SECTIONS OF N-DIMENSIONAL SPHERICAL CONES

by

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TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER I

INTRODUCTION

1.1 <u>Purpose and Scope:</u> Many proofs have been given for the proposition: "The intersection of a right circular cone and a plane is a second degree curve (conic)". Among the first mathematicians to have proved it was Apollonius, during the period 262-200 B.C.. In fact the first book in conic sections was written by Apollonius. In this thesis, first, we supply a vector proof of this proposition, thereby simplifying procedures in the proof. Then we generalize the theorem for sections of a spherical cone in an n-dimensional Euclidean space.

1.2 <u>Definitions and Notations:</u> All spaces are real Euclidean. A real Euclidean space is a vector space V_n such that to every vector in V_n there corresponds an ordered set of n real numbers for some fixed integer n, with a definition of addition of two such sets, of multiplication of a set by a number, and of inner product of two such sets. We denote an n-dimensional Euclidean space by E_n. The inner product of vectors $\alpha = (x_1,\ldots,x_n)$ and $\beta = (y_1,\ldots,y_n)$ will be denoted by (α,β) and defined as follows:

$$(\alpha,\beta) = x_1y_1+\ldots+ x_ny_n.$$

The norm of a vector $\alpha = (x_1,\ldots,x_n)$ is defined to be $\|\alpha\| = (\alpha,\alpha)^{\frac{1}{2}}$. Any subset S of E_n, for which the sum of any two vectors of S is a vector in S and the product of a vector S by a scalar is also in S, is called a subspace of E_n. The subspace spanned by the set of vectors $\{\alpha_1,\ldots,\alpha_k\}$ will be indicated by $[\alpha_1,\ldots,\alpha_k]$.

CHAPTER II

PLANE SECTIONS OF A CONE

2.1 <u>Theorem:</u> A plane and a right circular cone intersect in a second degree curve (conic section).

