapidly than $\{a_n\}$.

Lemma 2.2. Let

- (1) $a_n \rightarrow a$ and $b_n \rightarrow a$ with $\Delta a_n > 0$, $\Delta b_n > 0$, and
- (2) there exist constants A, B such that

$$0 < B < \frac{\Delta b_n}{\Delta a_n} < A,$$

then $\{b_n\}$ converges with the same order of rapidity as $\{a_n\}$.

Proof: Since

$$0 < B < \frac{\Delta b_n}{\Delta a_n} < A,$$

then

$$\frac{\Delta b_n}{\Delta a_n} < A \quad \text{and} \quad \frac{\Delta a_n}{\Delta b_n} < \frac{1}{B}.$$

Hence, by applying Lemma 2.1 twice, we have

$$|b_n - a| \leq A |a_n - a|$$

and

$$|a_n - a| \leq \frac{1}{B} |b_n - a|.$$

Thus, $\{b_n\}$ converges with the same order of rapidity as $\{a_n\}$.

Lemma 2.3. Let

(1) $a_n \rightarrow a$ and $b_n \rightarrow a$ with $\Delta a_n > 0$, $\Delta b_n \geq 0$, and

(2) $\frac{\Delta b_n}{\Delta a_n} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$

then $\{b_n\}$ converges more rapidly than $\{a_n\}$.

Proof: Let $\epsilon > 0$ be given, then there exists an integer N such that

$$\frac{\Delta b_n}{\Delta a_n} < \epsilon \quad \text{for all } n > N.$$

By Lemma 2.1, we have

$$|b_n - a| \leq \epsilon |a_n - a|,$$

and hence $\{b_n\}$ converges more rapidly than $\{a_n\}$.

Motivation: Consider the infinite series $\sum_{n=1}^{\infty} a_n$ and let S_n be the n th partial sum, with $S_n \rightarrow S$, then

$$S = S_n + (a_{n+1} + a_{n+2} + \dots)$$

$$= S_n + a_n \left[\frac{a_{n+1}}{a_n} + \frac{a_{n+2}}{a_{n+1}} + \frac{a_{n+3}}{a_n} + \dots \right].$$

Therefore, for some convergent series such that

$$\frac{a_{n+1}}{a_n} \rightarrow R, \text{ where } |R| < 1,$$

it seems reasonable to define a new series whose n th partial sums, say T_n , are given by

$$T_n = S_n + a_n \left[\left(\frac{a_{n+1}}{a_n} \right) + \left(\frac{a_{n+1}}{a_n} \right)^2 + \left(\frac{a_{n+1}}{a_n} \right)^3 + \dots \right]$$

$$= S_n + a_n \frac{\frac{a_{n+1}}{a_n}}{1 - \frac{a_{n+1}}{a_n}}$$

$$= S_n + \frac{\frac{a_{n+1}}{a_n}}{1 - \frac{a_{n+1}}{a_n}}$$

Thus, T_n gives S_n plus an estimate for $S - S_n$.

Definition 2.4. Let T be the transformation from a real sequence $\{a_n\}$ to a real sequence $\{b_n\}$ given by

$$b_n = T a_n = a_n + \frac{\Delta a_n}{1 - \frac{\Delta a_n}{\Delta a_{n-1}}} , \quad n \geq 2.$$

We note that T is a non-linear transformation and

$$b_n = a_n + \frac{\Delta a_n}{1 - \frac{\Delta a_n}{\Delta a_{n-1}}}$$

$$= \frac{a_{n-1} a_{n+1} - a_n^2}{a_{n-1} + a_{n+1} - 2a_n}$$

The following are some simple consequences of the transformation T.

Theorem 2.1. Let

- (1) $\{a_n\}$ converge to a, with

$$\frac{\Delta a_n}{\Delta a_{n-1}} \ll R < 1,$$

and

- (2) $\{b_n\}$ be a sequence such that

$$b_n = \frac{a_{n-1} a_{n+1} - a_n^2}{a_{n-1} + a_{n+1} - 2a_n},$$

then $\{b_n\}$ converges and furthermore $\{b_n\}$ converges to a.

Proof: Let $\epsilon > 0$ be given. Since $a_n \rightarrow a$ then there exists an integer N such that

$$|\Delta a_n| = |a_{n+1} - a_n| < \frac{1}{2}\epsilon |1 - R|$$

$$\ll \frac{1}{2}\epsilon \left| 1 - \frac{\Delta a_n}{\Delta a_{n-1}} \right|$$

and

$$|a_n - a| < \frac{\epsilon}{2} \quad \text{for all } n > N.$$

Consider

$$b_n = \frac{a_{n-1} a_{n+1} - a_n^2}{a_{n-1} + a_{n+1} - 2a_n}$$

$$= a_n + \frac{a_{n-1} a_{n+1} - a_n^2 - a_n a_{n-1} - a_{n+1} a_n + 2a_n^2}{a_{n-1} + a_{n+1} - 2a_n}$$

$$= a_n + \frac{a_{n-1} (a_{n+1} - a_n) - a_n (a_{n+1} - a_n)}{a_{n-1} + a_{n+1} - 2a_n}$$

$$= a_n + \frac{\Delta a_n (-\Delta a_{n-1})}{-\Delta a_{n-1} + \Delta a_n}$$

$$= a_n + \frac{\Delta a_n}{1 - \frac{\Delta a_n}{\Delta a_{n-1}}},$$

then

$$|b_n - a| = \left| a_n + \frac{\Delta a_n}{1 - \frac{\Delta a_n}{\Delta a_{n-1}}} - a \right|$$

$$\leq |a_n - a| + \left| \frac{\Delta a_n}{1 - \frac{\Delta a_n}{\Delta a_{n-1}}} \right|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{for all } n > N.$$

