P-GROUPS WITH ELEMENTS

OF INFINITE HEIGHT

by

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CHAPTER I

Introduction

Throughout this thesis, the word "group" will be understood to mean an additively written abelian group.

If G is a group, H a subgroup of G and n a positive integer, then nH is the collection of all elements in G of the form nh with h in H, and H[n] is the subset of H consisting of all elements h in H such that nh = 0. It is easy to show that nH and H[n] are subgroups of G.

If H and K are subgroups of G and x is an element of G, then {H,K} will denote the subgroup of G generated by the elements of H and K, {x,H} will denote the subgroup of G generated by x and the elements of H and {x} will denote the cyclic subgroup of G generated by x. If H and K are subgroups of G such that $H \cap K = 0$ and G = {H,K}, then G is said to be a direct sum of H and K. This is indicated by writing G = H + K.

A subgroup H of a group G is said to be a pure subgroup of G if $H \cap nG = nH$ for all positive n. If G = H + K, then both H and K are pure subgroups of G.

A group G is said to be a p-group if all its elements have order a power of the prime p. If G is a p-group, the following properties are noteworthy: (1) $G[p] \subseteq G[p^2] \subseteq \ldots$ and $G = \bigcup_{i=1}^{\infty} G[p^i]$; (2) if n is an integer relatively prime to p, then nG = G; and (3) a subgroup H of G is pure in G provided $H \cap p^iG = p^iH$ for all positive integers i. If G is a p-group, we set $G^1 = \bigcap_{i=1}^{\infty} p^iG$. From (2) it follows readily that $G^1 \subseteq nG$ for all positive integers n. If $G^1 = 0$, then the p-group G is said to be without elements of infinite height; and if $G^1 \neq 0$, G is said to be with elements of infinite height. A subgroup M of the p-group G is said to be a high subgroup of G if M is maximal in G with respect to $M \cap G^1 = 0$, that is, $M \cap G^1 = 0$ and if x is an element of G such that $x \notin M$, then $\{x,M\} \cap G^1 \neq 0$.

A group G is said to be divisible if $G = nG$ for all positive integers n. If E is a divisible group and if E is a subgroup of a group G, then E is a direct summand of G, that is, there is a subgroup H of G such that $G = H + E$ (see [1], p. 62). Every group can be imbedded in a divisible group, that is, if G is a group, then there is a divisible group D containing G as a subgroup (see [1], p. 65). If D is a divisible group containing the group G as a subgroup, then we say that D is minimal divisible containing G if no proper divisible subgroup of D contains G. Every group G is contained in a minimal divisible group D and if D is minimal divisible containing G, then $\{x\} \cap G \neq 0$ whenever x is a non-zero element of D (see [1], p. 66). It follows that if D is minimal divisible containing G then $D[p] = G[p]$ for each prime p. Finally, a group G is said to be reduced if G contains no non-zero divisible subgroup.

The following result seems to be well-known (see [3]):

If G and H are abelian groups containing the group A as a subgroup, then there exists a group K containing as subgroups G and $\overline{H}$ where $\overline{H}$ is isomorphic to H under a mapping which is the identity mapping on A such that $K = \{G,\overline{H}\}$ and $G \cap \overline{H} = A$. In particular, given a group G with a subgroup A, there is a group K such that $K = \{G,D\}$ and $G \cap D = A$ where D is minimal divisible containing A. The above construction seems to be a part of the folklore of abelian group theory; that is, the construction seems to be well-known to most mathematicians working in this field, though apparently it is described nowhere in the existing literature.

The main results of this thesis are Theorems 4 and 6 in Chapter III. The first of these theorems is an imbedding theorem for p-groups with elements of infinite height. It is shown that a p-group with elements of infinite height can be imbedded as a pure subgroup of a group K which is the direct sum of a p-group without elements of infinite height and a divisible p-group. The significance of such an imbedding is in the fact that the structure of the group K is considerably simpler than that of the original group G. The fact that G is pure in K is also important, since pure subgroups of a group inherit properties that other subgroups do not. Even when such imbeddings as this fail to clarify the structure of the original group G, they still provide some insight by allowing one to "visualize" G as a "well-behaved" subgroup of the simpler group K.

