THE FIRST FUNDAMENTAL THEOREM OF INVARIANT THEORY FOR THE UNIMODULAR AND ORTHOGONAL GROUPS

by

CHRISTOPHER G. MOURON, B.S.

A THESIS

IN

MATHEMATICS

Submitted to the Graduate Faculty
of Texas Tech University in
Partial Fulfillment of
the Requirements for
the Degree of

MASTER OF SCIENCE

Approved

December, 1996

ACKNOWLEDGEMENTS

I would first like to thank my co-advisor Dr. David Weinberg for his help, patience, motivation, and support, which without, this thesis would not be possible. Secondly, I would like to thank my other co-advisor Dr. Jeff Lee for his help in proving the harder theorems. Also, I would like to thank Dr. Hal Bennett for being on my committee and all of my teachers at Texas Tech for their support. Finally, I would like to thank my family for always being supportive in the bad times as well as the good.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

In order for a function to have geometric or physical significance, it must be invariant under some group. For example, the dot product of two vectors in $\Re^n$ and the cross product of two vectors in $\Re^3$ have geometric significance. It will be shown below that these two functions are invariant under the group of rotations. On the other hand, the vector formed by multiplying two vectors componentwise is not invariant under rotations, and thus has no geometric significance. In general, a function $f: X \to \Re$ is considered **invariant** under group *G* provided

$$f(gx) = f(x)$$

for every $x \in X$ and $g \in G$. (If *g* happens to act on the right of *x*, then *f* is invariant provided $f(xg) = f(x)$.)

Definition 1.1. A **rotation A** is a linear transformation $\mathbf{A}: \Re^n \to \Re^n$ such that $\mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{I}$ and det(**A**) = 1. We often call the group of rotations the **orthogonal** group.

Now it is intuitively obvious that the length of a vector in $\Re^n$ remains the same when it is rotated through the origin. Similarly, if 2 or more vectors are considered and the angle between the vectors is measured, we would expect that the angle between the vectors would remain unchanged. Length is simply a function $f: \Re^n \to \Re$, such that

$$f(\mathbf{x}) = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}.$$

Definition 1.2. A function $f:\Re^n \to \Re$ is called **invariant under the group of rotations** provided $f(\mathbf{Ax}) = f(\mathbf{x})$ for every rotation **A**. A function $f:\Re^n \times \Re^n \times ... \times \Re^n \to \Re$ is called **invariant** under the group of rotations provided

$$f(\mathbf{Ax}_1, \mathbf{Ax}_2, ..., \mathbf{Ax}_m) = f(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_m)$$

for every rotation **A**.

Example 1.1. *Invariance of length under rotation.*

Let **x** be any vector in $\Re^2$, $\mathbf{A} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$,and $f(\mathbf{x}) = (x_1^2 + x_2^2)^{1/2}$. Then

$$f(\mathbf{Ax}) = f\left(\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = f\left(\begin{bmatrix} x_1\cos\theta + x_2\sin\theta \\ -x_1\sin\theta + x_2\cos\theta \end{bmatrix}\right) =$$

$$((x_1\cos\theta + x_2\sin\theta)^2 + (-x_1\sin\theta + x_2\cos\theta)^2)^{1/2} =$$

$$(x_1^2\cos^2\theta + x_1x_2\cos\theta\sin\theta + x_2^2\sin^2\theta + x_1^2\sin^2\theta - x_1x_2\sin\theta\cos\theta + x_2^2\cos\theta)^{1/2} =$$

$$(x_1^2(\cos^2\theta + \sin^2\theta) + x_2^2(\cos^2\theta + \sin^2\theta))^{1/2} = (x_1^2 + x_2^2)^{1/2} = f(\mathbf{x}).$$

Also, if ϕ is the angle between 2 vectors **x** and **y** in $\Re^n$, then $\cos\phi = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{|\mathbf{x}||\mathbf{y}|}$ where $\langle \mathbf{x}, \mathbf{y} \rangle$ is the dot product defined by $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n$, and $|\mathbf{x}|$ is the length of vector **x** defined by $|\mathbf{x}| = (x_1^2 + x_2^2 + \cdots + x_2^2)^{1/2}$. Thus, we can define $g:\Re^n \times \Re^n \to [0, \pi)$ to be the function that takes two vectors to the angle between them

$$g(\mathbf{x}, \mathbf{y}) = \cos^{-1}\left(\frac{\langle \mathbf{x}, \mathbf{y} \rangle}{|\mathbf{x}||\mathbf{y}|}\right) = \phi. \qquad (1)$$

Example 1.2. *Invariance of the dot product under rotation in $\Re$.*

Let **x** and **y** be any vectors in $\Re$. and $\mathbf{A} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

Then

$$\langle \mathbf{Ax}, \mathbf{Ay} \rangle = \left\langle \begin{bmatrix} x_1 \cos\theta + x_2 \sin\theta \\ -x_1 \sin\theta + x_2 \cos\theta \end{bmatrix}, \begin{bmatrix} y_1 \cos\theta + y_2 \sin\theta \\ -y_1 \cos\theta + y_2 \sin\theta \end{bmatrix} \right\rangle =$$

$$(x_1 \cos\theta + x_2 \sin\theta)(y_1 \cos\theta + y_2 \sin\theta) + (-x_1 \sin\theta + x_2 \cos\theta)(-y_1 \sin\theta + y_2 \cos\theta) =$$

$$x_1 y_1 \cos^2\theta + x_1 y_2 \cos\theta \sin\theta + x_2 y_1 \cos\theta \sin\theta + x_2 y_2 \sin^2\theta +$$

$$x_1 y_1 \sin^2\theta - x_1 y_2 \cos\theta \sin\theta - x_2 y_1 \cos\theta \sin\theta + x_2 y_2 \cos^2\theta =$$

$$x_1 y_1 (\cos^2\theta + \sin^2\theta) + x_2 y_2 (\cos^2\theta + \sin^2\theta) = x_1 y_1 + x_2 y_2 = \langle \mathbf{x}, \mathbf{y} \rangle.$$

Example 1.3. *Invariance of the angle between vectors under rotation.*

Let **x** and **y** be any vectors in $\Re$ and $\mathbf{A} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

Notice from example 1 that $|\mathbf{x}| = f(\mathbf{x}) = f(\mathbf{Ax}) = |\mathbf{Ax}|$. So

$$g(\mathbf{Ax}, \mathbf{Ay}) = \cos^{-1}\left(\frac{\langle \mathbf{Ax}, \mathbf{Ay} \rangle}{|\mathbf{Ax}||\mathbf{Ay}|} \right) = \cos^{-1}\left(\frac{\langle \mathbf{Ax}, \mathbf{Ay} \rangle}{|\mathbf{x}||\mathbf{y}|} \right) =$$

$$\cos^{-1}\left(\frac{(x_1 \cos\theta + x_2 \sin\theta)(y_1 \cos\theta + y_2 \sin\theta) + (-x_1 \sin\theta + x_2 \cos\theta)(-y_1 \sin\theta + y_2 \cos\theta)}{|\mathbf{x}||\mathbf{y}|} \right) = = = =$$

$$\cos^{-1}\left(\frac{x_1 y_1 (\cos^2\theta + \sin^2\theta) + x_2 y_2 (\cos^2\theta + \sin^2\theta)}{|\mathbf{x}||\mathbf{y}|} \right) = \cos^{-1}\left(\frac{x_1 y_1 + x_2 y_2}{|\mathbf{x}||\mathbf{y}|} \right) = g(\mathbf{x}, \mathbf{y}). \quad (2)$$

Also, one would expect that rotation of a parallelepiped about a vertex preserves volume.

Example 1.4. *Invariance of area under rotation.*

Let **0**, **x**, **y**, and **x** + **y** be the vertices of a parallelogram in $\Re^2$. The area $V(\mathbf{x},\mathbf{y})$ of the parallelogram is thus given by $V(\mathbf{x},\mathbf{y}) = |\mathbf{x}||\mathbf{y}|\sin\phi$, where ϕ is the angle between vectors **x** and **y**. Let **A** be a rotation defined as in example 2. Then from equations (1) and (2) in examples 1 and 2 we have

$$V(\mathbf{Ax},\mathbf{Ay}) = |\mathbf{Ax}||\mathbf{Ay}|\sin\phi = |\mathbf{Ax}||\mathbf{Ay}|\sin(g(\mathbf{Ax},\mathbf{Ay})) =$$

$$|\mathbf{x}||\mathbf{y}|\sin(g(\mathbf{x},\mathbf{y})) = |\mathbf{x}||\mathbf{y}|\sin\phi = V(\mathbf{x},\mathbf{y}).$$

Since the volume of a parallelepiped with a vertex at the origin is the determinant of the matrix consisting of the vectors corresponding to the edges that meet at the origin, we can check the invariance of the determinant.

Example 1.5. *Invariance of the determinant under rotation.*

Let **A** be any rotation in $\Re^n$. Then by definition det(**A**) = 1. Thus, if **B** is any linear transformation from $\Re^n$ to $\Re^n$, then det(**AB**) = det(**A**)det(**B**) = (1)det(**B**) = det(**B**).

Notice that in the first three examples the work in showing the invariances of the functions was in showing the invariance of the polynomials $|\mathbf{x}|^2$ and $\langle \mathbf{x},\mathbf{y}\rangle$. This gives us motivation to formulate our first major question: "*What are all of the polynomials that are invariant under rotation?*"

Example 1.6. I want to describe all of the polynomials $f:\Re^2 \to \Re$ such that

$f(\mathbf{x}) = f(\mathbf{Ax})$, where

$$\mathbf{A} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } \mathbf{Ax} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}.$$

Definition 1.3. The **total degree** of a term of a polynomial to be the sum of the exponents of each variable in that term.