Therefore, $\{b_n\}$ converges to a .

Theorem 2.2. Let

- (1) $\{a_n\}$ be a convergent sequence with $\Delta a_n > 0$,
- (2) $\{b_n\}$ be a sequence such that

$$b_n = \frac{a_{n-1} a_{n+1} - a_n^2}{a_{n-1} + a_{n+1} - 2a_n},$$

and

$$(3) \frac{\Delta a_n}{\Delta a_{n-1}} < \frac{\Delta a_{n+1}}{\Delta a_n} \leq R < 1,$$

then $\{b_n\}$ converges no less rapidly than $\{a_n\}$.

Proof: By Theorem 2.1, we know $\{b_n\}$ converges to the same limit as $\{a_n\}$ does. Consider

$$b_{n+1} = \frac{a_n a_{n+2} - a_{n+1}^2}{a_n + a_{n+2} - 2a_{n+1}}$$

$$= a_n + \frac{a_n a_{n+2} - a_{n+1}^2 - a_n (a_n + a_{n+2} - 2a_{n+1})}{a_n + a_{n+2} - 2a_{n+1}}$$

$$= a_n + \frac{a_n a_{n+2} - a_{n+1}^2 - a_n^2 - a_n a_{n+2} + 2a_n a_{n+1}}{a_n + a_{n+2} - 2a_{n+1}}$$

$$= a_n + \frac{(a_n a_{n+1} - a_{n+1}^2) + (a_n a_{n+1} - a_n^2)}{(a_n - a_{n+1}) + (a_{n+2} - a_{n+1})}$$

$$= a_n + \frac{a_{n+1} (a_n - a_{n+1}) + a_n (a_{n+1} - a_n)}{-\Delta a_n + \Delta a_{n+1}}$$

$$= a_n + \frac{a_{n+1} (-\Delta a_n) + a_n \Delta a_n}{-\Delta a_n + \Delta a_{n+1}}$$

$$= a_n + \frac{\Delta a_n (a_n - a_{n+1})}{-\Delta a_n + \Delta a_{n+1}}$$

$$= a_n + \frac{\Delta a_n (-\Delta a_n)}{-\Delta a_n + \Delta a_{n+1}}$$

$$= a_n + \frac{\Delta a_n}{1 - \frac{\Delta a_{n+1}}{\Delta a_n}},$$

therefore,

$$\Delta b_n = b_{n+1} - b_n$$

$$\begin{aligned} &= \left[a_n + \frac{\Delta a_n}{1 - \frac{\Delta a_{n+1}}{\Delta a_n}} \right] - \left[a_n + \frac{\Delta a_n}{1 - \frac{\Delta a_n}{\Delta a_{n-1}}} \right] \\ &= \Delta a_n \left[\frac{1}{1 - \frac{\Delta a_{n+1}}{\Delta a_n}} - \frac{1}{1 - \frac{\Delta a_n}{\Delta a_{n-1}}} \right] > 0, \end{aligned}$$

furthermore,

$$\begin{aligned} \frac{\Delta b_n}{\Delta a_n} &= \frac{\left[1 - \frac{\Delta a_n}{\Delta a_{n-1}} \right] - \left[1 - \frac{\Delta a_{n+1}}{\Delta a_n} \right]}{\left[1 - \frac{\Delta a_{n+1}}{\Delta a_n} \right] \left[1 - \frac{\Delta a_n}{\Delta a_{n-1}} \right]} \\ &= \frac{\frac{\Delta a_{n+1}}{\Delta a_n} - \frac{\Delta a_n}{\Delta a_{n-1}}}{\left[1 - \frac{\Delta a_{n+1}}{\Delta a_n} \right] \left[1 - \frac{\Delta a_n}{\Delta a_{n-1}} \right]} \\ &\leq \frac{1}{(1 - R)^2}. \end{aligned}$$

Hence, by Lemma 2.1, we have $\{b_n\}$ converges no less rapidly than $\{a_n\}$.

Theorem 2.3. Let

- (1) $\{a_n\}$ be a convergent sequence such that $\Delta a_n > 0$,
- (2) $\{b_n\}$ be a sequence with

$$b_n = \frac{a_{n-1} a_{n+1} - a_n^2}{a_{n-1} + a_{n+1} - 2a_n},$$

and

$$(3) \quad 0 < R_2 \leq \frac{\Delta a_n}{\Delta a_{n-1}} < \frac{\Delta a_{n+1}}{\Delta a_n} \leq R_1 < 1,$$

$$\frac{\Delta a_{n+1}}{\Delta a_n} - \frac{\Delta a_n}{\Delta a_{n-1}} \geq R_3 > 0,$$

then $\{b_n\}$ converges with the same order of rapidity as $\{a_n\}$.