Theorem 6 may be viewed in some sense as a converse of Theorem 4, that is, Theorem 6 describes a construction whereby one obtains a p-group G with elements of infinite height from a p-group without elements of infinite height and a divisible p-group. The construction described in Theorem 6 is considerably simpler though not as powerful as the well-known construction due to Kulikov (see [1], p. 131). Indeed, there seems to be no general method for constructing reduced p-groups with elements of infinite height simpler than that of Theorem 6. For some applications of this construction see [4].

Since the concept of a subdirect sum is needed in both Theorems 4 and 6, Chapter II is devoted to a discussion of subdirect sums. Subdirect sums are extremely important in the study of abelian groups since they provide one (as is seen, for example, in Theorem 6) with a rather general technique for constructing complicated groups from simpler ones. Hence, though subdirect sums do not often play a significant role in the theory of abelian groups, they are virtually indispensable for invalidating the false conjectures of the mathematician doing research in abelian group theory.

CHAPTER II

SUBDIRECT SUMS

If G is the direct sum of G_1 and G_2, then Π_1 and Π_2 will denote the mappings of G onto G_1 and G_2 respectively defined as follows: if $g = g_1 + g_2$ with $g_1 \varepsilon G_1$ and $g_2 \varepsilon G_2$, then $\Pi_1(g) = g_1$ and $\Pi_2(g) = g_2$.

DEFINITION. If A is a subgroup of the group $G = G_1 + G_2$ such that $\Pi_1(A) = G_1$ and $\Pi_2(A) = G_2$, then A is said to be a subdirect sum of G_1 and G_2.

The following theorem gives us a convenient criterion for determining whether a given subgroup of the group $G = G_1 + G_2$ is indeed a subdirect sum of G_1 and G_2.

THEOREM 1. If A is a subgroup of the group $G = G_1 + G_2$, then A is a subdirect sum of G_1 and G_2 if and only if $\{G_1,A\} = G = \{A,G_2\}$.

PROOF: Suppose that $\{G_1,A\} = G = \{A,G_2\}$. We must show that for any $g_1 \varepsilon G_1$, there exists an element a contained in A such that $\Pi_1(a) = g_1$. Let g_1 be an element of G_1, then g_1 is contained in G and since $G = \{A,G_2\}$, we can write $g_1 = a + g_2$ where $a \varepsilon A$ and $g_2 \varepsilon G_2$. By applying Π_1 to g_1 we have $\Pi_1(g_1) = \Pi_1(a + g_2)$. However $\Pi_1(g_1) = g_1$ and $\Pi_1(a + g_2) = \Pi_1(a) + \Pi_1(g_2) = \Pi_1(a)$. Therefore, $g_1 = \Pi_1(a)$.

Similarly, we can show that for any $g_2 \varepsilon G_2$, there exists an element $a \varepsilon A$ such that $\Pi_2(a) = g_2$. This completes the proof that A is a subdirect sum of G_1 and G_2.

Conversely, suppose that A is a subdirect sum of G_1 and G_2. We then show that $\{G_1,A\} = G = \{A,G_2\}$. It is obvious that $\{A,G_2\} \subseteq G$ and $\{G_1,A\} \subseteq G$. Therefore, it suffices to show that $G \subseteq \{G_1,A\}$ and

$G \subseteq \{A, G_2\}$.

Let g be an element of G. Then $g = g_1 + g_2$, since $G = G_1 + G_2$. Since A is a subdirect sum of G_1 and G_2, $g_2 = \Pi_2(a)$ for some $a \in A$. However, since $A \subseteq G$, $a = g_1' + g_2'$ for some $g_1' \in G_1$ and $g_2' \in G_2$. Then, $\Pi_2(a) = \Pi_2(g_1' + g_2') = g_2'$. However, since $\Pi_2(a) = g_2$, we have $g_2 = g_2'$ and we can write $a = g_1' + g_2$, which gives us $g_2 = a - g_1'$. Substituting $a - g_1'$ in $g = g_1 + g_2$, we obtain $g = g_1 + (a - g_1') = (g_1 - g_1') + a$. Thus, g is an element of $\{G_1, A\}$, and we conclude that $G \subseteq \{G_1, A\}$.

A similar argument yields $G \subseteq \{A, G_2\}$.

We next establish certain useful isomorphisms for subdirect sums.