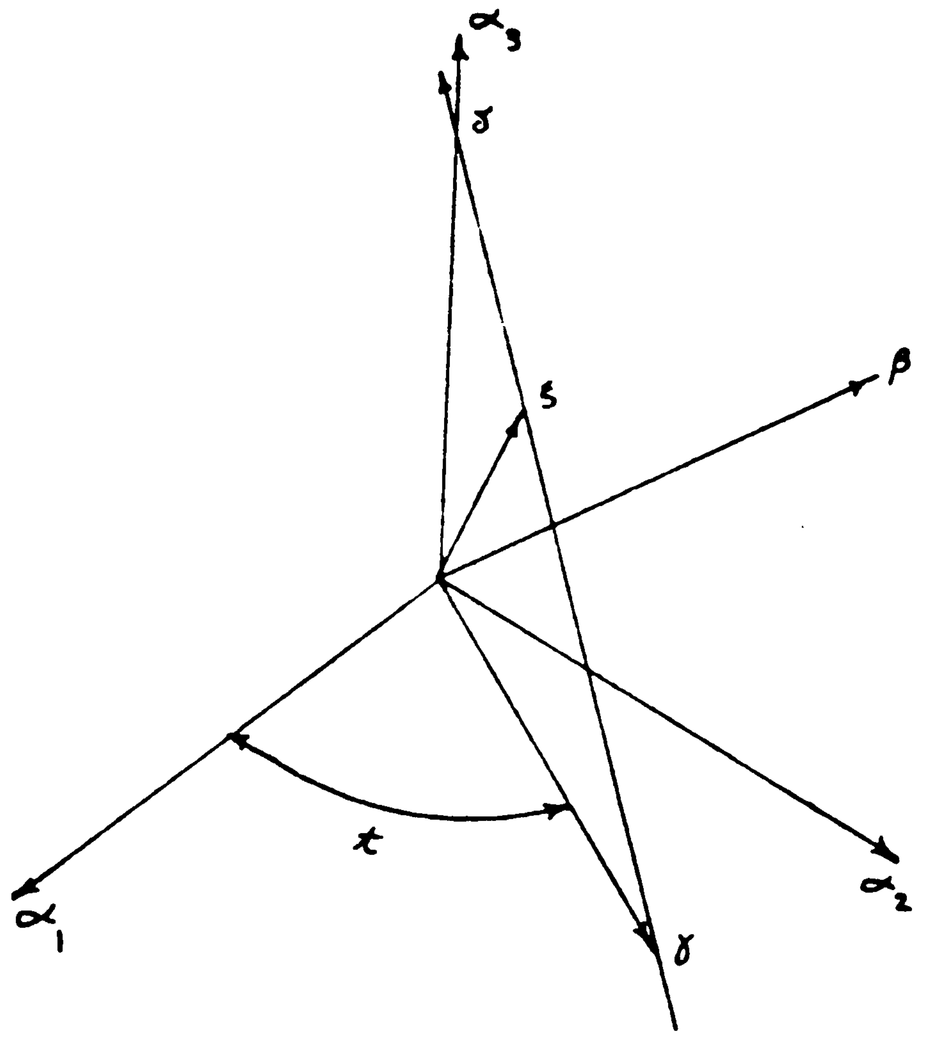

Fig. 1

Proof: Consider the orthonormal set $\{\alpha_1,\alpha_2,\alpha_3\}$ (Fig. 1). Let the vector $\gamma \in [\alpha_1,\alpha_2]$ such that

$$\gamma = (a \cos t)\alpha_1 + (a \sin t)\alpha_2, \qquad (1)$$

where a , a positive fixed real number, is the radius of the circle generated by γ. Choose the vector δ on α_3 such that $\delta = b\alpha_3$, where b is a fixed, real positive number. We can now consider the right circular cone generated by the line passing through the end points of γ and δ and about an axis whose unit vector is α_3.

Consider any vector ξ ending on the surface of the cone. By vector subtraction (Fig. 2), we have

$$\xi - \delta = k(\gamma-\delta), \text{ k a real number}$$

$$\xi = k\gamma + (1-k)\delta.$$

Let $p = k$ and $q = 1-k$. Then

$$p + q = k + (1-k) = 1.$$

Therefore

$$\xi = p\gamma + q\delta, \; p + q = 1.$$

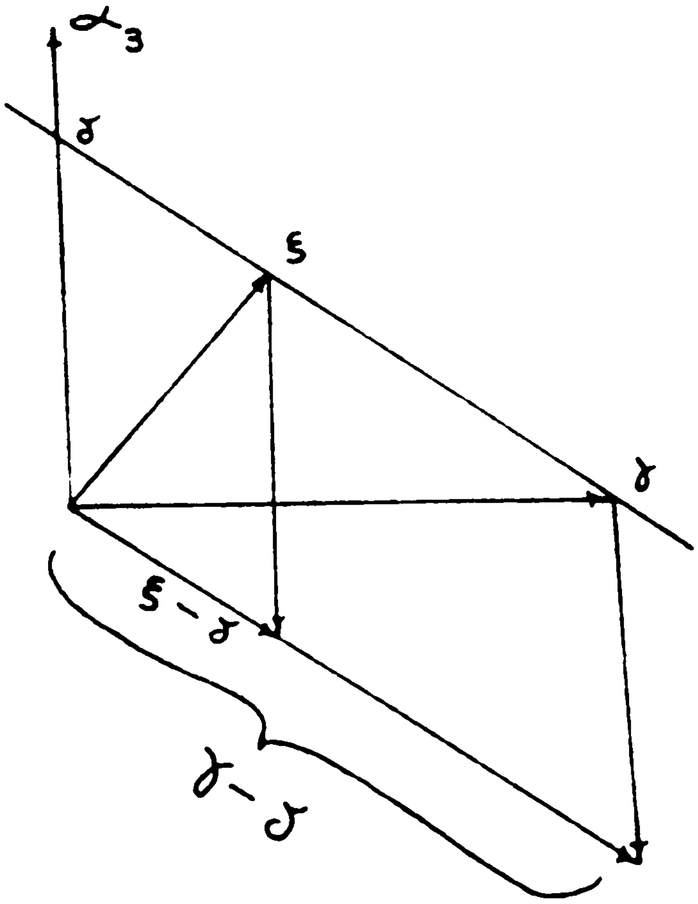

Fig. 2

Let β be a vector such that $\beta \in [\alpha_2, \alpha_3]$ and $||\beta|| = 1$. Then