Proof: By Theorem 2.1, we know $\{b_n\}$ converges to the same limit as $\{a_n\}$ does. Consider

$$\frac{\Delta b_n}{\Delta a_n} = \frac{\frac{\Delta a_{n+1}}{\Delta a_n} - \frac{\Delta a_n}{\Delta a_{n-1}}}{\left[1 - \frac{\Delta a_{n+1}}{\Delta a_n}\right] \left[1 - \frac{\Delta a_n}{\Delta a_{n-1}}\right]},$$

then

$$0 < \frac{R_3}{(1 - R_2)^2} \leq \frac{\Delta b_n}{\Delta a_n} \leq \frac{1}{(1 - R_1)^2}.$$

Hence, $\{b_n\}$ converges with the same order of rapidity as $\{a_n\}$.

Theorem 2.4. Let

- (1) $\{a_n\}$ be a convergent sequence with $\Delta a_n > 0$,
- (2) $\{b_n\}$ be a sequence such that

$$b_n = \frac{a_{n-1} a_{n+1} - a_n^2}{a_{n-1} + a_{n+1} - 2a_n},$$

$$(3) \quad \frac{\Delta a_n}{\Delta a_{n-1}} < \frac{\Delta a_{n+1}}{\Delta a_n} \leq R < 1, \text{ and}$$

$$(4) \quad \frac{\Delta a_{n+1}}{\Delta a_n} - \frac{\Delta a_n}{\Delta a_{n-1}} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

then $\{b_n\}$ converges more rapidly than $\{a_n\}$.

Proof: By Theorem 2.1, we know $\{b_n\}$ converges to the same limit as $\{a_n\}$ does. Consider

$$\frac{\Delta b_n}{\Delta a_n} = \frac{\frac{\Delta a_{n+1}}{\Delta a_n} - \frac{\Delta a_n}{\Delta a_{n-1}}}{\left[1 - \frac{\Delta a_{n+1}}{\Delta a_n}\right]\left[1 - \frac{\Delta a_n}{\Delta a_{n-1}}\right]}$$

then

$$\left| \frac{\Delta b_n}{\Delta a_n} \right| \leq \frac{\frac{\Delta a_{n+1}}{\Delta a_n} - \frac{\Delta a_n}{\Delta a_{n-1}}}{(1 - R)^2} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Hence, $\{b_n\}$ converges more rapidly than $\{a_n\}$.

Lemma 2.4. If $\frac{\Delta a_n}{\Delta a_{n-1}} \rightarrow R$, $|R| < 1$, and $\Delta a_n > 0$ for all n ,

then $\frac{a - a_n}{\Delta a_n} \rightarrow \frac{1}{1 - R}$, where $a_n \rightarrow a$.

Proof: Consider

$$\begin{aligned} & \frac{a - a_n}{\Delta a_n} \\ &= \frac{1}{\Delta a_n} [(a_{n+1} - a_n) + (a_{n+2} - a_{n+1}) + (a_{n+3} - a_{n+2}) + \dots] \\ &= \frac{1}{\Delta a_n} [\Delta a_n + \Delta a_{n+1} + \Delta a_{n+2} + \dots] \\ &= 1 + \frac{\Delta a_{n+1}}{\Delta a_n} + \frac{\Delta a_{n+1}}{\Delta a_n} \frac{\Delta a_{n+2}}{\Delta a_{n+1}} + \dots, \end{aligned}$$

then.

$$\frac{1}{1 - (R - \epsilon)} < \frac{a - a_n}{\Delta a_n} < \frac{1}{1 - (R + \epsilon)},$$

if ϵ is sufficiently small and n is sufficiently large, and hence

$$\frac{a - a_n}{\Delta a_n} \rightarrow \frac{1}{1 - R} \quad \text{as } n \rightarrow \infty.$$

Theorem 2.5. Let

$$(1) \frac{\Delta a_n}{\Delta a_{n-1}} \rightarrow R, |R| < 1, \text{ and}$$

$$(2) \Delta a_n > 0 \text{ for all } n,$$

then $\{b_n\}$ converges more rapidly than $\{a_n\}$, where

$$b_n = \frac{a_{n-1} a_{n+1} - a_n}{a_{n-1} + a_{n+1} - 2a_n}$$

Proof: Consider

$$|b_n - a|$$

$$= \left| a_n + \frac{\Delta a_n}{1 - \frac{\Delta a_n}{\Delta a_{n-1}}} - a \right|$$

$$= \left| a_n - a + \frac{\Delta a_n}{1 - \frac{\Delta a_n}{\Delta a_{n-1}}} \right|$$

$$= |a_n - a| \left| 1 + \frac{\Delta a_n}{a_n - a} - \frac{1}{1 - \frac{\Delta a_n}{\Delta a_{n-1}}} \right|$$

then

$$\begin{aligned} & \frac{|b_n - a|}{|a_n - a|} \\ &= \left| 1 + \frac{\Delta a_n}{a_n - a} - \frac{1}{1 - \frac{\Delta a_n}{\Delta a_{n-1}}} \right| \\ &= \left| 1 - \frac{\Delta a_n}{a - a_n} - \frac{1}{1 - \frac{\Delta a_n}{\Delta a_{n-1}}} \right|. \end{aligned}$$