THEOREM 2. If A is a subdirect sum of G_1 and G_2, then $\frac{G_1}{G_1 \cap A} \simeq \frac{G_2}{G_2 \cap A}$, $\frac{A}{G_1 \cap A} \simeq G_2$, and $\frac{A}{G_2 \cap A} \simeq G_1$.

PROOF. Since $G = \{G_1, A\}$, we have $\frac{G}{A} = \frac{\{G_1, A\}}{A}$. However, $\frac{\{G_1, A\}}{A} \simeq \frac{G_1}{G_1 \cap A}$, and therefore $\frac{G}{A} \simeq \frac{G_1}{G_1 \cap A}$. Similarly, since $G = \{A, G_2\}$, $\frac{G}{A} \simeq \frac{G_2}{G_2 \cap A}$. Since $\frac{G}{A}$ is isomorphic to both $\frac{G_1}{G_1 \cap A}$ and $\frac{G_2}{G_2 \cap A}$, we can conclude that $\frac{G_2}{G_2 \cap A} \simeq \frac{G_1}{G_1 \cap A}$.

Since $G = G_1 + G_2$, $\frac{G}{G_1} \simeq G_2$. But $\frac{G}{G_1} = \frac{\{G_1, A\}}{G_1} \simeq \frac{A}{G_1 \cap A}$ and we conclude that $\frac{A}{G_1 \cap A} \simeq G_2$. Similarly, we can show that $\frac{A}{G_2 \cap A} \simeq G_1$.

Since A being a subdirect sum of G_1 and G_2 implies that $\frac{G_1}{G_1 \cap A} \simeq \frac{G}{A}$

and $\frac{G}{A} \approx \frac{G_2}{G_2 \cap A}$, that is, implies the existence of homomorphisms of G_1 and G_2 onto $\frac{G}{A}$, it seems reasonable to investigate whether the existence of homomorphisms Φ_1 and Φ_2 of G_1 and G_2 onto some group H determines in a natural fashion a subdirect sum of G_1 and G_2.

THEOREM 3. Let Φ_1 and Φ_2 be homomorphisms of G_1 and G_2 respectively onto some group H. If A is the subset of $G = G_1 + G_2$ consisting of elements of the form $g_1 + g_2$ where $\Phi_1(g_1) = \Phi_2(g_2)$, then:

(i) A is a subgroup of G.

(ii) A is a subdirect sum of G_1 and G_2.

(iii) $G_1 \cap A = \ker \Phi_1$ and $G_2 \cap A = \ker \Phi_2$. (For this reason, the subgroups $G_1 \cap A$ and $G_2 \cap A$ are often referred to as kernels when A is a subdirect sum of G_1 and G_2.)

PROOF. (i) In order to show that A is a subgroup of G, it suffices to show that $a - b \in A$ whenever a and b are elements of A. Then $a - b = (g_1 - g_1') + (g_2 - g_2')$ where $a = g_1 + g_2$ and $b = g_1' + g_2'$. Since $\Phi_1(g_1 - g_1') = \Phi_1(g_1) - \Phi_1(g_1') = \Phi_2(g_2) - \Phi_2(g_2') = \Phi_2(g_2 - g_2')$, $a - b$ is an element of A.

(ii) To show that A is a subdirect sum of G_1 and G_2, we need only show that $G \subseteq \{G_1, A\}$ and $G \subseteq \{A, G_2\}$. Let $g \in G$ and write $g = g_1 + g_2$ with $g_1 \in G_1$ and $g_2 \in G_2$. Since Φ_1 is onto, there is a $g_1' \in G_1$ such that $\Phi_1(g_1') = \Phi_2(g_2)$. Then $g_1' + g_2 \in A$ and $g = (g_1 - g_1') + (g_1' + g_2) \in \{G_1, A\}$. We conclude that $G \subseteq \{G_1, A\}$. A similar argument establishes $G \subseteq \{A, G_2\}$.

(iii) Let $x \in G_1 \cap A$. We can write $x = x + 0$ and since $x \in A$, $\Phi_1(x) = \Phi_2(0)$. However, $\Phi_2(0) = 0$ and we conclude that $x \in \ker \Phi_1$.

Hence, $G_1 \cap A \subseteq \ker \phi_1$.

