TOPOLOGICAL CHARACTERIZATION OF AN INNER PRODUCT SPACE

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CONTENTS

Chapter I

History

The purpose of this work is to establish necessary and sufficient conditions that a topological linear space be an inner product space.

DEFINITION 1: A topological linear space X is innerproductable if there exists an inner product defined on X such that the topology induced by the inner product is equivalent to the given topology on X.

DEFINITION 2: A normed linear space X is innerproductable if there exists an inner product defined on X such that the norm induced by the inner product is equivalent to the norm on X.

Theorems already exist for the topological characterizations of metric linear spaces and normed linear spaces. For example:

THEOREM 1: Let X be a T_1 topological linear space. Then X is metrizable if and only if there exists a countable base at 0. [9, page 125].

THEOREM 2: A topological linear space X is normable if and only if (1) X is a T_1-space

(2) there exists in X a convex and bounded [see Def. 3 below] neighborhood of 0. [1, page 136].

DEFINITION 3: In a topological linear space, a set S is bounded if for each neighborhood U of 0, there exists a scalar α such that $S \subset \alpha U$.

In the past, several papers have appeared which characterize an inner product in terms of a given norm. A brief history of this development will be given here.

The first paper which characterizes an inner product in terms of a given norm seems to be that of Jordan and v. Newmann [2] in 1935 which states:

THEOREM 3: A normed linear space X is an inner product space if and only if

$$||f + g||^2 + ||f - g||^2 = 2(||f||^2 + ||g||^2)$$

for all f and g in X. The inner product is then defined by

$$\langle f, g\rangle = \frac{1}{4}(||f + g||^2 - ||f - g||^2 + i||f + ig||^2 - i||f - ig||^2).$$

This theorem also implies that every 2-dimensional subspace of an inner product space is isomorphic to the Euclidean plane.

In 1944 F. A. Ficken [3] proved:

THEOREM 4: A normed linear space X is an inner product space if and only if whenever $||f|| = ||g||$, f, g in X and a, b are real scalars then $||af + bg|| = ||bf + ag||$.

Also in 1944, M. M. Day [4] showed:

THEOREM 5: A normed linear space X is an inner product space if and only if $||f + g||^2 + ||f - g||^2 = 4$ whenever $||f|| = ||g|| = 1$.

I. J. Schoenberg [5] added the following theorem in 1952:

THEOREM 6: Let X be a real semi-normed space which is ptolemaic [see [5] for def.], then X is a real inner product space.

Again in 1959, Day [6] established:

THEOREM 7: If X is a normed linear space and there exist a and b such that $0 < a < 1$, $0 < b < 1$ and

$$(a + b - 2ab)(ab + (1 - a)(1 - b)) \leq$$

$$b(1 - b)||af + (1 - a)g||^2 + a(1 - a)||bf - (1 - b)g||^2$$

for every f, g in X with $||f|| = ||g|| = 1$, then X is an inner product space. The theorem remains true if $\leq$ is replaced by $\geq$ or =, however a different choice of a and b would result in each case.

The first paper which states the conditions for a normed linear space to be innerproductable [Def. 2] seems to be that of S. Kakutani and G. W. Mackey [10] in 1946.

THEOREM 8: If X is an infinite-dimensional complex Banach space, let A be its lattice of closed subspaces. If there exists an operation $M \to M'$ from A to A such that:

(1) if M_1 and M_2 are in A and $M_1 \subseteq M_2$ then $M_2' \subseteq M_1'$,

(2) if M is in A then $M'' = M$,

(3) if M is in A then $M' \cap M = 0$,

then X is innerproductable.

Then in 1961, J. G. de Lamadrid [11] proved:

THEOREM 9: A Banach space X is congruent to a Hilbert space if and only if there is a number $M > 0$ such that for every closed vector subspace $F \subset X$ there is a projection P_F: $X \to F$ with the property that the multiplicative semigroup S generated by $\{P_F\}_{F\subset X}$ is bounded by M.

In 1962, J. Lindenstrauss [12] published:

THEOREM 10: Let X be a Banach space, and define the modulus of smoothness $\rho_X(\tau)$ by

$$\rho_X(\tau) = \frac{1}{2} \sup_{\substack{||f||=1 \\ ||g||=\tau}} (||f + g|| + ||f - g|| - 2),$$

$\tau \geq 0$. Then X is an inner product space if

$$\rho_X(\tau) = (1 + \tau^2)^{1/2} - 1$$

for every τ such that $0 \leq \tau \leq 1$.