By Lemma 2.4,

$$\frac{|b_n - a|}{|a_n - a|} \rightarrow \left| 1 - (1 - R) \frac{1}{(1 - R)} \right| = 0, \text{ as } n \rightarrow \infty.$$

Therefore, $\{b_n\}$ converges more rapidly than $\{a_n\}$.

An Error Analysis: Assume

$$(1) \quad \left| \frac{\Delta a_n}{\Delta a_{n-1}} \right| \rightarrow R, \quad R < 1,$$

(2) let $\epsilon > 0$ such that $R + \epsilon < 1$, and

$$(3) \quad \left| R - \left| \frac{\Delta a_n}{\Delta a_{n-1}} \right| \right| < \epsilon, \quad n \geq m - 1,$$

then

$$(i) |a - b_m| \leq |\Delta a_m| \frac{2(R + \epsilon)}{1 - (R + \epsilon)}, \text{ and}$$

$$(ii) |a - b_m| \leq |\Delta a_m| \frac{2\epsilon}{(1 - R)^2 - \epsilon^2} \text{ if } \Delta a_n > 0, n \geq m - 1$$

or if Δa_n alternates in sign for $n \geq m - 1$.

Proof: Let

$$R_n = \frac{\Delta a_n}{\Delta a_{n-1}},$$

then

$$\begin{aligned} & |a - b_m| \\ &= \left| a - a_m - \frac{\Delta a_m}{1 - \frac{\Delta a_m}{\Delta a_{m-1}}} \right| \\ &= |(a_{m+1} - a_m) + (a_{m+2} - a_{m+1}) + (a_{m+3} - a_{m+2}) + \dots \\ &\quad - a_m (1 + R_m + R_m^2 + \dots)| \\ &= |\Delta a_m + \Delta a_{m+1} + \Delta a_{m+2} + \dots - \Delta a_m (1 + R_m + R_m^2 + \dots)| \end{aligned}$$

$$\begin{aligned}
 &= \left| \Delta a_m \left[1 + \frac{\Delta a_{m+1}}{\Delta a_m} + \frac{\Delta a_{m+1}}{\Delta a_m} \frac{\Delta a_{m+2}}{\Delta a_{m+1}} + \dots \right] \right. \\
 &\quad \left. - \Delta a_m (1 + R_m + R_m^2 + \dots) \right| \\
 &= \left| \Delta a_m (1 + R_{m+1} + R_{m+1} R_{m+2} + \dots) \right. \\
 &\quad \left. - \Delta a_m (1 + R_m + R_m^2 + \dots) \right|,
 \end{aligned}$$

or

$$\begin{aligned}
 (A) \quad & \left| a - b_m \right| \\
 &= \left| \Delta a_m \right| \left| (R_{m+1} - R_m) + (R_{m+2} R_{m+1} - R_m^2) + \dots \right| \\
 &\leq \left| \Delta a_m \right| \left| 2(R + \epsilon) + 2(R + \epsilon)^2 + \dots \right| \\
 &= \left| \Delta a_m \right| \frac{2(R + \epsilon)}{1 - (R + \epsilon)},
 \end{aligned}$$

which is (i). To obtain (ii), consider (A), then

$$\begin{aligned}
 & \left| a - b_m \right| \\
 &\leq \left| \Delta a_m \right| \left| [(R + \epsilon) - (R - \epsilon)] + [(R + \epsilon)^2 - (R - \epsilon)^2] \right. \\
 &\quad \left. + \dots \right| \\
 &= \left| \Delta a_m \right| \left| \frac{R + \epsilon}{1 - (R + \epsilon)} - \frac{R - \epsilon}{1 - (R - \epsilon)} \right| \\
 &= \left| \Delta a_m \right| \frac{2\epsilon}{(1 - R)^2 - \epsilon^2}.
 \end{aligned}$$

CHAPTER III

A NON-LINEAR TRANSFORMATION FOR IMPROPER INTEGRALS

In this chapter we assume f and g be real valued functions of a real variable such that f and g are continuous for $c \leq x < \infty$, and

$$\int_c^\infty f(x)dx, \quad \int_c^\infty g(x)dx,$$

converge, that is

$$A(t) = \int_c^t f(x)dx \rightarrow A$$

and

$$B(t) = \int_c^t g(x)dx \rightarrow B$$

as $t \rightarrow \infty$.