Definition 1.4. A **homogeneous** polynomial to be that which every term has the same total degree. Suppose $g(\mathbf{x})$ is an invariant, then if we break $g(\mathbf{x})$ into its homogeneous pieces $g_1(\mathbf{x}),\ldots,g_k(\mathbf{x})$, each homogeneous piece $g_i(\mathbf{x})$ must be invariant. Thus, it suffices to consider only homogeneous invariant polynomials. So let

$$g(\mathbf{x}) = a_m x^m + a_{m-1}x^{m-1}y + \ldots + a_i x^i y^{m-i} + \ldots + a_1 x^1 y^{m-1} + a_0 y^m.$$

Since $g(\mathbf{x})=g(\mathbf{Ax})$ for all θ, we may choose $\theta = \pi/2$. Then, $\bar{x} = y$ and $\bar{y} = -x$. Thus, $a_{m-i} = (-1)^i a_i$. So for each i odd, $a_i = 0$. Also, this implies that both *m* and *m/2* are even. Now take $\theta = \pi/4$, and let

$$\mathbf{x}_1 = \begin{pmatrix}1\\1\end{pmatrix} \quad \mathbf{x}_2 = \begin{pmatrix}2\\1\end{pmatrix}, \quad \cdots, \quad \mathbf{x}_{m/s-1} = \begin{pmatrix}m/2-1\\1\end{pmatrix}.$$

Then

$$\mathbf{Ax}_1 = \begin{pmatrix}\sqrt{2}\\0\end{pmatrix}, \quad \mathbf{Ax}_2 = \begin{pmatrix}3\sqrt{2}/2\\-\sqrt{2}/2\end{pmatrix}, \quad \cdots, \quad \mathbf{Ax}_{m/2-1} = \begin{pmatrix}m\sqrt{2}/4\\(2-m)\sqrt{2}/2\end{pmatrix}.$$

By plugging into $g(\mathbf{x}) = g(\mathbf{Ax})$, we get a *m* by *m/2-1* system of equations whose solution, in terms of a_m is:

$$a_i = \frac{m!}{(m-i)!\,i!}a_a \qquad \text{where } i = \{m/2,\ m/2+2,\ldots,\ m\}.$$

Therefore, by the binomial theorem, $g(\mathbf{x}) = a_m(x^2+y^2)^{m/2}$. Thus, by example 1, $x^2 + y^2$ is the only "real" invariant. This result is much harder for higher dimensions or multivector polynomials and different techniques must be used.

The next natural questions is "*Are certain polynomials invariant under other groups?*". Here we will find all of the polynomials invariant under the group of **unimodular** (having determinant one) matrices. First we will define two types of vectors: **covariant** or column vectors, and **contravariant** or row vectors. Notice that acts on a covariant vector on the right, and acts on a contravariant vector on the left. Also, if $\mathbf{x}$ is a covariant vector, and $\mathbf{y}$ is a contravariant vector, then for the dot product we have:

$$\mathbf{yx} = \mathbf{yA}^{-1}\mathbf{Ax}.$$

Thus, in order to have the dot product as an invariance, it will be necessary to define the **invariance of polynomials on the unimodular group** as:

$$f(\mathbf{x}_1,\ldots,\mathbf{x}_n;\mathbf{y}_1,\ldots\mathbf{y}_m) = f(\mathbf{Ax}_1,\ldots,\mathbf{Ax}_n;\mathbf{y}_1\mathbf{A}^{-1},\ldots,\mathbf{y}_m\mathbf{A}^{-1}),$$

for every $\mathbf{A} \in SL(n)$, and where $\{\mathbf{x}_1,\ldots,\mathbf{x}_n\}$ are n arbitrary covariant vectors and $\{\mathbf{y}_1,\ldots,\mathbf{y}_m\}$ are m arbitrary contravariant vectors.

Example 1.7. *The invariance of the determinant under the unimodular group.*

Let $f(\mathbf{x}_1,\ldots,\mathbf{x}_n)$ be the polynomial associated with the determinant of $n \times n$ matrix **M**. Each $\mathbf{x}_i$ is a vector in $\Re^n$ that corresponds to column i in matrix **M**. Also notice that if **A** is also an $n \times n$ matrix, then $\mathbf{Ax}_i$ is the *ith* column in matrix **AM**. Thus, if $\mathbf{A} \in Sl(n)$, then

$$f(\mathbf{Ax}_1,\ldots,\mathbf{Ax}_n) = \det(\mathbf{AM}) = \det(\mathbf{A})\det(\mathbf{M}) = (1)\det(\mathbf{M}) = f(\mathbf{x}_1,\ldots,\mathbf{x}_n).$$

Next, it is interesting to investigate the question: "*What are all of the vector operations that have geometric meaning?*".

Example 1.8. *The invariance of cross product in $\Re^3$ under rotation.*

We define cross-product as a mapping $\otimes$: $\Re^3 \times \Re^3 \rightarrow \Re^3$ such that if $\mathbf{x}, \mathbf{y} \in \Re^3$,

then

$$\mathbf{x} \otimes \mathbf{y} \;=\; \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \otimes \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \;=\; \begin{bmatrix} x_2 y_3 - y_2 x_3 \\ x_3 y_1 - y_3 x_1 \\ x_1 y_2 - y_1 x_2 \end{bmatrix}.$$

If $\mathbf{A}$ is a rotation in $\Re^3$, then we want to show $\mathbf{A}(\mathbf{x} \otimes \mathbf{y}) \;=\; \mathbf{Ax} \otimes \mathbf{Ay}$. Let

$$\mathbf{A} = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}.$$

Now define

$$\mathbf{i}' = \mathbf{Ai} = l_1\mathbf{i} + m_1\mathbf{j} + n_1\mathbf{k},$$

$$\mathbf{j}' = \mathbf{Aj} = l_2\mathbf{i} + m_2\mathbf{j} + n_2\mathbf{k},$$

$$\mathbf{k}' = \mathbf{Ak} = l_3\mathbf{i} + m_3\mathbf{j} + n_3\mathbf{k}.$$

Now since $\mathbf{A}$ is orthogonal,

$$\mathbf{AA}^{-1} = \mathbf{AA}^T = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n \\ l_3 & m_3 & n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5}$$

$$= \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} = \mathbf{A}^T\mathbf{A} = \mathbf{A}^{-1}\mathbf{A}.$$

Therefore, we may write: $\quad \mathbf{i} = \mathbf{A}^{-1}\mathbf{i}' = l_1\mathbf{i}' + l_2\mathbf{j}' + l_3\mathbf{k}'$,

$$\mathbf{j} = \mathbf{A}^{-1}\mathbf{j}' = m_1\mathbf{i}' + m_2\mathbf{j}' + m_3\mathbf{k}',$$

$$\mathbf{k} = \mathbf{A}^{-1}\mathbf{k}' = n_1\mathbf{i}' + n_2\mathbf{j}' + n_3\mathbf{k}'.$$

Since $\mathbf{j} \otimes \mathbf{k} = \mathbf{i}$,

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \otimes \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} m_2 n_3 - m_3 n_2 \\ m_3 n_1 - m_1 n_3 \\ m_1 n_2 - m_2 n_1 \end{pmatrix} = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}.$$

Thus, $l_1 = m_2 n_3 - m_3 n_2$, $l_2 = m_3 n_1 - m_1 n_3$, and , $l_3 = m_1 n_2 - m_2 n_1$. By the fact that $\mathbf{i} \otimes \mathbf{k} = -\mathbf{j}$, and $\mathbf{i} \otimes \mathbf{j} = \mathbf{k}$ and similar arguments as before, we get $m_1 = l_3 n_2 - l_2 n_3$ and $n_1 = l_2 m_3 - l_3 m_2$.Now, let

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \text{ and } \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

Let's compute the first component of $\mathbf{Ax} \otimes \mathbf{Ay}$. Since

$$\mathbf{Ax} = \begin{pmatrix} x_1 l_1 + x_2 m_1 + x_3 n_1 \\ x_1 l_2 + x_2 m_2 + x_3 n_2 \\ x_1 l_3 + x_2 m_3 + x_3 n_3 \end{pmatrix}, \text{ and } \mathbf{Ay} = \begin{pmatrix} y_1 l_1 + y_2 m_1 + y_3 n_1 \\ y_1 l_2 + y_2 m_2 + y_3 n_2 \\ y_1 l_3 + y_2 m_3 + y_3 n_3 \end{pmatrix},$$

the first component of $\mathbf{Ax} \otimes \mathbf{Ay}$ =

$$(x_1 l_2 + x_2 m_2 + x_3 n_2)(y_1 l_3 + y_2 m_3 + y_3 n_3) - (x_1 l_3 + x_2 m_3 + x_3 n_3)(y_1 l_2 + y_2 m_2 + y_3 n_2)$$

$$= (x_1 y_2 - x_2 y_1)(l_2 m_3 - l_3 m_2) + (x_3 y_1 - x_1 y_3)(l_3 n_2 - l_2 n_3) + (x_2 y_3 - x_3 y_2)(m_2 n_3 - m_3 n_2)$$

$$= (x_1 y_2 - x_2 y_1) n_1 + (x_3 y_1 - x_1 y_3) m_1 + (x_2 y_3 - x_3 y_2) l_1$$

$$= \text{ the first component of } \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{pmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{pmatrix}$$

$$= \text{ the first component of } \mathbf{A}(\mathbf{x} \otimes \mathbf{y}).$$

After a similar computation, it is found that the second and third components of $\mathbf{Ax} \otimes \mathbf{Ay}$ are equal to the respective second and third components of $\mathbf{A}(\mathbf{x} \otimes \mathbf{y})$. Thus,

$$\mathbf{Ax} \otimes \mathbf{Ay} = \mathbf{A}(\mathbf{x} \otimes \mathbf{y}) .$$