$$\beta = (\beta, \alpha_2)\alpha_2 + (\beta, \alpha_3)\alpha_3.$$

Consider the plane $[\alpha_1, \beta]$. Now in order to discuss the intersection of this plane and the cone we must also have $\xi \in [\alpha_1, \beta]$. Then

$$\xi = x\alpha_1 + y\beta.$$

Therefore

$$\xi = x\alpha_1 + y(\beta, \alpha_2)\alpha_2 + y(\beta, \alpha_3)\alpha_3$$

for ξ in the plane, and

$$\xi = p(a \cos t)\alpha_1 + p(a \sin t)\alpha_2 + qb\alpha_3$$

for ξ ending on the cone. These equalities imply that

$$\begin{cases} x = ap \cos t \\ y(\beta, \alpha_2) = ap \sin t \\ y(\beta, \alpha_3) = qb, \text{ where } p + q = 1. \end{cases}$$

Suppose sin t and cos t are different from zero. Then

$$\begin{cases} p = \dfrac{x}{a \cos t} \\ p = \dfrac{y(\beta, \alpha_2)}{a \sin t} \\ q = \dfrac{y(\beta, \alpha_3)}{b}. \end{cases}$$

Therefore we have the following equations

$$\begin{cases} \dfrac{x}{a\cos t} + \dfrac{y(\beta,\alpha_3)}{b} = 1 \\ \dfrac{y(\beta,\alpha_2)}{a\sin t} + \dfrac{y(\beta,\alpha_3)}{b} = 1. \end{cases}$$

Thus

$$\begin{cases} \dfrac{x}{a\cos t} = 1 - \dfrac{y(\beta,\alpha_3)}{b} \\ \dfrac{y(\beta,\alpha_2)}{a\sin t} = 1 - \dfrac{y(\beta,\alpha_3)}{b}. \end{cases} \qquad (2)$$

This set of equations implies that

$$\frac{x}{a\cos t} = \frac{y(\beta,\alpha_2)}{a\sin t}.$$

From (2) we have

$$\begin{cases} \dfrac{x}{a\cos t} = \dfrac{b - y(\beta,\alpha_3)}{b} \\ \dfrac{y(\beta,\alpha_2)}{a\sin t} = \dfrac{b - y(\beta,\alpha_3)}{b}. \end{cases}$$

Therefore

$$\begin{cases} \dfrac{a}{b}\cos t = \dfrac{x}{b-y(\beta,\alpha_3)} \\ \dfrac{a}{b}\sin t = \dfrac{y(\beta,\alpha_2)}{b-y(\beta,\alpha_3)}, \quad b-y(\beta,\alpha_3) \neq 0. \end{cases}$$

This set of equations implies that

$$\begin{cases} \dfrac{a^2}{b^2}\cos^2 t = \dfrac{x^2}{[b-y(\beta,\alpha_3)]^2} \\ \dfrac{a^2}{b^2}\sin^2 t = \dfrac{y^2(\beta,\alpha_2)^2}{[b-y(\beta,\alpha_3)]^2}, \quad b-y(\beta,\alpha_3) \neq 0. \end{cases}$$

Therefore we have

$$\frac{a^2}{b^2}(\cos^2 t + \sin^2 t) = \frac{x^2 + y^2(\beta,\alpha_2)^2}{[b-y(\beta,\alpha_3)]^2}.$$

Consequently

$$b^2x^2 + b^2(\beta,\alpha_2)^2y^2 = a^2[b-y(\beta,\alpha_3)]^2$$

which proves the theorem.

2.2 <u>Special cases:</u> Cases where either cos t or sin t or $b-y(\beta,\alpha_3)$ is zero are now discussed.