Now let $x \in \ker \phi_1$. Then $x \in G_1$, since $\ker \phi_1 \subseteq G_1$. Since $x = x + 0$ and $\phi_1(x) = 0 = \phi_2(0)$ we conclude that $x \in A$. Therefore, $x \in G_1 \cap A$ and we conclude that $\ker \phi_1 \subseteq G_1 \cap A$. Hence, $\ker \phi_1 = G_1 \cap A$.

Similarly, it can be shown that $G_2 \cap A = \ker \phi_2$.

We conclude this chapter with two lemmas giving conditions which insure the purity of a subdirect sum.

LEMMA 1. If A is a subdirect sum of G_1 and G_2 such that either $G_1 \cap A$ is pure in G_1 or $G_2 \cap A$ is pure in G_2, then A itself is pure in $G = G_1 + G_2$.

PROOF. Let A be a subdirect sum of G_1 and G_2 and suppose that $G_1 \cap A$ is a pure subgroup of G_1. We wish to show $A \cap nG \subseteq nA$ for all positive integers n.

Let $x \in A \cap nG$. Then $x = ng$ for some g in G. Since A is a subdirect sum of G_1 and G_2, $G = \{G_1, A\}$ and we can write $g = g_1' + a$ for some $g_1' \in G_1$ and $a \in A$. Then $ng = n(g_1' + a) = ng_1' + na$, and since $x = ng$, we have $x = ng_1' + na$. Then $x - na = ng_1'$ and therefore $x - na \in (G_1 \cap A) \cap nG_1$. Since $G_1 \cap A$ is pure in G_1, there is a $y \in G_1 \cap A$ such that $x - na = ny$. Thus, $x = na + ny = n(a + y)$ and since a and y are elements of A, $x \in nA$. Therefore, $A \cap nG \subseteq nA$ and we conclude that A is a pure subgroup of G.

A similar argument shows that A is pure in G when $G_2 \cap A$ is pure in G_2.

LEMMA 2. If A and G_2 are subgroups of a group G such that $G = \{A, G_2\}$ and $G_2 \cap A \subseteq nA$ for each positive integer n, then A is pure in G.

PROOF. It suffices to show that $A \cap nG \subseteq nA$. Let $x \in A \cap nG$. Then $x = ng$ for some $g \in G$. Since $G = \{A,G_2\}$, $g = a + g_2$ where $a \in A$ and $g_2 \in G_2$. Therefore, $x = na + ng_2$ and we have $x - na = ng_2 \in A \cap G_2$. Since $A \cap G_2 \subseteq nA$, we can write $x - na = na'$ for some $a' \in A$. Thus, $x = na + na' = n(a + a') \in nA$. We conclude that $A \cap nG \subseteq nA$ and hence that A is pure in G.

CHAPTER III

P-GROUPS

We now restrict our attention to p-groups; in particular, we shall be concerned with p-groups G for which $G^1 \neq 0$. We shall first establish an imbedding theorem for such groups using the construction discussed in the Introduction.

THEOREM 4. Let G be a p-group and let D be minimal divisible containing G^1. If K is a group containing G and D as subgroups such that $K = \{G,D\}$ and $G \cap D = G^1$, then

(i) G is a pure subgroup of K;

(ii) $K = H + D$ with $H \approx \frac{G}{G^1}$;

(iii) $H \cap G$ is a high subgroup of G;

(iv) $K[p^n] = \{H[p^n], G[p^n]\}$ for all positive integers n;

(v) $H \cap G$ is pure in G;

(vi) $K = \{H,G\}$; and

(vii) K is a subdirect sum of H and D.

PROOF. (i) Since G and D are subgroups of K with $K = \{G,D\}$, by Lemma 2 it suffices to observe that $D \cap G = G^1 \subseteq nG$ for all positive integers n in order to show that G is pure in K.

(ii) Since D is divisible, D is a direct summand of K and we can write $K = H + D$. Then $H \approx \frac{K}{D} = \frac{\{G,D\}}{D} \approx \frac{G}{G \cap D} = \frac{G}{G^1}$.

(iii) To show that $H \cap G = M$ is a high subgroup of G, we must first show that $(H \cap G) \cap G^1 = 0$. Let $x \in (H \cap G) \cap G^1$, then $x \in H$ and

$x \in G^1 = G \cap D \subseteq D$. Hence, $x \in H \cap D = 0$ and therefore we conclude that $(H \cap G) \cap G^1 = 0$.