Example 1.9. *The invariance of vector addition.*

Here we want to show that vector addition makes sense in $\Re^3$ as an operation since it is invariant under rotation. We define vector addition in the usual way as a mapping $+ : \Re^3 \times \Re^3 \to \Re^3$ performed by adding the components. The invariance of vector addition is simply true by the distributive law of matrix multiplication:

$$\mathbf{A}(\mathbf{x}+\mathbf{y}) = (\mathbf{Ax})+(\mathbf{Ay}) .$$

Example 1.10. *Non-invariance of vector multiplication.*

Now we will examine why component-wise multiplication of a vectors does not make geometric sense. Let $* : \Re^2 \times \Re^2 \to \Re^2$ be the operation that multiplies the components of one vector to another. If **A** is a rotation, we would like

$$(\mathbf{Ax}) * (\mathbf{Ay}) \;=\; \mathbf{A}(\mathbf{x} * \mathbf{y}) .$$

However, if we let

$$\mathbf{A} \;=\; \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}, \quad \mathbf{x} \;=\; \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \text{ and } \mathbf{y} \;=\; \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

then

$$(\mathbf{Ax}) * (\mathbf{Ay}) \;=\; \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} * \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \;=\; \begin{bmatrix} \sqrt{3}/2 - 1 \\ 1/2 + \sqrt{3} \end{bmatrix} * \begin{bmatrix} \sqrt{3} - 1/2 \\ 1 + \sqrt{3}/2 \end{bmatrix} \;=\; \begin{bmatrix} 2 - 5\sqrt{3}/4 \\ 2 + 5\sqrt{3}/4 \end{bmatrix}$$

which is not equal to

$$\begin{bmatrix} \sqrt{3}-1 \\ 1+\sqrt{3} \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} * \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \mathbf{A}(\mathbf{x} * \mathbf{y}).$$

Since in each category, the number of invariant polynomials is infinite, it would impossible to write them all down. Therefore, the goal is to find a way to describe different types of polynomials that are invariant. This will be done by showing that every polynomial can be written as the sum, product, and scalar multiple of a finite list of certain polynomials. This list is called an **integrity basis** for the invariant polynomials. The existence of this integrity basis is called the **First Fundamental Theorem of Invariant Theory** . Our goal, therefore, is to show this theorem is true for the orthogonal and the unimodular groups. To do this, we must first show that any invariant polynomial of degree n with m vectors can be expressed in terms of our integrity basis if any invariant polynomial of degree n and n vectors can be expressed in terms of our integrity basis. Furthermore, if the determinant is in our basis or can be expressed by our basis, then only the invariant polynomials of n-1 vectors need to shown to be expressible by our integrity basis. Proof of this result relies on the very powerful Capelli identity, which will be shown in Chapter II. The Capelli identity preserves invariance, and enables us to inductively cut down the number of needed vectors from m to n-1.

The next step in the proof of the First Fundamental Theorem of Invariant Theory is to show inductively for each group that the only necessary invariants in the integrity basis for invariant polynomials in n-1 vectors of degree n are the dot product, and the determinant. This again will be done by using Capelli's identity to break down the degree of the polynomial. The method that will be presented is based on the work done in

Herman Weyl's text, "The Classical Groups." However, in this thesis, all of the missing details are filled in and many useful examples are presented.

Before moving on, a theorem on algebraic inequalities is needed.

Theorem 1.1. *Principle of the Irrelevance of Algebraic Inequalities*: A polynomial $F(x_1, x_2, \ldots, x_n)$ vanishes identically if it vanishes numerically for all points $(x_1, \ldots, x_n) = (\alpha_1, \ldots, \alpha_n)$ such that $R_1(\alpha_1, \ldots, \alpha_n) \neq 0, \ldots, R_j(\alpha_1, \ldots, \alpha_n) \neq 0$, where $R_1, \ldots, R_j$ are polynomials.

Proof. The proof is by induction on the number of variables *n*.

Case $n = 1$: Notice that the set $\{\alpha :: R_1(\alpha) \neq 0, \ldots, R_j(\alpha) \neq 0\}$ is an open subset of $\Re$. Therefore, this subset contains a nonempty open interval. Since $F(x_1)$ vanishes on this infinite set, it follows by the Fundamental Theorem of Algebra that $F(x_1) \equiv 0$.

Induction step: Assume that the statement is true for *n* variables. Let

$$F(x_1, \ldots, x_{n+1}) = G_k(x_1, \ldots, x_n)x_{n+1}^k + G_{k-1}(x_1, \ldots, x_n)x_{n+1}^{k-1} + \ldots + G_0(x_1, \ldots, x_n)$$

be a polynomial in $n+1$ variables that vanishes numerically for all points $(\alpha_1, \ldots, \alpha_{n+1})$ such that $R_1(\alpha_1, \ldots, \alpha_{n+1}) \neq 0, \ldots, R_j(\alpha_1, \ldots, \alpha_{n+1}) \neq 0$. Now observe that

$\{(\alpha_1, \ldots, \alpha_{n+1}) : R_1(\alpha_1, \ldots, \alpha_{n+1}) \neq 0, \ldots, R_j(\alpha_1, \ldots, \alpha_{n+1}) \neq 0\}$ is an open subset of $\Re^{n+1}$.

Fix a point $(\beta_1, \ldots, \beta_{n+1})$ in this subset. It follows from the Fundamental Theorem of Algebra that $G_k, G_{k-1}, \ldots, G_0$ vanish numerically for all points $(\alpha_1, \ldots, \alpha_n)$ such that $R_1(\alpha_1, \ldots, \alpha_n, \beta_{n+1}) \neq 0, \ldots, R_j(\alpha_1, \ldots, \alpha_n, \beta_{n+1}) \neq 0$. Therefore, it follows from the induction hypothesis that $G_k, G_{k-1}, \ldots, G_0$ vanish identically. Therefore, $F(x_1, \ldots, x_{n+1}) \equiv 0$. **Q.E.D.**

CHAPTER II

THE CAPELLI IDENTITY

Let **x, y** and **z** be vectors in $\Re^n$. Let f be a polynomial in m vectors. Define the polarization $D_{\mathbf{yx}}$ of f as

$$D_{\mathbf{yx}} f = \sum_{i=1}^{n} y_i \frac{\partial f}{\partial x_i}.$$

Let $\Delta_{\mathbf{yx}} f$ be defined in a similar fashion with one exception on composite polarizations. Notice that by using the chain rule

$$D_{\mathbf{yx}} D_{\mathbf{xz}} f = D_{\mathbf{yx}} \sum_{i=1}^{n} x_i \frac{\partial f}{\partial z_i} = \sum_{j,i=1}^{n} y_j x_i \frac{\partial^2 f}{\partial x_j \partial z_i} + \sum_{i=1}^{n} y_i \frac{\partial f}{\partial x_i}. \qquad (3)$$

However, we will define $\Delta_{\mathbf{yx}} \Delta_{\mathbf{xz}}$ by

$$\Delta_{\mathbf{yx}} \Delta_{\mathbf{xz}} f = \sum_{j,i=1}^{n} y_j x_i \frac{\partial^2 f}{\partial x_j \partial z_i}.$$

We see that, in general, $D_{\mathbf{yx}} D_{\mathbf{xz}}$ is not commutative since

$$D_{\mathbf{xz}} D_{\mathbf{yx}} f = D_{\mathbf{xz}} \sum_{i=1}^{n} y_i \frac{\partial f}{\partial x_i} = \sum_{j,i=1}^{n} x_j y_i \frac{\partial^2 f}{\partial z_j \partial x_i} \neq D_{\mathbf{yx}} D_{\mathbf{xz}} f.$$

However, $\Delta_{\mathbf{yx}} \Delta_{\mathbf{xz}}$ is commutative. Therefore, it is easy to write composite polarization operations in terms of composite Δ operations. For example, equation (3) gives us

$$D_{\mathbf{yx}} D_{\mathbf{xz}} f = \Delta_{\mathbf{yx}} \Delta_{\mathbf{xz}} f + \Delta_{\mathbf{yz}} f.$$

Notice that we copy the subscripts of the polarization operators and if the second subscript of an operator matches the first subscript of an operator to the right (in this case, **x**), then we add a Δ with the subscripts coming from the outside subscripts of the polar operators. A more complicated example:

$$D_{yx}D_{xz}D_{zy}f = D_{yx}(\Delta_{xz}\Delta_{zy} + \Delta_{xy})f = D_{yx}\Delta_{xz}\Delta_{zy}f + D_{yx}\Delta_{xy}f =$$

$$\Delta_{yx}\Delta_{xz}\Delta_{zy}f + \Delta_{yz}\Delta_{zy}f + \Delta_{yx}\Delta_{xy}f + \Delta_{yy}f\ .$$

Now let f be a function of m vectors $\{\mathbf{x}^1, \mathbf{x}^2, \ldots\ldots, \mathbf{x}^m\}$. We want to show:

Theorem 2.1.

$$\begin{vmatrix} D_{x^m x^m} + (m-1) & \cdots & D_{x^m x^2} & D_{x^m x^1} \\ \cdots & \cdots & \cdots & \cdots \\ D_{x^2 x^m} & \cdots & D_{x^2 x^2} + 1 & D_{x^2 x^1} \\ D_{x^1 x^m} & \cdots & D_{x^1 x^2} & D_{x^1 x^1} \end{vmatrix} = \begin{vmatrix} \Delta_{x^m x^m} & \cdots & \Delta_{x^m x^2} & \Delta_{x^m x^1} \\ \cdots & \cdots & \cdots & \cdots \\ \Delta_{x^2 x^m} & \cdots & \Delta_{x^2 x^2} & \Delta_{x^2 x^1} \\ \Delta_{x^1 x^m} & \cdots & \Delta_{x^1 x^2} & \Delta_{x^1 x^1} \end{vmatrix}. \quad (3)$$