Definition 3.1. Let

$$A(t) = \int_c^t f(x)dx \rightarrow A$$

and

$$B(t) = \int_c^t g(x)dx \rightarrow A$$

as $t \rightarrow \infty$, then $B(t)$ is said to converge no less rapidly than $A(t)$ if and only if, there exists a constant $K > 0$ such that

$$|B(t) - A| \leq K |A(t) - A|.$$

Definition 3.2. Let

$$A(t) = \int_c^t f(x)dx \rightarrow A$$

and

$$B(t) = \int_c^t g(x)dx \rightarrow A$$

as $t \rightarrow \infty$, then $B(t)$ is said to converge with the same order of rapidity as $A(t)$ if and only if, there exist positive constants K_1 and K_2 such that

$$|A(t) - A| \leq K_1 |B(t) - A|$$

and

$$|B(t) - A| \leq K_2 |A(t) - A|.$$

Definition 3.3. Let

$$A(t) = \int_c^t f(x)dx \rightarrow A$$

and

$$B(t) = \int_c^t g(x)dx \rightarrow A$$

as $t \rightarrow \infty$, then $B(t)$ is said to converge more rapidly than $A(t)$ if and only if, for every $\epsilon > 0$, there exists an integer N such that

$$|B(t) - A| \leq \epsilon |A(t) - A|$$

for all $t > N$.

The following lemmas are useful for investigating the relative rate of convergence of two improper integrals.

Lemma 3.1. Let

$$(1) \quad A(t) = \int_c^t f(x)dx \rightarrow A, \text{ as } t \rightarrow \infty,$$

$$(2) \quad A(t;k) = \int_c^{t+k} f(x)dx, \text{ where } k \in I \text{ (set of non-negative integers)},$$

and

$$(3) \quad \Delta A(t;k) = \int_c^{t+(k+1)} f(x)dx - \int_c^{t+k} f(x)dx,$$

then $\Delta A(t;k) \rightarrow 0$ as $t \rightarrow \infty$.

Proof: Since

$$A(t) = \int_c^t f(x)dx \rightarrow A \text{ as } t \rightarrow \infty$$

implies

$$A(t;k) = \int_c^{t+k} f(x)dx \rightarrow A \text{ as } t \rightarrow \infty ,$$

then

$$\Delta A(t;k) = \int_c^{t+(k+1)} f(x)dx - \int_c^{t+k} f(x)dx$$

$$\rightarrow A - A = 0 \text{ as } t \rightarrow \infty .$$

Lemma 3.2. Let

$$(1) \quad A(t) = \int_c^t f(x)dx \rightarrow A \text{ and } B(t) = \int_c^t g(x)dx \rightarrow A$$

as $t \rightarrow \infty$,

$$\Delta A(t;k) = \int_c^{t+k+1} f(x)dx - \int_c^{t+k} f(x)dx > 0$$

$$\Delta B(t;k) = \int_c^{t+k+1} g(x)dx - \int_c^{t+k} g(x)dx \geq 0$$

and

$$(2) \quad \frac{\Delta B(t;k)}{\Delta A(t;k)} < M, \text{ where } M \text{ is a positive constant and } k$$

is any non-negative integer,

then $B(t)$ converges no less rapidly than $A(t)$.

Proof: Let m, n be two non-negative integers such that $n > m$. Since

$$\frac{\Delta B(t;k)}{\Delta A(t;k)} < M \text{ and } \Delta A(t;k) > 0,$$

we have

$$B(t;k) - B(t;k-1) < M [A(t;k) - A(t;k-1)].$$

Consider

$$0 \leq B(t;n) - B(t;n-1) < M [A(t;n) - A(t;n-1)]$$

$$0 \leq B(t;n-1) - B(t;n-2) < M [A(t;n-1) - A(t;n-2)]$$

⋮
⋮

$$0 \leq B(t;m+2) - B(t;m+1) < M [A(t;m+2) - A(t;m+1)]$$

$$0 \leq B(t;m+1) - B(t;m) < M [A(t;m+1) - A(t;m)],$$

this implies

$$0 \leq B(t;n) - B(t;m) < M [A(t;n) - A(t;m)]$$

or

$$|B(t;n) - B(t;m)| < M |A(t;n) - A(t;m)| ;$$

thus,

$$|B(t; m) - A| \leq M |A(t; m) - A| \quad \text{as } n \rightarrow \infty$$

or

$$|B(t) - A| \leq M |A(t) - A|.$$

Therefore, by Definition 3.1, we have $B(t)$ converges no less rapidly than $A(t)$.

Lemma 3.3. Let

$$(1) \quad A(t) = \int_C^t f(x)dx \longrightarrow A \text{ and } B(t) = \int_C^t g(x)dx \longrightarrow A$$

as $t \rightarrow \infty$, $\Delta A(t; k) > 0$, $\Delta B(t; k) > 0$, and

(2) there exist constants M_1 , M_2 such that

$$0 < M_1 < \frac{\Delta B(t; k)}{\Delta A(t; k)} < M_2$$

where k is any non-negative integers,

then $B(t)$ converges with the same order of rapidity as $A(t)$.