A paper by J. T. Joichi [7] in 1966 shows that:

THEOREM 11: A normed linear space X is innerproductable if and only if there exists a constant $k \geq 1$ such that for each finite dimensional subspace M of X, there exists a linear mapping T_M of M into H, a Hilbert space, with the property that

$$\frac{1}{k}||f|| \leq ||T_M f|| \leq k||f||$$

for each f in M.

Chapter II

Innerproductability

The significant purpose of this chapter is to give necessary and sufficient conditions for a topological linear space to be innerproductable in terms of its topology, rather than in terms of the properties of some one specific norm. It should be remarked that with the majority of the propositions cited in Chapter I, the failure of the particular norm under consideration to possess the indicated properties in no way precludes the possible existence of some other norm with an equivalent topology which might induce an inner product on the space. To give emphasis to this point, a trivial example is now given.

Example: Let X be the Euclidean plane. It is well known that equivalent norms can be defined on X by

$$||x||_p = (|a|^p + |b|^p)^{1/p}, \text{ where } x = (a, b).$$

It is an easy matter to verify, by the use of Theorem 3 of Chapter 1, that, if p = 2 the norm $||x||_p$ induces an inner product on the space, while for p = 1 the norm $||x||_p$ cannot be used to define an inner product.

The principal conclusion of this research is contained in the following Theorems 12, 14 and 15. It is interesting to compare these theorems with Theorems 1 and 2 of Chapter I.

THEOREM 12: Let X be a topological linear space such that there exists a Hamel basis H with the property that the set

$$C = \{x: x = \sum_{i=1}^{n} \alpha_i x_i, \quad x_i \in H, \quad \sum_{i=1}^{n} |\alpha_i|^2 < 1, \quad n \text{ arbitrary}\}$$

is open and bounded, then X is innerproductable.

Proof: First it will be proven that C is a convex set. Let x, y ε C, then the indexing of the basis elements can be arranged so that

$$x = \sum_{i=1}^{n} \alpha_i x_i, \quad x_i \in H, \quad \sum_{i=1}^{n} |\alpha_i|^2 < 1 \qquad \text{and}$$

$$y = \sum_{i=1}^{n} \beta_i x_i, \quad x_i \in H, \quad \sum_{i=1}^{n} |\beta_i|^2 < 1.$$

For any t, $0 < t < 1$,

$$tx + (1 - t)y = t \sum_{i=1}^{n} \alpha_i x_i + (1 - t) \sum_{i=1}^{n} \beta_i x_i$$

$$= \sum_{i=1}^{n} (t\alpha_i + [1 - t]\beta_i)x_i.$$

Now $\left(\sum_{i=1}^{n} |t\alpha_i + [1 - t]\beta_i|^2\right)^{1/2}$

$$\leq \left(\sum_{i=1}^{n} |t\alpha_i|^2\right)^{1/2} + \left(\sum_{i=1}^{n} |[1 - t]\beta_i|^2\right)^{1/2}$$

$$= t \left(\sum_{i=1}^{n} |\alpha_i|^2\right)^{1/2} + (1 - t) \left(\sum_{i=1}^{n} |\beta_i|^2\right)^{1/2}$$

$$< t + (1 - t) = 1.$$

Hence $\sum_{i=1}^{n} |t\alpha_i + [1 - t]\beta_i|^2 < 1.$

It is immediately evident that if $|t| \leq 1$ and $x \in C$ then $tx \in C$. For $tx = \sum_{i=1}^{n} t\alpha_i x_i$ and

$$\sum_{i=1}^{n} |t\alpha_i|^2 = |t|^2 \sum_{i=1}^{n} |\alpha_i|^2 < 1.$$

Thus C is also balanced.