Suppose cos t is zero. This implies that sin t is plus or minus one and that (1) in 2.1 is as follows:

$$y = a(\pm 1)\alpha_2.$$

Therefore ξ in 2.1 becomes

$$\xi = x\alpha_1 + y(\beta,\alpha_2)\alpha_2 + y(\beta,\alpha_3)\alpha_3$$

for ξ in the plane, and

$$\xi = pa(0)\alpha_1 + pa(\pm 1)\alpha_2 + qb\alpha_3$$

for ξ ending on the cone. These equalities imply that

$$\begin{cases} x = 0 \\ y(\beta,\alpha_2) = \pm ap \\ y(\beta,\alpha_3) = qb, \text{ where } p + q = 1. \end{cases}$$

Then

$$\begin{cases} p = \pm\dfrac{y(\beta,\alpha_2)}{a} \\ q = \dfrac{y(\beta,\alpha_3)}{b}. \end{cases}$$

Therefore

$$\pm\frac{y(\beta,\alpha_2)}{a} + \frac{y(\beta,\alpha_3)}{b} = 1.$$

This equation implies that

$$\pm\frac{y(\beta,\alpha_2)}{a} = 1 - \frac{y(\beta,\alpha_3)}{b}$$

Thus

$$\pm\frac{y(\beta,\alpha_2)}{a} = \frac{b-y(\beta,\alpha_3)}{b}.$$

Therefore, we have

$$\frac{y^2(\beta,\alpha_2)^2}{a^2} = \frac{[b-y(\beta,\alpha_3)]^2}{b^2}.$$

Consequently

$$b^2(\beta,\alpha_2)^2y^2 = a^2[b-y(\beta,\alpha_3)]^2$$

which gives the y-intercepts of the conic section.

Suppose sin t is zero. This implies that cos t is plus or minus one and that 2.1 (1) is

$$\gamma = a(\pm 1)\alpha_1.$$

Therefore ξ in 2.1 becomes

$$\xi = x\alpha_1 + y(\beta,\alpha_2)\alpha_2 + y(\beta,\alpha_3)\alpha_3$$

for ξ in the plane, and

$$\xi = pa(\pm 1)\alpha_1 + pa(0)\alpha_2 + qb\alpha_3$$

for ξ ending on the cone. These equalities imply that

$$\begin{cases} x = ap(\pm 1) \\ y(\beta,\alpha_2) = 0 \\ y(\beta,\alpha_3) = qb, \text{ where } p + q = 1. \end{cases}$$

Consider the equation

$$y(\beta,\alpha_2) = 0.$$

This equation implies that either y is zero or (β,α_2) is zero.

Suppose y is zero. Then qb is zero also. Since b is a fixed positive number this implies that q is zero. We know p + q = 1. Therefore p is one. This implies that

$$x = \pm a$$

And

$$x^2 = a^2$$

which gives the x-intercepts of the conic section.

Now suppose (β,α_2) is zero. Then β is α_3. This implies that

$$y(\beta,\alpha_3) = y = qb.$$

Therefore by (1) of this section we have

$$\begin{cases} p = \pm \dfrac{x}{a} \\ q = \dfrac{y}{b}, \text{ where } p + q = 1. \end{cases}$$

This set of equations implies that

$$\pm\frac{x}{a} + \frac{y}{b} = 1.$$

Therefore

$$\pm\frac{x}{a} = \frac{b-y}{b}.$$

This equation implies that

$$\frac{x^2}{a^2} = \frac{(b-y)^2}{b^2}.$$

Consequently

$$b^2x^2 = a^2(b-y)^2$$

which means the section is two intersecting lines.