Proof: Since

$$0 < M_1 < \frac{\Delta B(t; k)}{\Delta A(t; k)} < M_2,$$

then

$$\frac{\Delta B(t;k)}{\Delta A(t;k)} < M_2 \quad \text{and} \quad \frac{\Delta A(t;k)}{\Delta B(t;k)} < \frac{1}{M_1}.$$

Hence, by applying Lemma 3.2, twice, we get

$$|B(t) - A| < M_2 |A(t) - A|$$

and

$$|A(t) - A| < \frac{1}{M_1} |B(t) - A|.$$

Thus, $B(t)$ converges with the same order of rapidity as $A(t)$.

Lemma 3.4. Let

$$(1) \quad A(t) = \int_C^t f(x)dx \rightarrow A \quad \text{and} \quad B(t) = \int_C^t g(x)dx \rightarrow A$$

as $t \rightarrow \infty$, $\Delta A(t;k) > 0$, $\Delta B(t;k) \geq 0$, and

$$(2) \quad \frac{\Delta B(t;k)}{\Delta A(t;k)} \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

then $B(t)$ converges more rapidly than $A(t)$.

Proof: Let $\epsilon > 0$ be given, then there exists an integer N such that

$$\frac{\Delta B(t;k)}{\Delta A(t;k)} < \epsilon \quad \text{for } t > N.$$

By Lemma 3.2, we have

$$|B(t) - A| < \epsilon \quad |A(t) - A|$$

and hence $B(t)$ converges more rapidly than $A(t)$.

The following theorems are simple consequences of the preceding lemmas.

Theorem 3.1. Let

$$(1) \quad A(t) = \int_c^t f(x)dx \rightarrow A \quad \text{as } t \rightarrow \infty,$$

$$(2) \quad B(t;k) = \frac{A(t;k-1) A(t;k+1) - A(t;k)^2}{A(t;k-1) + A(t;k+1) - 2A(t;k)}, \quad \text{and}$$

$$(3) \quad \frac{\Delta A(t;k)}{\Delta A(t;k-1)} \leq R < 1,$$

then $B(t;k)$ converges; furthermore $B(t;k) \rightarrow A$.

Proof: Since

$$B(t;k) = \frac{A(t;k-1) A(t;k+1) - A(t;k)^2}{A(t;k-1) + A(t;k+1) - 2A(t;k)}$$

$$= A(t;k) + \frac{\Delta A(t;k)}{1 - \frac{\Delta A(t;k)}{\Delta A(t;k-1)}}$$

$$\leq A(t;k) + \frac{\Delta A(t;k)}{1-R}$$

$$\longrightarrow A \quad \text{as } k \rightarrow \infty ,$$

hence $B(t;k)$ converges and $B(t;k) \rightarrow A$.

Theorem 3.2. Let

$$(1) \quad A(t) = \int_c^t f(x)dx \rightarrow A \quad \text{as } t \rightarrow \infty ,$$

$$(2) \quad B(t;k) = \frac{A(t;k-1) A(t;k+1) - A(t;k)^2}{A(t;k-1) + A(t;k+1) - 2A(t;k)} , \text{ and}$$

$$(3) \quad \frac{\Delta A(t;k)}{\Delta A(t;k-1)} < \frac{\Delta A(t;k+1)}{\Delta A(t;k)} \leq R < 1, \quad \Delta A(t;k) > 0,$$

then $B(t;k)$ converges no less rapidly than $A(t)$.

Proof: By Theorem 3.1, we know

$$B(t;k) \rightarrow A \quad \text{as } t \rightarrow \infty .$$

Since $\Delta A(t;k) > 0$ and $\Delta B(t;k) > 0$, we have

$$0 < \frac{\Delta B(t;k)}{\Delta A(t;k)} = \frac{\frac{\Delta A(t;k+1)}{\Delta A(t;k)} - \frac{\Delta A(t;k)}{\Delta A(t;k-1)}}{\left[1 - \frac{\Delta A(t;k+1)}{\Delta A(t;k)}\right] \left[1 - \frac{\Delta A(t;k)}{\Delta A(t;k-1)}\right]}$$

$$\leq \frac{1}{(1-R)^2}.$$

Hence, by Lemma 3.2, we have that $B(t)$ converges no less rapidly than $A(t)$.

Theorem 3.3. Let

$$(1) \quad A(t) = \int_C^t f(x)dx \rightarrow A \text{ as } t \rightarrow \infty, \Delta A(t;k) > 0,$$

$$(2) \quad B(t;k) = \frac{A(t;k-1), A(t;k+1) - A(t;k)^2}{A(t;k-1) + A(t;k+1) - 2A(t;k)}, \text{ and}$$

$$(3) \quad 0 < R_2 \leq \frac{\Delta A(t;k)}{\Delta A(t;k-1)} < \frac{\Delta A(t;k+1)}{\Delta A(t;k)} \leq R_1 < 1,$$

$$\frac{\Delta A(t;k+1)}{\Delta A(t;k)} - \frac{\Delta A(t;k)}{\Delta A(t;k-1)} \geq R_3 > 0,$$

then $B(t)$ converges with the same order of rapidity as $A(t)$.