For example, if $m = 2$:

$$\begin{vmatrix} D_{x^2x^2} + 1 & D_{x^2x^1} \\ D_{x^1x^2} & D_{x^1x^1} \end{vmatrix} = (D_{x^2x^2} + 1)D_{x^1x^1} - D_{x^1x^2}D_{x^2x^1} = D_{x^2x^2}D_{x^1x^1} + D_{x^1x^1} - D_{x^1x^2}D_{x^2x^1} =$$

$$\Delta_{x^2x^2}\Delta_{x^1x^1} + \Delta_{x^1x^1} - (\Delta_{x^1x^2}\Delta_{x^2x^1} + \Delta_{x^1x^1}) = \Delta_{x^2x^2}\Delta_{x^1x^1} - \Delta_{x^1x^2}\Delta_{x^2x^1} = \begin{vmatrix} \Delta_{x^2x^2} & \Delta_{x^2x^1} \\ \Delta_{x^1x^2} & \Delta_{x^1x^1} \end{vmatrix}.$$

If $m = 3$:

$$\begin{vmatrix} D_{x^3x^3} + 2 & D_{x^3x^2} & D_{x^3x^1} \\ D_{x^2x^3} & D_{x^2x^2} + 1 & D_{x^2x^1} \\ D_{x^1x^3} & D_{x^1x^2} & D_{x^1x^1} \end{vmatrix}$$

$$= (D_{x^3x^3} + 2)\begin{vmatrix} D_{x^2x^2} + 1 & D_{x^2x^1} \\ D_{x^1x^2} & D_{x^1x^1} \end{vmatrix} - D_{x^2x^3}\begin{vmatrix} D_{x^3x^2} & D_{x^3x^1} \\ D_{x^1x^2} & D_{x^1x^1} \end{vmatrix} + D_{x^1x^3}\begin{vmatrix} D_{x^3x^2} & D_{x^3x^1} \\ D_{x^2x^2} & D_{x^2x^1} \end{vmatrix} =$$

$$(D_{x^3x^3} + 2)\begin{vmatrix} \Delta_{x^2x^2} & \Delta_{x^2x^1} \\ \Delta_{x^1x^2} & \Delta_{x^1x^1} \end{vmatrix} - D_{x^2x^3}(D_{x^3x^2}D_{x^1x^1} - D_{x^1x^2}D_{x^3x^1})$$

$$+D_{x^1x^3}(D_{x^3x^2}D_{x^2x^1} - (D_{x^2x^2} + 1)D_{x^3x^1}) =$$

$$D_{x^3x^3}(\Delta_{x^2x^2}\Delta_{x^1x^1}-\Delta_{x^1x^2}\Delta_{x^2x^1})+2\begin{vmatrix}\Delta_{x^2x^2} & \Delta_{x^2x^1}\\ \Delta_{x^1x^2} & \Delta_{x^1x^1}\end{vmatrix}-D_{x^2x^3}(\Delta_{x^3x^2}\Delta_{x^1x^1}-\Delta_{x^1x^2}\Delta_{x^3x^1})+$$

$$D_{x^1x^3}(\Delta_{x^3x^2}\Delta_{x^2x^1}+\Delta_{x^3x^1}-(\Delta_{x^2x^2}\Delta_{x^3x^1}+\Delta_{x^3x^1}))$$

$$=D_{x^3x^3}\Delta_{x^2x^2}\Delta_{x^1x^1}-D_{x^3x^3}\Delta_{x^1x^2}\Delta_{x^2x^1})+2\begin{vmatrix}\Delta_{x^2x^2} & \Delta_{x^2x^1}\\ \Delta_{x^1x^2} & \Delta_{x^1x^1}\end{vmatrix}-$$

$$D_{x^2x^3}\Delta_{x^3x^2}\Delta_{x^1x^1}+D_{x^2x^3}\Delta_{x^3x^1}\Delta_{x^1x^2}+D_{x^1x^3}\Delta_{x^3x^2}\Delta_{x^2x^1}-D_{x^1x^3}\Delta_{x^3x^1}\Delta_{x^2x^2}=$$

$$\Delta_{x^3x^3}\Delta_{x^2x^2}\Delta_{x^1x^1}-\Delta_{x^3x^3}\Delta_{x^1x^2}\Delta_{x^2x^1}+2\begin{vmatrix}\Delta_{x^2x^2} & \Delta_{x^2x^1}\\ \Delta_{x^1x^2} & \Delta_{x^1x^1}\end{vmatrix}-\Delta_{x^2x^3}\Delta_{x^3x^2}\Delta_{x^1x^1}$$

$$-\Delta_{x^2x^2}\Delta_{x^1x^1}+\Delta_{x^2x^3}\Delta_{x^3x^1}\Delta_{x^1x^2}+\Delta_{x^2x^1}\Delta_{x^1x^2}+\Delta_{x^1x^3}\Delta_{x^3x^2}\Delta_{x^2x^1}$$

$$+\Delta_{x^1x^2}\Delta_{x^2x^1}-\Delta_{x^1x^3}\Delta_{x^3x^1}\Delta_{x^2x^2}-\Delta_{x^1x^1}\Delta_{x^2x^2}=$$

$$\Delta_{x^3x^3}\Delta_{x^2x^2}\Delta_{x^1x^1}-\Delta_{x^3x^3}\Delta_{x^1x^2}\Delta_{x^2x^1}+\Delta_{x^2x^3}\Delta_{x^1x^3}\Delta_{x^3x^1}-$$

$$\Delta_{x^2x^3}\Delta_{x^3x^2}\Delta_{x^1x^1}+\Delta_{x^1x^3}\Delta_{x^3x^2}\Delta_{x^2x^1}-\Delta_{x^1x^3}\Delta_{x^2x^2}\Delta_{x^3x^1}+$$

$$2\begin{vmatrix}\Delta_{x^2x^2} & \Delta_{x^2x^1}\\ \Delta_{x^1x^2} & \Delta_{x^1x^1}\end{vmatrix}-2\Delta_{x^2x^2}\Delta_{x^1x^1}+2\Delta_{x^1x^2}\Delta_{x^2x^1}=$$

$$\begin{vmatrix}\Delta_{x^3x^3} & \Delta_{x^3x^2} & \Delta_{x^3x^1}\\ \Delta_{x^2x^3} & \Delta_{x^2x^2} & \Delta_{x^2x^1}\\ \Delta_{x^1x^3} & \Delta_{x^1x^2} & \Delta_{x^1x^1}\end{vmatrix}+2\begin{vmatrix}\Delta_{x^2x^2} & \Delta_{x^2x^1}\\ \Delta_{x^1x^2} & \Delta_{x^1x^1}\end{vmatrix}-2\begin{vmatrix}\Delta_{x^2x^2} & \Delta_{x^2x^1}\\ \Delta_{x^1x^2} & \Delta_{x^1x^1}\end{vmatrix}=$$

$$\begin{vmatrix}\Delta_{x^3x^3} & \Delta_{x^3x^2} & \Delta_{x^3x^1}\\ \Delta_{x^2x^3} & \Delta_{x^2x^2} & \Delta_{x^2x^1}\\ \Delta_{x^1x^3} & \Delta_{x^1x^2} & \Delta_{x^1x^1}\end{vmatrix}.$$

To prove theorem 2.1, we first need a lemma.

Lemma 2.2.

$$\begin{vmatrix} D_{x^m x^m} + (m-1) & \Delta_{x^m x^{m-1}} & \cdots & \Delta_{x^m x^2} & \Delta_{x^m x^1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ D_{x^2 x^m} & \Delta_{x^2 x^{m-1}} & \cdots & \Delta_{x^2 x^2} & \Delta_{x^2 x^1} \\ D_{x^1 x^m} & \Delta_{x^1 x^{m-1}} & \cdots & \Delta_{x^1 x^2} & \Delta_{x^1 x^1} \end{vmatrix} =$$

$$\begin{vmatrix} \Delta_{x^m x^m} & \Delta_{x^m x^{m-1}} & \cdots & \Delta_{x^m x^2} & \Delta_{x^m x^1} \\ \cdots & \cdots & \cdots & \cdots & \\ \Delta_{x^2 x^m} & \Delta_{x^2 x^{m-1}} & \cdots & \Delta_{x^2 x^2} & \Delta_{x^2 x^1} \\ \Delta_{x^1 x^m} & \Delta_{x^1 x^{m-1}} & \cdots & \Delta_{x^1 x^2} & \Delta_{x^1 x^1} \end{vmatrix} \tag{3}$$

First, for motivation, the case $m = 4$ is proved.