Suppose $b-y(\beta,\alpha_3)$ is zero. This implies that

$$b = y(\beta,\alpha_3).$$

Therefore ξ in 2.1 becomes

$$\xi = x\alpha_1 + y(\beta,\alpha_2)\alpha_2 + b\alpha_3$$

for ξ in the plane, and

$$\xi = pa(\cos t)\alpha_1 + pa(\sin t)\alpha_2 + qb\alpha_3$$

for ξ ending on the cone. These equalities imply that

$$\begin{cases} x = pa \cos t \\ y(\beta,\alpha_2) = pa \sin t \\ b = qb, \text{ where } p+q=1. \end{cases}$$

This set of equations implies that q is one. And since $p + q = 1$, this implies that p is zero. Therefore we have the following equations:

$$\begin{cases} x = 0 \\ y(\beta, \alpha_2) = 0. \end{cases}$$

Since b is not zero, then y is not zero. Therefore

$$\begin{cases} (\beta, \alpha_2) = 0 \\ \beta = \alpha_3. \end{cases}$$

This set of equations implies that the plane is $[\alpha_1, \alpha_3]$ and in this plane we obtain the point (0,b), i.e. we have the intersection of a line through the origin and the cone.

CHAPTER III

SPHERICAL CONES

3. Spherical cones: Let E_n be an n-dimensional Euclidean space. Choose the orthonormal basis $\{\alpha_1,\ldots,\alpha_n\}$ in E_n. Let $\gamma \in [\alpha_1,\ldots,\alpha_{n-1}]$ with $||\gamma|| = r > 0$. Then there are real numbers a_i, $i=1,\ldots,n-1$, for which

$$\gamma = \sum_{i=1}^{n-1} a_i\alpha_i \text{ and } \sum_{i=1}^{n-1} a_i^2 = r^2.$$

We can choose real numbers $t_1,\ldots,t_{n-2}$ such that

$$\begin{aligned} a_1 &= r \cos t_1 \\ a_2 &= r \sin t_1 \cos t_2 \\ \ldots &\quad \ldots \quad \ldots \\ a_{n-2} &= r \sin t_1 \ldots \sin t_{n-3} \cos t_{n-2} \\ a_{n-1} &= r \sin t_1 \ldots \sin t_{n-2}. \end{aligned}$$

Choose $\delta = b\alpha_n$, where b is a fixed, real positive number. Let $\xi = p\gamma + q\delta$, $p + q = 1$. The locus of ξ (end point of ξ) as γ changes, i.e., as $t_1,\ldots,t_{n-2}$ change, is called an (n-1)-dimensional right spherical cone.

CHAPTER IV

SECTIONS OF N-DIMENSIONAL SPHERICAL CONES

4. <u>Theorem:</u> A hyperplane intersects a right spherical cone in a quadric.

Proof: Let $\{\alpha_1,\ldots,\alpha_n\}$ be the orthonormal set of 3. Choose the orthonormal set $\{\beta_2,\ldots,\beta_{n-1}\}$ such that $\beta_i \in [\alpha_2,\ldots,\alpha_n]$ for $i = 2,\ldots,n-1$. Then

$$\beta_i = (\beta_i,\alpha_2)\alpha_2 + \ldots + (\beta_i,\alpha_n)\alpha_n, \quad i=2,\ldots,n-1. \qquad (1)$$

Consider the hyperplane $[\alpha_1,\beta_2,\ldots,\beta_{n-1}]$. Let ξ be in the intersection of this hyperplane and the cone. Then $\xi \in [\alpha_1,\beta_2,\ldots,\beta_{n-1}]$ and ξ satisfies the definition in 3. Thus

$$\begin{cases} \xi = x_1\alpha_1 + x_2\beta_2 + \ldots + x_{n-1}\beta_{n-1} \\ \xi = p(a_1\alpha_1 + \ldots + a_{n-1}\alpha_{n-1}) + qb\alpha_n, \ p + q = 1. \end{cases}$$

However, by (1) we have

$$\begin{aligned} \xi &= x_1\alpha_1 + x_2[(\beta_2.\alpha_2)\alpha_2 + \ldots + (\beta_2,\alpha_n)\alpha_n] \\ &\quad + \ldots + x_{n-1}[(\beta_{n-1},\alpha_2)\alpha_2 + \ldots + (\beta_{n-1},\alpha_n)\alpha_n] \\ &\equiv p(a_1\alpha_1 + \ldots + a_{n-1}\alpha_{n-1}) + qb\alpha_n, \ p + q = 1. \end{aligned}$$