Proof: By Theorem 3.1, we know

$$B(t;k) \rightarrow A \text{ as } t \rightarrow \infty.$$

Consider

$$0 < \frac{\Delta B(t;k)}{\Delta A(t;k)}$$

$$\begin{aligned}
 & \frac{\Delta A(t;k+1)}{\Delta A(t;k)} - \frac{\Delta A(t;k)}{\Delta A(t;k-1)} \\
 = & \left[1 - \frac{\Delta A(t;k+1)}{\Delta A(t;k)} \right] \left[1 - \frac{\Delta A(t;k)}{\Delta A(t;k-1)} \right]
 \end{aligned}$$

then

$$0 < \frac{R_3}{(1 - R_2)^2} \leq \frac{\Delta B(t;k)}{\Delta A(t;k)} \leq \frac{1}{(1 - R_1)^2}$$

Hence, $B(t)$ converges with the same order of rapidity as $A(t)$.

Theorem 3.4. Let

$$(1) \quad A(t) = \int_C^t f(x) dx \rightarrow A \text{ as } t \rightarrow \infty, \quad \Delta A(t;k) > 0,$$

$$(2) \quad B(t;k) = \frac{A(t;k-1) A(t;k+1) - A(t;k)^2}{A(t;k-1) + A(t;k+1) - 2A(t;k)}$$

$$(3) \quad \frac{\Delta A(t;k)}{\Delta A(t;k-1)} < \frac{\Delta A(t;k+1)}{\Delta A(t;k)} \leq R < 1, \text{ and}$$

$$(4) \quad \frac{\Delta A(t;k+1)}{\Delta A(t;k)} - \frac{\Delta A(t;k)}{\Delta A(t;k-1)} \rightarrow 0 \text{ as } t \rightarrow \infty,$$

then $B(t)$ converges more rapidly than $A(t)$.

Proof: By Theorem 3.1, we know $B(t) \rightarrow A$ as $t \rightarrow \infty$.

Consider

$$\frac{\Delta B(t;k)}{\Delta A(t;k)} = \frac{\frac{\Delta A(t;k+1)}{\Delta A(t;k)} - \frac{\Delta A(t;k)}{\Delta A(t;k-1)}}{\left[1 - \frac{\Delta A(t;k+1)}{\Delta A(t;k)}\right] \left[1 - \frac{\Delta A(t;k)}{\Delta A(t;k-1)}\right]}$$

then

$$\left| \frac{\Delta B(t;k)}{\Delta A(t;k)} \right| \leq \frac{\frac{\Delta A(t;k+1)}{\Delta A(t;k)} - \frac{\Delta A(t;k)}{\Delta A(t;k-1)}}{(1-R)^2}$$

$$\rightarrow 0 \quad \text{as } t \rightarrow \infty ,$$

Hence, $B(t)$ converges more rapidly than $A(t)$.

Lemma 3.5. If

$$\frac{\Delta A(t;n)}{\Delta A(t;n-1)} \rightarrow R$$

and $\Delta A(t;n) > 0$ where $|R| < 1$, then

$$\frac{A - A(t;n)}{\Delta A(t;n)} \rightarrow \frac{1}{1-R} , \text{ where } A(t) \rightarrow A.$$

Proof: Consider

$$\frac{A - A(t; n)}{\Delta A(t; n)}$$

$$= \frac{1}{\Delta A(t; n)} \left[[A(t; n+1) - A(t; n)] + [A(t; n+2) - A(t; n+1)] + [A(t; n+3) - A(t; n+2)] + \dots \right]$$

$$= \frac{1}{\Delta A(t; n)} \left[\Delta A(t; n) + \Delta A(t; n+1) + \Delta A(t; n+2) + \dots \right]$$

$$= 1 + \frac{\Delta A(t; n+1)}{\Delta A(t; n)} + \frac{\Delta A(t; n+2)}{\Delta A(t; n)} + \frac{\Delta A(t; n+3)}{\Delta A(t; n+1)} + \dots ,$$

then

$$\frac{1}{1 - (R - \epsilon)} < \frac{A - A(t; n)}{\Delta A(t; n)} < \frac{1}{1 - (R + \epsilon)} .$$

If ϵ is sufficiently small and n is sufficiently large,

and hence

$$\frac{A - A(t; n)}{\Delta A(t; n)} \rightarrow \frac{1}{1 - R} .$$

Theorem 3.5. Let

$$(1) \frac{\Delta A(t; n)}{\Delta A(t; n-1)} \longrightarrow R, |R| < 1, \text{ and } \Delta A(t; n) > 0, \text{ and}$$

$$(2) \Delta A(t; n) > 0 \text{ for all } n,$$

then $B(t)$ converges more rapidly than $A(t)$.