$$\begin{vmatrix} D_{x^4 x^4} + 3 & \Delta_{x^4 x^3} & \Delta_{x^4 x^2} & \Delta_{x^4 x^1} \\ D_{x^3 x^4} & \Delta_{x^3 x^3} & \Delta_{x^3 x^2} & \Delta_{x^3 x^1} \\ D_{x^2 x^4} & \Delta_{x^2 x^3} & \Delta_{x^2 x^2} & \Delta_{x^2 x^1} \\ D_{x^1 x^4} & \Delta_{x^1 x^3} & \Delta_{x^1 x^2} & \Delta_{x^1 x^1} \end{vmatrix} =$$

$$(D_{x^4 x^4} + 3)\begin{vmatrix} \Delta_{x^3 x^3} & \Delta_{x^3 x^2} & \Delta_{x^3 x^1} \\ \Delta_{x^2 x^3} & \Delta_{x^2 x^2} & \Delta_{x^2 x^1} \\ \Delta_{x^1 x^3} & \Delta_{x^1 x^2} & \Delta_{x^1 x^1} \end{vmatrix} - D_{x^3 x^4}\begin{vmatrix} \Delta_{x^4 x^3} & \Delta_{x^4 x^2} & \Delta_{x^4 x^1} \\ \Delta_{x^2 x^3} & \Delta_{x^2 x^2} & \Delta_{x^2 x^1} \\ \Delta_{x^1 x^3} & \Delta_{x^1 x^2} & \Delta_{x^1 x^1} \end{vmatrix} +$$

$$D_{x^2 x^4}\begin{vmatrix} \Delta_{x^4 x^3} & \Delta_{x^4 x^2} & \Delta_{x^4 x^1} \\ \Delta_{x^3 x^3} & \Delta_{x^3 x^2} & \Delta_{x^3 x^1} \\ \Delta_{x^1 x^3} & \Delta_{x^1 x^2} & \Delta_{x^1 x^1} \end{vmatrix} - D_{x^1 x^4}\begin{vmatrix} \Delta_{x^4 x^3} & \Delta_{x^4 x^2} & \Delta_{x^4 x^1} \\ \Delta_{x^3 x^3} & \Delta_{x^3 x^2} & \Delta_{x^3 x^1} \\ \Delta_{x^2 x^3} & \Delta_{x^2 x^2} & \Delta_{x^2 x^1} \end{vmatrix} =$$

$$D_{x^4 x^4}\begin{vmatrix} \Delta_{x^3 x^3} & \cdots & \Delta_{x^3 x^1} \\ \vdots & \ddots & \vdots \\ \Delta_{x^1 x^3} & \cdots & \Delta_{x^1 x^1} \end{vmatrix} + 3\begin{vmatrix} \Delta_{x^3 x^3} & \cdots & \Delta_{x^3 x^1} \\ \vdots & \ddots & \vdots \\ \Delta_{x^1 x^3} & \cdots & \Delta_{x^1 x^1} \end{vmatrix} -$$

$$D_{x^3 x^4}\Delta_{x^4 x^3}\begin{vmatrix} \Delta_{x^2 x^2} & \Delta_{x^2 x^1} \\ \Delta_{x^1 x^2} & \Delta_{x^1 x^1} \end{vmatrix} + D_{x^3 x^4}\Delta_{x^4 x^2}\begin{vmatrix} \Delta_{x^2 x^3} & \Delta_{x^2 x^1} \\ \Delta_{x^1 x^3} & \Delta_{x^1 x^1} \end{vmatrix} - D_{x^4 x^3}\Delta_{x^3 x^1}\begin{vmatrix} \Delta_{x^2 x^3} & \Delta_{x^2 x^2} \\ \Delta_{x^1 x^3} & \Delta_{x^1 x^3} \end{vmatrix} +$$

$$D_{x^2x^4}\Delta_{x^4x^3}\begin{vmatrix}\Delta_{x^3x^2} & \Delta_{x^3x^1}\\ \Delta_{x^1x^2} & \Delta_{x^1x^1}\end{vmatrix} - D_{x^2x^4}\Delta_{x^4x^2}\begin{vmatrix}\Delta_{x^3x^3} & \Delta_{x^3x^1}\\ \Delta_{x^1x^3} & \Delta_{x^1x^1}\end{vmatrix} +$$

$$D_{x^2x^4}\Delta_{x^4x^1}\begin{vmatrix}\Delta_{x^3x^3} & \Delta_{x^3x^2}\\ \Delta_{x^1x^3} & \Delta_{x^1x^2}\end{vmatrix} - D_{x^1x^4}\Delta_{x^4x^3}\begin{vmatrix}\Delta_{x^3x^2} & \Delta_{x^3x^1}\\ \Delta_{x^2x^2} & \Delta_{x^2x^1}\end{vmatrix} +$$

$$D_{x^1x^4}\Delta_{x^4x^2}\begin{vmatrix}\Delta_{x^3x^3} & \Delta_{x^3x^1}\\ \Delta_{x^2x^3} & \Delta_{x^2x^1}\end{vmatrix} - D_{x^1x^4}\Delta_{x^4x^1}\begin{vmatrix}\Delta_{x^3x^3} & \Delta_{x^3x^2}\\ \Delta_{x^2x^3} & \Delta_{x^2x^2}\end{vmatrix} =$$

$$\Delta_{x^4x^4}\begin{vmatrix}\Delta_{x^3x^3} & \cdots & \Delta_{x^3x^1}\\ \vdots & \ddots & \vdots\\ \Delta_{x^1x^3} & \cdots & \Delta_{x^1x^1}\end{vmatrix} + 3\begin{vmatrix}\Delta_{x^3x^3} & \cdots & \Delta_{x^3x^1}\\ \vdots & \ddots & \vdots\\ \Delta_{x^1x^3} & \cdots & \Delta_{x^1x^1}\end{vmatrix} - \Delta_{x^3x^4}\Delta_{x^4x^3}\begin{vmatrix}\Delta_{x^2x^2} & \Delta_{x^2x^1}\\ \Delta_{x^1x^2} & \Delta_{x^1x^1}\end{vmatrix} -$$

$$\Delta_{x^3x^3}\begin{vmatrix}\Delta_{x^2x^2} & \Delta_{x^2x^1}\\ \Delta_{x^1x^2} & \Delta_{x^1x^1}\end{vmatrix} + \Delta_{x^3x^4}\Delta_{x^4x^2}\begin{vmatrix}\Delta_{x^2x^3} & \Delta_{x^2x^1}\\ \Delta_{x^1x^3} & \Delta_{x^1x^1}\end{vmatrix} + \Delta_{x^3x^2}\begin{vmatrix}\Delta_{x^2x^3} & \Delta_{x^2x^1}\\ \Delta_{x^1x^3} & \Delta_{x^1x^1}\end{vmatrix} -$$

$$\Delta_{x^3x^4}\Delta_{x^4x^1}\begin{vmatrix}\Delta_{x^2x^3} & \Delta_{x^2x^2}\\ \Delta_{x^1x^3} & \Delta_{x^1x^2}\end{vmatrix} - \Delta_{x^3x^1}\begin{vmatrix}\Delta_{x^2x^3} & \Delta_{x^2x^2}\\ \Delta_{x^1x^3} & \Delta_{x^1x^2}\end{vmatrix} + \Delta_{x^2x^4}\Delta_{x^4x^3}\begin{vmatrix}\Delta_{x^3x^2} & \Delta_{x^3x^1}\\ \Delta_{x^1x^2} & \Delta_{x^1x^1}\end{vmatrix} +$$

$$\Delta_{x^2x^3}\begin{vmatrix}\Delta_{x^3x^2} & \Delta_{x^3x^1}\\ \Delta_{x^1x^2} & \Delta_{x^1x^1}\end{vmatrix} - \Delta_{x^2x^4}\Delta_{x^4x^2}\begin{vmatrix}\Delta_{x^3x^3} & \Delta_{x^3x^1}\\ \Delta_{x^1x^3} & \Delta_{x^1x^1}\end{vmatrix} - \Delta_{x^2x^2}\begin{vmatrix}\Delta_{x^3x^3} & \Delta_{x^3x^1}\\ \Delta_{x^1x^3} & \Delta_{x^1x^1}\end{vmatrix} +$$

$$\Delta_{x^2x^4}\Delta_{x^4x^1}\begin{vmatrix}\Delta_{x^3x^3} & \Delta_{x^3x^2}\\ \Delta_{x^1x^3} & \Delta_{x^1x^2}\end{vmatrix} + \Delta_{x^2x^1}\begin{vmatrix}\Delta_{x^3x^3} & \Delta_{x^3x^2}\\ \Delta_{x^1x^3} & \Delta_{x^1x^2}\end{vmatrix} - \Delta_{x^1x^4}\Delta_{x^4x^3}\begin{vmatrix}\Delta_{x^3x^2} & \Delta_{x^3x^1}\\ \Delta_{x^2x^2} & \Delta_{x^2x^1}\end{vmatrix} -$$

$$-\Delta_{x^1x^3}\begin{vmatrix}\Delta_{x^3x^2} & \Delta_{x^3x^1}\\ \Delta_{x^2x^2} & \Delta_{x^2x}\end{vmatrix} + \Delta_{x^1x^4}\Delta_{x^4x^2}\begin{vmatrix}\Delta_{x^3x^3} & \Delta_{x^3x^1}\\ \Delta_{x^2x^3} & \Delta_{x^2x^1}\end{vmatrix} + \Delta_{x^1x^2}\begin{vmatrix}\Delta_{x^3x^3} & \Delta_{x^3x^1}\\ \Delta_{x^2x^3} & \Delta_{x^2x^1}\end{vmatrix}$$

$$-\Delta_{x^1x^4}\Delta_{x^4x^1}\begin{vmatrix}\Delta_{x^3x^3} & \Delta_{x^3x^2}\\ \Delta_{x^2x^3} & \Delta_{x^2x^2}\end{vmatrix} - \Delta_{x^1x^1}\begin{vmatrix}\Delta_{x^3x^3} & \Delta_{x^3x^2}\\ \Delta_{x^2x^3} & \Delta_{x^2x^2}\end{vmatrix} =$$