Next it will be shown that X is a T_1 - space. Since C is open and bounded it follows that $\{\frac{1}{k} C\}_{k=1}^{\infty}$ is a countable basis at 0, for if U is any neighborhood of 0 there exists a real number k such that $C \subset kU$ [Def. 3]; thus $\frac{1}{k} C \subset U$. It can easily be verified that

$$\frac{1}{k} C = \{x: x = \sum_{i=1}^{n} \alpha_i x_i, \quad x_i \in H, \quad \sum_{i=1}^{n} |\alpha_i|^2 < \frac{1}{k^2} \}.$$

For any x, y ε X, with $x \neq y$ express $x - y$ as $\sum_{i=1}^{n} \alpha_i x_i$, $x_i \in H$. Put $\sum_{i=1}^{n} |\alpha_i|^2 = a$ when $0 < a < \infty$. Pick k so that $\frac{1}{k^2} < a$, then $\frac{1}{k} C$ is a neighborhood of 0 and $x - y \not\in \frac{1}{k} C$. The set $\frac{1}{k} C + y$ is a neighborhood of y and $x \not\in \frac{1}{k} C + y$. The argument is symmetric, thus X is T_1.

It is well known that [1, page 136] the Minkowski functional on a bounded, convex and balanced neighborhood of 0 in a T_1 - space defines a norm on the space; therefore let the Minkowski functional of C induce the norm $||\cdot||$ on X.

Observe that if $x = \sum_{i=1}^{n} \alpha_i x_i$ and $||x|| = \frac{1}{a}$, then

$$1 = a||x|| = ||\sum_{i=1}^{n} a\alpha_i x_i||.$$

This implies that ax is on the boundary of C, which in turn implies $\sum_{i=1}^{n} |a\alpha_i|^2 = 1$. Hence $\sum_{i=1}^{n} |\alpha_i|^2 = \frac{1}{a^2} = ||x||^2$.

It can now be shown that for any $x, y \in X$,

$$||x + y||^2 + ||x - y||^2 = 2(||x||^2 + ||y||^2).$$

Suppose that $x \neq 0$, $y \neq 0$, $x \neq \pm y$, for equality is obvious otherwise. As before let $x = \sum_{i=1}^{n} \alpha_i x_i$, $y = \sum_{i=1}^{n} \beta_i x_i$, $x_i \in H$. Since the limits of summation are finite,

$$||x + y||^2 + ||x - y||^2 = \sum_{i=1}^{n} |\alpha_i + \beta_i|^2 + \sum_{i=1}^{n} |\alpha_i - \beta_i|^2$$

$$= \sum_{i=1}^{n} (|\alpha_i + \beta_i|^2 + |\alpha_i - \beta_i|^2)$$

Replacing the α's and β's by their complex equivalents, (u, v) and (r, s) respectively, and noting that

$$|\alpha + \beta|^2 + |\alpha - \beta|^2$$

$$= |(u + r, v + s)|^2 + |(u - r, v - s)|^2$$

$$= (u + r)^2 + (v + s)^2 + (u - r)^2 + (v - s)^2$$

$$= 2(u^2 + v^2 + r^2 + s^2)$$

$$= 2(|\alpha|^2 + |\beta|^2),$$

it follows that

$$||x + y||^2 + ||x - y||^2$$

$$= \sum_{i=1}^{n} (|\alpha_i + \beta_i|^2 + |\alpha_i - \beta_i|^2)$$

$$= 2(\sum_{i=1}^{n} |\alpha_i|^2 + \sum_{i=1}^{n} |\beta_i|^2)$$

$$= 2(||x||^2 + ||y||^2).$$

Therefore by Theorem 3 above, X is innerproductable. The proof of the theorem is complete.

COROLLARY 1: If X is a topological linear space with an inner product induced by the set C of Theorem 12, then the set H of Theorem 12 is an orthonormal basis with respect to this inner product.