Now we must show that $\{M,x\} \cap G^1 \neq 0$ when $x \notin M$. Since $x \notin M$ and $x \in G$, $x \notin H$. We can write $x = h + d$ since $K = H + D$. It must be true that $d \neq 0$, for if this were not true, then $x \in G \cap H = M$. Since $d \neq 0$, there exists a positive integer n such that $nd \in G^1$. Then $nx = n(h + d) = nh + nd$. Certainly $nx \neq 0$, for if this were not the case, then $nh = - nd$, but these elements would be contained in $D \cap H$ and would imply that $D \cap H \neq 0$ since $nd \neq 0$. Since nx and nd are in G and nh is in H, it follows that $nh \in H \cap G = M$. Hence $nd = nx - nh$ and we conclude that nd is a non-zero element in $\{M,x\} \cap G^1$.

(iv) Since $H[p^n] \subseteq K[p^n]$ and $G[p^n] \subseteq K[p^n]$, $\{H[p^n], G[p^n]\} \subseteq K[p^n]$. Furthermore, since $K = H + D$ the proof that $K[p^n] \subseteq \{H[p^n], G[p^n]\}$ for a given n reduces to showing that $D[p^n] \subseteq \{H[p^n], G[p^n]\}$.

Before starting the proof by induction on n, we need to recall that $D[p] = G^1[p] \subseteq G[p]$.

Let $d \in D[p]$, and by the above observation we have $d \in G[p]$. Therefore, $D[p] \subseteq \{H[p], G[p]\}$ and completes the proof for $n = 1$.

Now assume that $K[p^i] \subseteq \{H[p^i], G[p^i]\}$. Again we need only show that $D[p^{i+1}] \subseteq \{H[p^{i+1}], G[p^{i+1}]\}$. Let $d \in D[p^{i+1}]$, then $p^{i+1}d = p(p^i d) = 0$. Since $p(p^i d) = 0$, $p^i d \in D[p] = G^1[p] \subseteq G[p]$. Then $p^i d = g$ for some $g \in G^1[p]$ or $p^i d = p^i g'$ for some $g' \in G^1$. Since $p^i d = p^i g'$, we have $p^i(d - g') = 0$. However, by our inductive assumption that $K[p^i] \subseteq \{G[p^i], H[p^i]\}$ we can write $d - g' = g'' + h$ for some $g'' \in G[p^i]$ and $h \in H[p^i]$. Since $d - g' = g'' + h$, we have $d = g' + g'' + h$. It remains only to observe that $p^{i+1}g' = 0$ in order to conclude that $d \in \{G[p^{i+1}], H[p^{i+1}]\}$. However, $p^{i+1}g' = p^{i+1}d = 0$ since $d \in D[p^{i+1}]$. We conclude that $D[p^{i+1}] \subseteq \{H[p^{i+1}], G[p^{i+1}]\}$.

By mathematical induction, we have $K[p^n] = \{H[p^n], G[p^n]\}$ for all positive integers n.

(v) Since G is a p-group, it suffices to show that $(H \cap G) \cap p^nG \subseteq p^n(H \cap G)$ in order to establish the purity of $H \cap G$. Since $(H \cap G) \cap p^nG = H \cap p^nG$, the proof reduces further to showing that $H \cap p^nG \subseteq p^n(H \cap G)$.

Let $x \in H \cap p^nG$, then $x = p^ng$ for some $g \in G$. Moreover, by (ii) we see that H is pure in K since H is a direct summand. This gives us $x \in H \cap p^nK = p^nH$, and we can write $x = p^nh'$ for some $h' \in H$. Thus, $p^nh' = p^ng$ and we have $p^n(h' - g) = 0$. Therefore, $(h' - g) \in K[p^n] = \{H[p^n], G[p^n]\}$. Then $h' - g = h_1 + g_1$ for some $h_1 \in H[p^n]$ and $g_1 \in G[p^n]$. Then $g + g_1 = h_1 - h' \in G \cap H$ and since $x = p^ng = p^n(g + g_1)$, we have $x \in p^n(G \cap H)$.

(vi) To show that $K = \{H,G\}$, it suffices to show that $K \subseteq \{H,G\}$.

In showing that $K \subseteq \{H,G\}$, we need to observe that K is a p-group. This follows immediately since $K = H + D$ where D and H are p-groups (H is a p-group since it is the homomorphic image of a p-group).