After collecting similar terms

$$\Delta_{x^4x^4}\begin{vmatrix}\Delta_{x^3x^3} & \Delta_{x^3x^2} & \Delta_{x^3x^1}\\ \Delta_{x^2x^3} & \Delta_{x^2x^2} & \Delta_{x^2x^1}\\ \Delta_{x^1x^3} & \Delta_{x^1x^2} & \Delta_{x^1x^1}\end{vmatrix} - \Delta_{x^4x^3}\begin{vmatrix}\Delta_{x^3x^4} & \Delta_{x^3x^2} & \Delta_{x^3x^1}\\ \Delta_{x^2x^4} & \Delta_{x^2x^2} & \Delta_{x^2x^1}\\ \Delta_{x^1x^4} & \Delta_{x^1x^2} & \Delta_{x^1x^1}\end{vmatrix} + \Delta_{x^4x^2}\begin{vmatrix}\Delta_{x^3x^4} & \Delta_{x^3x^3} & \Delta_{x^3x^1}\\ \Delta_{x^2x^4} & \Delta_{x^2x^3} & \Delta_{x^2x^1}\\ \Delta_{x^1x^4} & \Delta_{x^1x^3} & \Delta_{x^1x^1}\end{vmatrix} - \quad -$$

$$\Delta_{x^4x^1}\begin{vmatrix}\Delta_{x^3x^4} & \Delta_{x^3x^3} & \Delta_{x^3x^2}\\ \Delta_{x^2x^4} & \Delta_{x^2x^3} & \Delta_{x^2x^2}\\ \Delta_{x^1x^4} & \Delta_{x^1x^3} & \Delta_{x^1x^2}\end{vmatrix} + 3\begin{vmatrix}\Delta_{x^3x^3} & \Delta_{x^3x^2} & \Delta_{x^3x^1}\\ \Delta_{x^2x^3} & \Delta_{x^2x^2} & \Delta_{x^2x^1}\\ \Delta_{x^1x^3} & \Delta_{x^1x^2} & \Delta_{x^1x^1}\end{vmatrix} - 3\begin{vmatrix}\Delta_{x^3x^3} & \Delta_{x^3x^2} & \Delta_{x^3x^1}\\ \Delta_{x^2x^3} & \Delta_{x^2x^2} & \Delta_{x^2x^1}\\ \Delta_{x^1x^3} & \Delta_{x^1x^2} & \Delta_{x^1x^1}\end{vmatrix} =$$

$$\begin{vmatrix}\Delta_{x^4x^4} & \Delta_{x^4x^3} & \Delta_{x^4x^2} & \Delta_{x^4x^1}\\ \Delta_{x^3x^4} & \Delta_{x^3x^3} & \Delta_{x^3x^2} & \Delta_{x^3x^1}\\ \Delta_{x^2x^4} & \Delta_{x^2x^3} & \Delta_{x^2x^2} & \Delta_{x^2x^1}\\ \Delta_{x^1x^4} & \Delta_{x^1x^3} & \Delta_{x^1x^2} & \Delta_{x^1x^1}\end{vmatrix}.$$

Now, to prove the general formula, assume m is even for ease of notation. (Proof is similar if m is odd.) We must evaluate the determinant on the left by cofactor expansion down the first column:

$$(D_{x^mx^m} + m - 1)\begin{vmatrix}\Delta_{x^{m-1}x^{m-1}} & \cdots & \Delta_{x^{m-1}x^1}\\ \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots\\ \Delta_{x^1x^{m-1}} & \cdots & \Delta_{x^1x^1}\end{vmatrix} - \sum_{j\,odd} D_{x^jx^m}\begin{vmatrix}\Delta_{x^mx^{m-1}} & \cdots & \Delta_{x^mx^1}\\ \cdots & \cdots & \cdots\\ \Delta_{x^{j+1}x^{m-1}} & \cdots & \Delta_{x^{j+1}x^1}\\ \Delta_{x^{j-1}x^{m-1}} & \cdots & \Delta_{x^{j-1}x^1}\\ \cdots & \cdots & \cdots\\ \Delta_{x^1x^{m-1}} & \cdots & \Delta_{x^1x^1}\end{vmatrix}$$

$$+\sum_{j\,even} D_{x^jx^m}\begin{vmatrix}\Delta_{x^mx^{m-1}} & \cdots & \Delta_{x^mx^1}\\ \cdots & \cdots & \cdots\\ \Delta_{x^{j+1}x^{m-1}} & \cdots & \Delta_{x^{j+1}x^1}\\ \Delta_{x^{j-1}x^{m-1}} & \cdots & \Delta_{x^{j-1}x^1}\\ \cdots & \cdots & \cdots\\ \Delta_{x^1x^{m-1}} & \cdots & \Delta_{x^1x^1}\end{vmatrix}.$$

Let

$$\mathbf{C}_{ij} = \begin{vmatrix}\Delta_{x^{m-1}x^{m-1}} & \cdots & \Delta_{x^{m-1}x^{i+1}} & \Delta_{x^{m-1}x^{i-1}} & \cdots & \Delta_{x^{m-1}x^1}\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ \Delta_{x^{j+1}x^{m-1}} & \cdots & \Delta_{x^{j+1}x^{i+1}} & \Delta_{x^{j+1}x^{i-1}} & \cdots & \Delta_{x^{j+1}x^1}\\ \Delta_{x^{j-1}x^{m-1}} & \cdots & \Delta_{x^{j-1}x^{i+1}} & \Delta_{x^{j-1}x^{i-1}} & \cdots & \Delta_{x^{j-1}x^1}\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ \Delta_{x^1x^{m-1}} & \cdots & \Delta_{x^1x^{i+1}} & \Delta_{x^1x^{i-1}} & \cdots & \Delta_{x^1x^1}\end{vmatrix},$$

then by expanding the determinants under the finite sums by the cofactors across the top row, we get

$$(D_{\mathbf{x}^m\mathbf{x}^m}-m+1)\begin{vmatrix}\Delta_{\mathbf{x}^{m-1}\mathbf{x}^{m-1}} & \cdots & \Delta_{\mathbf{x}^{m-1}\mathbf{x}^1}\\ \vdots & \ddots & \vdots\\ \Delta_{\mathbf{x}^1\mathbf{x}^{m-1}} & \cdots & \Delta_{\mathbf{x}^1\mathbf{x}^1}\end{vmatrix}-\sum_{jodd,iodd}D_{\mathbf{x}^j\mathbf{x}^m}\Delta_{\mathbf{x}^m\mathbf{x}^i}\mathbf{C}_{ij}+\sum_{jodd,ieven}D_{\mathbf{x}^j\mathbf{x}^m}\Delta_{\mathbf{x}^m\mathbf{x}^i}\mathbf{C}_{ij}-$$

$$\sum_{jeven,ieven}D_{\mathbf{x}^j\mathbf{x}^m}\Delta_{\mathbf{x}^m\mathbf{x}^i}\mathbf{C}_{ij}+\sum_{jeven,iodd}D_{\mathbf{x}^j\mathbf{x}^m}\Delta_{\mathbf{x}^m\mathbf{x}^i}\mathbf{C}_{ij}.$$

Since there is no Δ with $\mathbf{x}^\alpha$ as a left subscript in the first term, we can write $D_{\mathbf{x}^m\mathbf{x}^m}$ as $\Delta_{\mathbf{x}^m\mathbf{x}^m}$. However, since $D_{\mathbf{x}^j\mathbf{x}^m}\Delta_{\mathbf{x}^m\mathbf{x}^i}=\Delta_{\mathbf{x}^j\mathbf{x}^m}\Delta_{\mathbf{x}^m\mathbf{x}^i}+\Delta_{\mathbf{x}^j\mathbf{x}^i}$, we can write the above in terms of all Δ's:

$$(\Delta_{\mathbf{x}^m\mathbf{x}^m}+m-1)\begin{vmatrix}\Delta_{\mathbf{x}^{m-1}\mathbf{x}^{m-1}} & \cdots & \Delta_{\mathbf{x}^{m-1}\mathbf{x}^1}\\ \cdots & \cdots & \cdots\\ \Delta_{\mathbf{x}^1\mathbf{x}^{m-1}} & \cdots & \Delta_{\mathbf{x}^1\mathbf{x}^1}\end{vmatrix}-\sum_{jodd,iodd}(\Delta_{\mathbf{x}^j\mathbf{x}^m}\Delta_{\mathbf{x}^m\mathbf{x}^i}\mathbf{C}_{ij}+\Delta_{\mathbf{x}^j\mathbf{x}^i}\mathbf{C}_{ij})+$$

$$\sum_{jodd,ieven}(\Delta_{\mathbf{x}^j\mathbf{x}^m}\Delta_{\mathbf{x}^m\mathbf{x}^i}\mathbf{C}_{ij}+\Delta_{\mathbf{x}^j\mathbf{x}^i}\mathbf{C}_{ij})-\sum_{jeven,iodd}(\Delta_{\mathbf{x}^j\mathbf{x}^m}\Delta_{\mathbf{x}^m\mathbf{x}^i}\mathbf{C}_{ij}+\Delta_{\mathbf{x}^j\mathbf{x}^i}\mathbf{C}_{ij})+$$

$$\sum_{jeven,iodd}(\Delta_{\mathbf{x}^j\mathbf{x}^m}\Delta_{\mathbf{x}^m\mathbf{x}^i}\mathbf{C}_{ij}+\Delta_{\mathbf{x}^j\mathbf{x}^i}\mathbf{C}_{ij}).$$