Proof: Let $H = \{x_i\}$, x_i's independent in X. If

$$x = \sum_{i=1}^{n} a_i x_i,$$

then

$$||x||^2 = \sum_{i=1}^{n} |a_i|^2$$

from the proof of Theorem 12. By Theorem 3

$$<x_j, x_k> = \frac{1}{4} (||x_j + x_k||^2 - ||x_j - x_k||^2 + i||x_j + ix_k||^2 - i||x_j - ix_k||^2).$$

It follows readily from these two facts that:

(1) If $j \neq k$,

$$<x_j, x_k> = \frac{1}{4} ([1^2 + 1^2] - [1^2 + |-1|^2] + i[1^2 + |i|^2] - i[1^2 + |-i|^2])$$

$$= \frac{1}{4} (2 - 2 + 2i - 2i) = 0.$$

(2) If $j = k$,

$$<x_j, x_k> = \frac{1}{4} (2^2 - 0^2 + i[1^2 + |i|^2] - i[1^2 + |-i|^2])$$

$$= \frac{1}{4} (4 + 2i - 2i) = 1.$$

DEFINITION 4: A sequence $\{x_n\}$ in a topological linear space is Cauchy if given any neighborhood U of 0, there exists a number N such that if n, m > N then $x_n - x_m \in U$ [8, page 60].

DEFINITION 5: A topological linear space X is complete if every Cauchy sequence in X converges to a point in X.

THEOREM 13: A normed linear space is complete as a normed linear space if and only if it is complete as a topological linear space.

The trivial proof is suppressed.

In view of Corollary I, X of Theorem 12 is either a finite dimensional linear space or a pre-Hilbert space. To see this recall that [13, page 252] every Hilbert space, with a non-zero element, has a complete orthonormal set in X and that [14, page 150] an infinite complete orthonormal set is never a Hamel basis in a Hilbert space.

THEOREM 14: If X is an infinite dimensional Hilbert space, then there exists a pre-Hilbert space $M \subset X$ with a Hamel basis H such that the set

$$C = \{x: x = \sum_{i=1}^{n} a_i x_i, \quad x_i \in H, \quad \sum_{i=1}^{n} |a_i|^2 < 1\}$$

is open and bounded [Def. 3] in the relative topology of M.

Proof: Since X is infinite dimensional, X has at least one non-zero element so there exists in X a complete orthonormal set K. Let M be the linear manifold generated by K, i.e. $x \varepsilon M$ if and only if x is a finite linear combination of the elements of K. Hence K is a Hamel basis for M and is the H of the theorem. The closure of M is X [1, page 114]. It is easily verified that the inner product on X when restricted to M is an inner product on M. Thus M is a pre-Hilbert space contained in X.

In determining the relative topology on M, it will suffice to restrict the discussion to a neighborhood base at 0 [1, page 124]. Denote by $||\cdot||$ and $||\cdot||_M$ the norms induced by the inner product on X and the inner product on X restricted to M respectively. One neighborhood base at 0 in X is of the form

$$\{x \varepsilon X: \ ||x|| < \tfrac{1}{n}\}_{n=1}^{\infty}.$$

Similarly

$$\{x \varepsilon M: \ ||x||_M < \tfrac{1}{n}\}_{n=1}^{\infty}$$

is a neighborhood base at 0 in M. For each n, the intersection of the above two sets is just the second set. Thus the inner product on X restricted to M induces the relative topology of M.

Since $K \equiv H$ is a complete orthonormal set in X and $M \subset X$, for each x in M, with

$$x = \sum_{i=1}^{n} a_i x_i, \quad x_i \in H,$$

$$||x||_M^2 = ||x||^2 = <x, x> = \sum_{i=1}^{n} |a_i|^2.$$

This implies that

$$C = \{x: x = \sum_{i=1}^{n} a_i x_i, \quad x_i \in H, \quad \sum_{i=1}^{n} |a_i|^2 < 1\}$$

$$\equiv \{x: \quad ||x||_M < 1\} = S_1(0),$$

where the notation $S_1(0)$ denotes the open sphere in M with center at 0 and radius 1. Since the set of open spheres $\{S_{\frac{1}{k}}(0)\}$ with centers at 0 and radius $\frac{1}{k}$ form a base at 0, given any neighborhood U of 0, there exists a k such that $S_{\frac{1}{k}}(0) \subset U$. Thus $C = S_1(0) \subset kU$ and C is bounded in the sense of Definition 3. This completes the proof.

THEOREM 15: An infinite dimensional complete topological linear space X which (a) is T_1 and (b) has a countable base at 0 is a Hilbert space if and only if (c) there exists in X a dense subset M with a Hamel basis H such that the set

$$C = \{x: x = \sum_{i=1}^{n} a_i x_i, \quad x_i \in H, \quad \sum_{i=1}^{n} |a_i|^2 < 1\}$$

is open and bounded in the relative topology on M.