This identity implies that

$$\begin{cases} x_1 = pa_1 \\ x_2(\beta_2,\alpha_2) + \ldots + x_{n-1}(\beta_{n-1},\alpha_2) = pa_2 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \\ x_2(\beta_2,\alpha_{n-1}) + \ldots + x_{n-1}(\beta_{n-1},\alpha_{n-1}) = pa_{n-1} \\ x_2(\beta_2,\alpha_n) + \ldots + x_{n-1}(\beta_{n-1},\alpha_n) = qb, \; p + q = 1. \end{cases}$$

Suppose $a_1,\ldots,a_{n-1}$ are all different from zero. Then

$$\begin{cases} p = \dfrac{x_1}{a_1} \\ p = \dfrac{x_2(\beta_2,\alpha_2) + \ldots + x_{n-1}(\beta_{n-1},\alpha_2)}{a_2} \\ \ldots \quad \ldots \quad \ldots \\ p = \dfrac{x_2(\beta_2,\alpha_{n-1}) + \ldots + x_{n-1}(\beta_{n-1},\alpha_{n-1})}{a_{n-1}} \\ q = \dfrac{x_2(\beta_2,\alpha_n) + \ldots + x_{n-1}(\beta_{n-1},\alpha_n)}{b}, \text{ where } p + q = 1. \end{cases}$$

Therefore we have the following equations

$$\begin{cases} \dfrac{x_1}{a_1} + \dfrac{x_2(\beta_2,\alpha_n) + \ldots + x_{n-1}(\beta_{n-1},\alpha_n)}{b} = 1 \\ \dfrac{x_2(\beta_2,\alpha_2) + \ldots + x_{n-1}(\beta_{n-1},\alpha_2)}{a_2} \\ \qquad\qquad + \dfrac{x_2(\beta_2,\alpha_n) + \ldots + x_{n-1}(\beta_{n-1},\alpha_n)}{b} = 1 \\ \qquad \ldots \qquad \ldots \qquad \ldots \\ \dfrac{x_2(\beta_2,\alpha_{n-1}) + \ldots + x_{n-1}(\beta_{n-1},\alpha_{n-1})}{a_{n-1}} \\ \qquad\qquad + \dfrac{x_2(\beta_2,\alpha_n) + \ldots + x_{n-1}(\beta_{n-1},\alpha_n)}{b} = 1. \end{cases}$$

This set of equations implies that

$$\begin{cases} \dfrac{x_1}{a_1} = 1 - \dfrac{x_2(\beta_2,\alpha_n) + \ldots + x_{n-1}(\beta_{n-1},\alpha_n)}{b} \\ \dfrac{x_2(\beta_2,\alpha_i) + \ldots + x_{n-1}(\beta_{n-1},\alpha_i)}{a_i} \\ \qquad\qquad = 1 - \dfrac{x_2(\beta_2,\alpha_n) + \ldots + x_{n-1}(\beta_{n-1},\alpha_n)}{b}, \end{cases}$$

where $i = 2,\ldots,n-1$.

Therefore

$$\begin{cases} \dfrac{x_1}{a_1} = \dfrac{b - [x_2(\beta_2,\alpha_n) + \ldots + x_{n-1}(\beta_{n-1},\alpha_n)]}{b} \\ \dfrac{x_2(\beta_2,\alpha_i) + \ldots + x_{n-1}(\beta_{n-1},\alpha_i)}{a_i} \\ \qquad\qquad = \dfrac{b - [x_2(\beta_2,\alpha_n) + \ldots + x_{n-1}(\beta_{n-1},\alpha_n)]}{b}, \end{cases}$$

where $i=2,\ldots n-1$.