Proof: Consider

$$\begin{aligned} & |B(t; n) - A| \\ &= \left| A(t; n) + \frac{\Delta A(t; n)}{1 - \frac{\Delta A(t; n)}{\Delta A(t; n-1)}} - A \right| \\ &= \left| A(t; n) - A + \frac{\Delta A(t; n)}{1 - \frac{\Delta A(t; n)}{\Delta A(t; n-1)}} \right| \\ &= |A(t; n) - A| \left| 1 + \frac{\Delta A(t; n)}{A(t; n) - A} \frac{1}{1 - \frac{\Delta A(t; n)}{\Delta A(t; n-1)}} \right| \end{aligned}$$

then

$$\frac{|B(t; n) - A|}{|A(t; n) - A|} = \left| 1 - \frac{\Delta A(t; n)}{A - A(t; n)} \frac{1}{1 - \frac{\Delta A(t; n)}{\Delta A(t; n-1)}} \right|$$

By Lemma 3.5,

$$\frac{|B(t; n) - A|}{|A(t; n) - A|} \rightarrow \left| 1 - (1 - R) \frac{1}{1 - R} \right| = 0$$

as $n \rightarrow \infty$. Therefore, $B(t)$ converges more rapidly than $A(t)$.

Remark: If we assume that $f(t) > 0$ and continuous for $t \geq c$, then hypothesis (1) in Theorem 3.5 is implied by assuming that

$$\lim_{t \rightarrow \infty} \frac{f(t+1)}{f(t)} = R, \quad 0 \leq R < 1.$$

For

$$\lim_{t \rightarrow \infty} \frac{\Delta A(t; n)}{\Delta A(t; n-1)}$$

$$= \lim_{t \rightarrow \infty} \frac{\int_{t+n}^{t+n+1} f(x) dx}{\int_{t+n-1}^{t+n} f(x) dx}$$

$$= \lim_{t \rightarrow \infty} \frac{f(t+n+1) - f(t+n)}{f(t+n) - f(t+n-1)}$$

$$= \lim_{t \rightarrow \infty} \frac{\frac{f(t+n+1)}{f(t+n)} - 1}{1 - \frac{f(t+n-1)}{f(t+n)}}$$

$$= \frac{\frac{R - 1}{1 - \frac{1}{R}}}{R} = R.$$

An Error Analysis: Assume

$$(1) \left| \frac{\Delta A(t; n)}{\Delta A(t; n-1)} \right| \rightarrow R, \quad R < 1,$$

(2) let $\epsilon > 0$ such that $R + \epsilon < 1$, and

$$(3) \left| R - \left| \frac{\Delta A(t; n)}{\Delta A(t; n-1)} \right| \right| < \epsilon, \quad n \geq m - 1,$$

then

$$(i) |A - B(t; m)| \leq \frac{2(R + \epsilon)}{1 - (R + \epsilon)} |\Delta A(t; m)|$$

$$(ii) |A - B(t; m)| \leq \frac{2}{(1 - R)^2 - \epsilon^2} |\Delta A(t; m)|$$

if $\Delta A(t; m) > 0$, $n \geq m - 1$ or if $\Delta A(t; n)$ alternates in sign
for $n \geq m - 1$.

Proof: Let

$$R_n = \frac{\Delta A(t; n)}{\Delta A(t; n-1)},$$

then

$$\begin{aligned} & |A - B(t; m)| \\ &= \left| A - A(t; m) - \frac{\Delta A(t; m)}{1 - \frac{\Delta A(t; m)}{\Delta A(t; m-1)}} \right| \\ &= |[A(t; m+1) - A(t; m)] + [A(t; m+2) - A(t; m+1)] + \dots \\ &\quad - \Delta A(t; m) (1 + R_m + R_m^2 + \dots)| \\ &= |\Delta A(t; m) + \Delta A(t; m+1) + \Delta A(t; m+2) + \dots \\ &\quad - \Delta A(t; m) (1 + R_m + R_m^2 + \dots)| \\ &= |\Delta A(t; m) (1 + R_{m+1} + R_{m+1} R_{m+2} + \dots) \\ &\quad - \Delta A(t; m) (1 + R_m + R_m^2 + \dots)|, \end{aligned}$$

or,

$$\begin{aligned} (A) \quad & |A - B(t; m)| \\ &= |\Delta A(t; m)| |(R_{m+1} - R_m) + (R_{m+2} R_{m+1} - R_m^2) + \dots| \\ &\leq |\Delta A(t; m)| |2(R + \epsilon) + 2(R + \epsilon)^2 + \dots| \end{aligned}$$

$$= |\Delta A(t; m)| \left| \frac{2(R + \epsilon)}{1 - (R + \epsilon)} \right|,$$

which is (i). To obtain (ii), consider (A), then

$$\begin{aligned} & |A - B(t; m)| \\ & \leq |\Delta A(t; m)| \left| [(R + \epsilon) - (R - \epsilon)] \right. \\ & \quad \left. + [(R + \epsilon)^2 - (R - \epsilon)^2] + \dots \right| \\ & = |\Delta A(t; m)| \left| \frac{R + \epsilon}{1 - (R + \epsilon)} - \frac{R - \epsilon}{1 - (R - \epsilon)} \right| \\ & = |\Delta A(t; m)| \frac{2}{(1 - R)^2 - \epsilon^2}. \end{aligned}$$

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