Proof: The necessity of (a), (b) is by Theorem 1 and that of (c) by Theorem 14. For sufficiency, let M be the topological linear space of Theorem 12. Then by Theorem 12, M is innerproductable. By (a), (b) X is metrizable [Th. 1]. Since M has a completion as a metric space which is unique up to an isometry [14, page 54] and the closure of M is X, X can be considered as the completion of M. Therefore, X is a Hilbert space. This completes the proof.

It may be that one wishes to test a given norm to see if this norm can be used to define an inner product on the space. Theorems 12, 14 and 15 make it possible, in certain cases, to test not only the given norm, but all equivalent norms to see if there exists one which can be used to define an inner product on the space. This chapter is concluded with such an example.

Example: Let $X = L^p(a, b) = \{x: \int_a^b |x(s)|^p ds < \infty\}$, where $0 < p < 1$ and x is a real valued function on the real interval [a, b]. If we define

$$||x|| = \int_a^b |x(s)|^p ds,$$

X becomes a complete normed linear space and moreover the only nonempty open convex set in X is the whole space [1, page 158].

If it is assumed that X is innerproductable, there would exist a norm $||\cdot||^*$ equivalent to $||\cdot||$ and a set K of vectors in X which is a complete orthonormal basis with respect to the inner product defined from $||\cdot||^*$. Thus the set

$$C = \{x: x \in \sum_{i=1}^{n} \alpha_i x_i, \quad x_i \in K, \quad \sum_{i=1}^{n} |\alpha_i|^2 < 1\}$$

would be a convex neighborhood of 0 in the linear manifold M generated by K. The topology of M being the relative topology induced by the norm $||\cdot||^*$ on X. The closure of C, Cl(C), in X is also a convex set in X, for assume otherwise. Then there exist x and y in Cl(C) and a number t between 0 and 1 such that $z = tx + (1 - t)y \notin Cl(C)$. This implies that there is an $\varepsilon > 0$ such that

$$\{w: ||z - w||^* < \varepsilon\} \cap C = \emptyset.$$

Since x and y are in Cl(C), a p and q can be picked from C such that

$$||p - x||^* < \frac{\varepsilon}{2t + 1} \quad \text{and} \quad ||q - y||^* < \frac{\varepsilon}{2t + 1}.$$

Consider the point $r = tp + (1 - t)q$ in M. Then

$$||r - z||^* = ||tp + (1 - t)q - tx - (1 - t)y||^*$$

$$\leq t||p - x||^* + ||q - y||^* + t||q - y||^*$$

$$< \frac{t\varepsilon}{2t + 1} + \frac{\varepsilon}{2t + 1} + \frac{t\varepsilon}{2t + 1} = \varepsilon$$

Hence $r \notin C$ and C is not convex in M, a contradiction.

Also the interior of Cl(C) is convex in X. Denote this set by S. If S is not convex, there exists a t, $0 < t < 1$, such that for some x, y in S, $z = tx + (1 - t)y$ is not in S. But since Cl(S) = Cl(C) and Cl(S) is convex, z is in Cl(C) and must be on the boundary of Cl(C). This implies $||z||^* = 1$. Evidently $||x||^* < 1$ and $||y||^* < 1$, so an $\varepsilon > 0$ may be chosen such that $||(1 + \varepsilon)x||^* < 1$ and $||(1 + \varepsilon)y||^* < 1$ so that $(1 + \varepsilon)x$ and $(1 + \varepsilon)y$ are in S. But the point

$$t(1 + \varepsilon)x + (1 - t)(1 + \varepsilon)y = (1 + \varepsilon)z$$

is not in Cl(C) since $||(1 + \varepsilon)z||^* > 1$. Thus a contradiction.

Hence S is an open convex set in X. But, as was pointed out initially, the only nonempty open convex set in X is X itself. Therefore $S \equiv X$. However if x_1 and $x_2 \ \varepsilon \ H$, then $x = 3x_1 + 2x_2 \not\varepsilon S$ since

$$||3x_1 + 2x_2||^* = 3^2 + 2^4 = 13.$$

But clearly $x \ \varepsilon \ X$ and this contradiction forces the conclusion that X is not innerproductable.

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