Thus

$$\begin{cases} \dfrac{a_1}{b} = \dfrac{x_1}{b-[x_2(\beta_2,\alpha_n)+\ldots+x_{n-1}(\beta_{n-1},\alpha_n)]} \\ \dfrac{a_i}{b} = \dfrac{x_2(\beta_2,\alpha_i)+\ldots+x_{n-1}(\beta_{n-1},\alpha_i)}{b-[x_2(\beta_2,\alpha_n)+\ldots+x_{n-1}(\beta_{n-1},\alpha_n)]}, \quad i=2,\ldots,n-1. \end{cases}$$

This set of equations implies that

$$\begin{cases} \dfrac{a_1^2}{b^2} = \dfrac{x_1^2}{\{b-[x_2(\beta_2,\alpha_n)+\ldots+x_{n-1}(\beta_{n-1},\alpha_n)]\}^2} \\ \dfrac{a_i^2}{b^2} = \dfrac{[x_2(\beta_2,\alpha_i)+\ldots+x_{n-1}(\beta_{n-1},\alpha_i)]^2}{\{b-[x_2(\beta_2,\alpha_n)+\ldots+x_{n-1}(\beta_{n-1},\alpha_n)]\}^2}, \quad i=2,\ldots,n-1. \end{cases}$$

Therefore

$$\begin{cases} \dfrac{a_1^2}{b^2}+\ldots+\dfrac{a_{n-1}^2}{b^2} \\ = \dfrac{x_1^2+[x_2(\beta_2,\alpha_2)+\ldots+x_{n-1}(\beta_{n-1},\alpha_2)]^2}{\{b-[x_2(\beta_2,\alpha_n)+\ldots+x_{n-1}(\beta_{n-1},\alpha_n)]\}^2} \\ \quad +\ldots+\dfrac{[x_2(\beta_2,\alpha_{n-1})+\ldots+x_{n-1}(\beta_{n-1},\alpha_{n-1})]^2}{\{b-[x_2(\beta_2,\alpha_n)+\ldots+x_{n-1}(\beta_{n-1},\alpha_n)]\}^2}, \end{cases}$$

where $b-[x_2(\beta_2,\alpha_n)+\ldots+x_{n-1}(\beta_{n-1},\alpha_n)] \neq 0$ and

$\sum_{i=1}^{n-1} a_i^2 = r^2$ as defined in 3. Consequently

$$r^2 \; b-[x_2(\beta_2,\alpha_n) + \dots + x_{n-1}(\beta_{n-1},\alpha_n)]^2$$
$$= b^2 x_1^2 + [x_2(\beta_2,\alpha_2) + \dots + x_{n-1}(\beta_{n-1},\alpha_2)]^2$$
$$+ \dots + [x_2(\beta_2,\alpha_{n-1}) + \dots + x_{n-1}(\beta_{n-1},\alpha_{n-1})]^2$$

which proves the theorem.

Special cases where $a_1,\dots,a_{n-1}$ are zero or $b- [x_2(\beta_2,\alpha_n) + \dots + x_{n-1}(\beta_{n-1},\alpha_n)]$ is zero should be discussed. We omit the discussion.

CHAPTER V

A DISCUSSION OF THE SECTIONS

5. <u>Discussion:</u> In 4 if we set $\beta_i=\alpha_i$, $i=2,\ldots,k$, $k \leq n-1$, then the hyperplane becomes $[\alpha_1,\ldots,\alpha_k,\beta_{k+1},\ldots,\beta_{n-1}]$. Thus the equation of the intersection will be

$$r^2\{b-[x_{k+1}(\beta_{k+1},\alpha_n) +\ldots+ x_{n-1}(\beta_{n-1},\alpha_n)]\}^2$$

$$= b^2\{x_1^2 +\ldots+ x_k^2$$

$$+ [x_{k+1}(\beta_{k+1},\alpha_{k+1}) +\ldots+ x_{n-1}(\beta_{n-1},\alpha_{k+1})]^2$$

$$+\ldots+ [x_{k+1}(\beta_{k+1},\alpha_{n-1}) +\ldots+ x_{n-1}(\beta_{n-1},\alpha_{n-1})]^2\}.$$

In this case the quadric which is in an (n-1)-dimensional subspace has spherical sections.

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