Which after a rearrangement of terms equals

$$\Delta_{\mathbf{x}^m\mathbf{x}^m}\begin{vmatrix}\Delta_{\mathbf{x}^{m-1}\mathbf{x}^{m-1}} & \cdots & \Delta_{\mathbf{x}^{m-1}\mathbf{x}^1}\\ \vdots & \ddots & \vdots\\ \Delta_{\mathbf{x}^1\mathbf{x}^{m-1}} & \cdots & \Delta_{\mathbf{x}^1\mathbf{x}^1}\end{vmatrix}-\sum_{jodd,iodd}\Delta_{\mathbf{x}^j\mathbf{x}^m}\Delta_{\mathbf{x}^m\mathbf{x}^i}\mathbf{C}_{ij}+\sum_{jodd,ieven}\Delta_{\mathbf{x}^j\mathbf{x}^m}\Delta_{\mathbf{x}^m\mathbf{x}^i}\mathbf{C}_{ij}-$$

$$\sum_{jeven,ieven}\Delta_{\mathbf{x}^j\mathbf{x}^m}\Delta_{\mathbf{x}^m\mathbf{x}^i}\mathbf{C}_{ij}-\sum_{jeven,iodd}\Delta_{\mathbf{x}^j\mathbf{x}^m}\Delta_{\mathbf{x}^m\mathbf{x}^i}\mathbf{C}_{ij}+$$

$$(m-1)\begin{vmatrix}\Delta_{\mathbf{x}^{m-1}\mathbf{x}^{m-1}} & \cdots & \Delta_{\mathbf{x}^{m-1}\mathbf{x}^1}\\ \cdots & \cdots & \cdots\\ \Delta_{\mathbf{x}^1\mathbf{x}^{m-1}} & \cdots & \Delta_{\mathbf{x}^1\mathbf{x}^1}\end{vmatrix}-\sum_{jodd,iodd}\Delta_{\mathbf{x}^j\mathbf{x}^i}\mathbf{C}_{ij}+$$

$$\sum_{jodd,ieven}\Delta_{\mathbf{x}^j\mathbf{x}^i}\mathbf{C}_{ij}-\sum_{jeven,ieven}\Delta_{\mathbf{x}^j\mathbf{x}^i}\mathbf{C}_{ij}+\sum_{jeven,iodd}\Delta_{\mathbf{x}^j\mathbf{x}^i}\mathbf{C}_{ij}=$$

$$\Delta_{\mathbf{x}^m\mathbf{x}^m}\begin{vmatrix}\Delta_{\mathbf{x}^{m-1}\mathbf{x}^{m-1}} & \cdots & \Delta_{\mathbf{x}^{m-1}\mathbf{x}^1}\\ \vdots & \ddots & \vdots\\ \vdots & \ddots & \vdots\\ \Delta_{\mathbf{x}^1\mathbf{x}^{m-1}} & \cdots & \Delta_{\mathbf{x}^1\mathbf{x}^1}\end{vmatrix}-\sum_{jodd}\Delta_{\mathbf{x}^j\mathbf{x}^m}\begin{vmatrix}\Delta_{\mathbf{x}^m\mathbf{x}^{m-1}} & \cdots & \Delta_{\mathbf{x}^m\mathbf{x}^1}\\ \vdots & \ddots & \vdots\\ \Delta_{\mathbf{x}^{j+1}\mathbf{x}^{m-1}} & \cdots & \Delta_{\mathbf{x}^{j+1}\mathbf{x}^1}\\ \Delta_{\mathbf{x}^{j-1}\mathbf{x}^{m-1}} & \cdots & \Delta_{\mathbf{x}^{j-1}\mathbf{x}^1}\\ \vdots & \ddots & \vdots\\ \Delta_{\mathbf{x}^1\mathbf{x}^{m-1}} & \cdots & \Delta_{\mathbf{x}^1\mathbf{x}^1}\end{vmatrix}$$

$$+\sum_{jeven}\Delta_{\mathbf{x}^j\mathbf{x}^m}\begin{vmatrix}\Delta_{\mathbf{x}^m\mathbf{x}^{m-1}} & \cdots & \Delta_{\mathbf{x}^m\mathbf{x}^1}\\ \vdots & \ddots & \vdots\\ \Delta_{\mathbf{x}^{j+1}\mathbf{x}^{m-1}} & \cdots & \Delta_{\mathbf{x}^{j+1}\mathbf{x}^1}\\ \Delta_{\mathbf{x}^{j-1}\mathbf{x}^{m-1}} & \cdots & \Delta_{\mathbf{x}^{j-1}\mathbf{x}^1}\\ \vdots & \ddots & \vdots\\ \Delta_{\mathbf{x}^1\mathbf{x}^{m-1}} & \cdots & \Delta_{\mathbf{x}^1\mathbf{x}^1}\end{vmatrix}+$$

$$+(m-1)\begin{vmatrix}\Delta_{\mathbf{x}^{m-1}\mathbf{x}^{m-1}} & \cdots & \Delta_{\mathbf{x}^{m-1}\mathbf{x}^1}\\ \vdots & \ddots & \vdots\\ \Delta_{\mathbf{x}^1\mathbf{x}^{m-1}} & \cdots & \Delta_{\mathbf{x}^1\mathbf{x}^1}\end{vmatrix}-\sum_{jodd}\begin{vmatrix}\Delta_{\mathbf{x}^{m-1}\mathbf{x}^{m-1}} & \cdots & \Delta_{\mathbf{x}^{m-1}\mathbf{x}^1}\\ \vdots & \ddots & \vdots\\ \Delta_{\mathbf{x}^1\mathbf{x}^{m-1}} & \cdots & \Delta_{\mathbf{x}^1\mathbf{x}^1}\end{vmatrix}-\sum_{jeven}\begin{vmatrix}\Delta_{\mathbf{x}^{m-1}\mathbf{x}^{m-1}} & \cdots & \Delta_{\mathbf{x}^{m-1}\mathbf{x}^1}\\ \vdots & \ddots & \vdots\\ \Delta_{\mathbf{x}^1\mathbf{x}^{m-1}} & \cdots & \Delta_{\mathbf{x}^1\mathbf{x}^1}\end{vmatrix}$$

$$=\begin{vmatrix}\Delta_{\mathbf{x}^m\mathbf{x}^m} & \cdots & \Delta_{\mathbf{x}^m\mathbf{x}^1}\\ \vdots & \ddots & \vdots\\ \Delta_{\mathbf{x}^1\mathbf{x}^m} & \cdots & \Delta_{\mathbf{x}^1\mathbf{x}^1}\end{vmatrix}+(m-1)\begin{vmatrix}\Delta_{\mathbf{x}^{m-1}\mathbf{x}^{m-1}} & \cdots & \Delta_{\mathbf{x}^{m-1}\mathbf{x}^1}\\ \vdots & \ddots & \vdots\\ \Delta_{\mathbf{x}^1\mathbf{x}^{m-1}} & \cdots & \Delta_{\mathbf{x}^1\mathbf{x}^1}\end{vmatrix}-(m-1)\begin{vmatrix}\Delta_{\mathbf{x}^{m-1}\mathbf{x}^{m-1}} & \cdots & \Delta_{\mathbf{x}^{m-1}\mathbf{x}^1}\\ \vdots & \ddots & \vdots\\ \Delta_{\mathbf{x}^1\mathbf{x}^{m-1}} & \cdots & \Delta_{\mathbf{x}^1\mathbf{x}^1}\end{vmatrix}.$$

Which equals the determinant on the right in (4). Notice that by exactly the same argument (6)

$$\begin{vmatrix}D_{\mathbf{x}^m\mathbf{x}^m}+m-1 & \Delta_{\mathbf{x}^m\mathbf{y}^{m-1}} & \cdots & \Delta_{\mathbf{x}^m\mathbf{y}^2} & \Delta_{\mathbf{x}^m\mathbf{y}^1}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ D_{\mathbf{x}^2\mathbf{x}^m} & \Delta_{\mathbf{x}^2\mathbf{y}^{m-1}} & \cdots & \Delta_{\mathbf{x}^2\mathbf{y}^2} & \Delta_{\mathbf{x}^2\mathbf{y}^1}\\ D_{\mathbf{x}^1\mathbf{x}^m} & \Delta_{\mathbf{x}^1\mathbf{y}^{m-1}} & \cdots & \Delta_{\mathbf{x}^1\mathbf{y}^2} & \Delta_{\mathbf{x}^1\mathbf{y}^1}\end{vmatrix}=\begin{vmatrix}\Delta_{\mathbf{x}^m\mathbf{x}^m} & \Delta_{\mathbf{x}^m\mathbf{y}^{m-1}} & \cdots & \Delta_{\mathbf{x}^m\mathbf{y}^2} & \Delta_{\mathbf{x}^m\mathbf{y}^1}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \Delta_{\mathbf{x}^2\mathbf{x}^m} & \Delta_{\mathbf{x}^2\mathbf{y}^{m-1}} & \cdots & \Delta_{\mathbf{x}^2\mathbf{y}^2} & \Delta_{\mathbf{x}^2\mathbf{y}^1}\\ \Delta_{\mathbf{x}^1\mathbf{x}^m} & \Delta_{\mathbf{x}^1\mathbf{y}^{m-1}} & \cdots & \Delta_{\mathbf{x}^1\mathbf{y}^2} & \Delta_{\mathbf{x}^1\mathbf{y}}\end{vmatrix}.$$

where $\mathbf{x}^j$ may or may not be equal to $\mathbf{y}^j$.