We have from (iv) that $K[p^n] = \{H[p^n], G[p^n]\}$ for all positive integers n. Since $\{H[p^n], G[p^n]\} \subseteq \{H,G\}$, $K = \bigcup_{n=1}^{\infty} K[p^n] = \bigcup_{n=1}^{\infty} \{H[p^n], G[p^n]\} \subseteq \{H,G\}$,

(vii) We now have both $K = \{G,D\}$ and $K = \{G,H\}$. Applying Theorem 1 we have immediately that G is a subdirect sum of H and D.

Using Theorem 4, we can now give a rather simple proof of a theorem of Irwin and Walker [2].

THEOREM 5. If M is a high subgroup of the p-group G, then M is pure in G and $\frac{G}{M}$ is isomorphic to a minimal divisible group containing G^1.

PROOF. Form the group K as in Theorem 4. Then $G^1 = G \cap D$ and if M is a high subgroup of G, $M \cap D = (M \cap G) \cap D = M \cap (G \cap D) = M \cap G^1 = 0$. Since $M \cap D = 0$, we can find a subgroup H of G that contains M and is maximal in G with respect to $H \cap D = 0$. Then $K = H + D$ (see [1], p. 63) and clearly $M \subseteq H \cap G$. Since $H \cap D = 0$, $(H \cap G) \cap G^1 = (H \cap G) \cap (G \cap D)$ and $(H \cap G) \cap (G \cap D) = (H \cap G) \cap D = 0$. However, since M is maximal with respect to $M \cap G^1 = 0$ and since $M \subseteq H \cap G$, we must have $M = H \cap G$. It follows from Theorem 4 that M is pure in G and that $\frac{G}{M} = \frac{G}{H \cap G}$ which is isomorphic to $\frac{\{H,G\}}{H} = \frac{K}{H}$ and since $\frac{K}{H} \approx D$, then $\frac{G}{M} \approx D$.

The properties of the imbedding described in Theorem 4 suggest the possibility of constructing as a subdirect sum a p-group G with $\frac{G}{G^1}$ and G^1 prescribed up to reasonable limitations; that is, if H is a p-group without elements of infinite height and if A is a p-group such that H and A are related by certain structural and cardinality conditions, then we should hope to construct a p-group G as a subdirect sum of H and D, where D is minimal divisible containing A, such that $\frac{G}{G^1} \approx H$ and $G^1 = A$. In particular, the conditions between H and A described in the following theorem are satisfied if both H and A are countably infinite, or if H is countably infinite and A has a countably infinite basic subgroup (see [1], p. 98) or if H has the cardinality of the continuum and the cardinality of A does not exceed that of the continuum.

THEOREM 6. Let D be a minimal divisible group containing the p-group A and let H be a p-group without elements of infinite height. If Φ is a homomorphism of H onto $\frac{D}{A}$ with ker Φ pure in H, then any subdirect sum G of H and D with kernels M = ker Φ and A has the following properties:

(i) G is pure in H + D.

(ii) $G^1 = A$.

(iii) $\frac{G}{G^1} \approx H$.

(iv) M is a high subgroup of G.

PROOF. (i) Since ker Φ = M is pure in G, by Lemma 1 we conclude that G is pure in H + D.

(ii) Since A is a kernel of the subdirect sum, $A = G \cap D$. But, since G is pure in K, $G^1 = \bigcap_{n=1}^{\infty} p^n G = \bigcap_{n=1}^{\infty} (p^n K \cap G) = \bigcap_{n=1}^{\infty} (p^n K) \cap G$ $= K^1 \cap G = D \cap G = A$.

(iii) Since $G^1 = A$, $\frac{G}{G^1} = \frac{G}{A} = \frac{G}{G \cap D}$. However, by Theorem 2 we have $\frac{G}{G \cap D} \approx H$. Therefore, $\frac{G}{G^1} \approx H$.

(iv) Since M is a kernel of the subdirect sum G, $M = G \cap H$. Then certainly $M \cap G^1 = (G \cap H) \cap G^1 = 0$, since $G^1 \subseteq D$ and $D \cap H = 0$.

Now assume that $x \in G$ and $x \notin M$. Applying the same argument as the one in the proof of part (iii) of Theorem 4, we can show that $\{M,x\} \cap G^1 \neq 0$. This completes the proof that M is a high subgroup of G.

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