Also, if $\mathbf{x}^m \neq \mathbf{y}^m$, then

$$\begin{vmatrix} D_{\mathbf{x}^m\mathbf{y}^m} & \Delta_{\mathbf{x}^m\mathbf{y}^{m-1}} & \cdots & \Delta_{\mathbf{x}^m\mathbf{y}^2} & \Delta_{\mathbf{x}^m\mathbf{y}^1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ D_{\mathbf{x}^2\mathbf{y}^m} & \Delta_{\mathbf{x}^2\mathbf{y}^{m-1}} & \cdots & \Delta_{\mathbf{x}^2\mathbf{y}^2} & \Delta_{\mathbf{x}^2\mathbf{y}^1} \\ D_{\mathbf{x}^1\mathbf{y}^m} & \Delta_{\mathbf{x}^1\mathbf{y}^{m-1}} & \cdots & \Delta_{\mathbf{x}^1\mathbf{y}^2} & \Delta_{\mathbf{x}^1\mathbf{y}^1} \end{vmatrix} = \begin{vmatrix} \Delta_{\mathbf{x}^m\mathbf{y}^m} & \Delta_{\mathbf{x}^m\mathbf{y}^{m-1}} & \cdots & \Delta_{\mathbf{x}^m\mathbf{y}^2} & \Delta_{\mathbf{x}^m\mathbf{y}^1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Delta_{\mathbf{x}^2\mathbf{y}^m} & \Delta_{\mathbf{x}^2\mathbf{y}^{m-1}} & \cdots & \Delta_{\mathbf{x}^2\mathbf{y}^2} & \Delta_{\mathbf{x}^2\mathbf{y}^1} \\ \Delta_{\mathbf{x}^1\mathbf{y}^m} & \Delta_{\mathbf{x}^1\mathbf{y}^{m-1}} & \cdots & \Delta_{\mathbf{x}^1\mathbf{y}^2} & \Delta_{\mathbf{x}^1\mathbf{y}^1} \end{vmatrix}. \quad (7)$$

Now assume for some j<m,

$$\begin{vmatrix} \Delta_{\mathbf{x}^m\mathbf{x}^m} & \cdots & \cdots & \Delta_{\mathbf{x}^m\mathbf{x}^1} \\ \vdots & & & \vdots \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \vdots & & & \vdots \\ \Delta_{\mathbf{x}^1\mathbf{x}^m} & \cdots & \cdots & \Delta_{\mathbf{x}^1\mathbf{x}^1} \end{vmatrix} =$$

$$\begin{vmatrix} D_{\mathbf{x}^m\mathbf{x}^m}+m-1 & D_{\mathbf{x}^m\mathbf{x}^{m-1}} & \cdots & D_{\mathbf{x}^m\mathbf{x}^j} & \Delta_{\mathbf{x}^m\mathbf{x}^{j-1}} & \cdots & \Delta_{\mathbf{x}^m\mathbf{x}^1} \\ D_{\mathbf{x}^{m-1}\mathbf{x}^m} & D_{\mathbf{x}^{m-1}\mathbf{x}^{m-1}}+m-2 & \cdots & D_{\mathbf{x}^{m-1}\mathbf{x}^j} & \Delta_{\mathbf{x}^{m-1}\mathbf{x}^{j-1}} & \cdots & \Delta_{\mathbf{x}^{m-1}\mathbf{x}^1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & & D_{\mathbf{x}^j\mathbf{x}^j}+j-1 & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ D_{\mathbf{x}^1\mathbf{x}^m} & D_{\mathbf{x}^1\mathbf{x}^{m-1}} & \cdots & D_{\mathbf{x}^1\mathbf{x}^j} & \Delta_{\mathbf{x}^1\mathbf{x}^{j-1}} & \cdots & \Delta_{\mathbf{x}^1\mathbf{x}^1} \end{vmatrix}. (8)$$

Expand the determinant on the right by its cofactors down the first *m-j*+1 columns. Then the determinant on the right equals

$$\sum_{\Psi}\lambda(D_{\mathbf{x}^{\sigma(m)}\mathbf{x}^m}+\delta(m-1))(D_{\mathbf{x}^{\sigma(m-1)}\mathbf{x}^{m-1}}+\delta(m-2))\ldots(D_{\mathbf{x}^{\sigma(j)}\mathbf{x}^j}+\delta(j-1))\begin{vmatrix} \Delta_{\mathbf{x}^{\sigma(j-1)}\mathbf{x}^{j-1}} & \cdots & \Delta_{\mathbf{x}^{\sigma(j-1)}\mathbf{x}^1} \\ \vdots & \ddots & \vdots \\ \Delta_{\mathbf{x}^{\sigma(1)}\mathbf{x}^{j-1}} & \cdots & \Delta_{\mathbf{x}^{\sigma(1)}\mathbf{x}^1} \end{vmatrix}$$

Where Ψ ranges over all permutations of $\{m, m-1, \ldots, 1\}$, $\{\sigma(m), \sigma(m-1), \ldots, \sigma(1)\}$ is some permutation of $\{m, m-1, \ldots, 1\}$,

$$\delta = \begin{cases} 1 & \text{if} \quad \sigma(k) = k \\ 0 & \text{if} \quad \sigma(k) \neq k \end{cases}, \quad k \in \{m, m-1, \ldots 1\} \text{ ,and } \lambda = \begin{cases} 1 & \text{odd permutation} \\ \text{-1} & \text{even permutation} \end{cases}.$$

Since (5), (6), and (7) are true for all $m > 0$ and that $j-1 = \sigma(i)$ for at most one $i \in \{j-1, \ldots, 1\}$, we get

$$\sum_{\Psi} \lambda (D_{x^{\sigma(m)}x^m} + \delta(m-1)) \ldots (D_{x^{\sigma(j)}x^j} + \delta(j-1)) \begin{vmatrix} D_{x^{\sigma(j-1)}x^{j-1}} + \delta(j-2) & \cdots & \Delta_{x^{\sigma(j-1)}x^1} \\ \vdots & \ddots & \vdots \\ D_{x^{\sigma(1)}x^1} + \delta(j-2) & \cdots & \Delta_{x^{\sigma(1)}x^1} \end{vmatrix}$$

$$= \begin{vmatrix} D_{x^m x^m} + m-1 & D_{x^m x^{m-1}} & \cdots & D_{x^m x^{j-1}} & \Delta_{x^m x^{j-2}} & \cdots & \Delta_{x^m x^1} \\ D_{x^{m-1} x^m} & D_{x^{m-1} x^{m-1}} + m-2 & \cdots & D_{x^{m-1} x^{j-1}} & \Delta_{x^{m-1} x^{j-2}} & \cdots & \Delta_{x^{m-1} x^1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ D_{x^{j-1} x^m} & D_{x^{j-1} x^{m-1}} & & D_{x^{j-1} x^{j-1}} + j-2 & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ D_{x^1 x^m} & D_{x^1 x^{m-1}} & \cdots & D_{x^1 x^{j-1}} & \Delta_{x^1 x^{j-2}} & \cdots & \Delta_{x^1 x^1} \end{vmatrix}.$$

Therefore, by induction on the columns of the determinant, we get (4). We can write

$$\begin{bmatrix} x_1^m & x_2^m & \cdots & x_n^m \\ x_1^{m-1} & x_2^{m-1} & \cdots & x_n^{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^1 & x_2^1 & \cdots & x_n^1 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x_1^m} & \frac{\partial}{\partial x_1^{m-1}} & \cdots & \frac{\partial}{\partial x_1^1} \\ \frac{\partial}{\partial x_2^m} & \frac{\partial}{\partial x_2^{m-1}} & \cdots & \frac{\partial}{\partial x_2^1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_n^m} & \frac{\partial}{\partial x_n^{m-1}} & \cdots & \frac{\partial}{\partial x_n^1} \end{bmatrix}$$

$$= \begin{bmatrix} \Delta_{x^m x^m} & \Delta_{x^m x^{m-1}} & \cdots & \Delta_{x^m x^1} \\ \Delta_{x^{m-1} x^m} & \Delta_{x^{m-1} x^{m-1}} & \cdots & \Delta_{x^{m-1} x^1} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{x^1 x^m} & \Delta_{x^1 x^{m-1}} & \cdots & \Delta_{x^1 x^1} \end{bmatrix}.$$

If $m > n$ then

$$\begin{vmatrix} \Delta_{x^m x^m} & \cdots & \Delta_{x^m x^1} \\ \vdots & \ddots & \vdots \\ \Delta_{x^1 x^m} & \cdots & \Delta_{x^1 x^1} \end{vmatrix} = 0,$$

since if $\mathbf{A} = \mathbf{BC}$, then $\text{rank}(\mathbf{A}) \le \min\{\text{rank}(\mathbf{B}), \text{rank}(\mathbf{C})\}$ which, in this case means $\text{rank}(\mathbf{A}) \le n < m$. Therefore, $|\mathbf{A}| = 0$. If $m = n$, then let

$$[\mathbf{x}^m, \mathbf{x}^{m-1}, \ldots\ldots \mathbf{x}^1] = \begin{vmatrix} x_1^m & \cdots & x_1^1 \\ \vdots & \ddots & \vdots \\ x_n^m & \cdots & x_n^1 \end{vmatrix}, \qquad \text{and } \Omega = \begin{vmatrix} \dfrac{\partial}{\partial x_1^m} & \cdots & \dfrac{\partial}{\partial x_1^1} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial}{\partial x_n^m} & \cdots & \dfrac{\partial}{\partial x_n^1} \end{vmatrix}.$$

Thus,

$$[\mathbf{x}^m, \mathbf{x}^{m-1}, \ldots., \mathbf{x}^1]\Omega = \begin{vmatrix} \Delta_{\mathbf{x}^m\mathbf{x}^m} & \cdots & \Delta_{\mathbf{x}^m\mathbf{x}^1} \\ \vdots & \ddots & \cdots \\ \Delta_{\mathbf{x}^1\mathbf{x}^m} & \cdots & \Delta_{\mathbf{x}^1\mathbf{x}^1} \end{vmatrix}.$$

So combining with (6) we get

Theorem 2.3 *The Capelli Identity*

$$\begin{vmatrix} D_{\mathbf{x}^m\mathbf{x}^m} + m - 1 & D_{\mathbf{x}^m\mathbf{x}^{m-1}} & \cdots & D_{\mathbf{x}^m\mathbf{x}^1} \\ D_{\mathbf{x}^{m-1}\mathbf{x}^m} & D_{\mathbf{x}^{m-1}\mathbf{x}^{m-1}} + m - 2 & \cdots & D_{\mathbf{x}^{m-1}\mathbf{x}^1} \\ \vdots & \vdots & \ddots & \vdots \\ D_{\mathbf{x}^1\mathbf{x}^m} & D_{\mathbf{x}^1\mathbf{x}^{m-1}} & \cdots & D_{\mathbf{x}^1\mathbf{x}^1} \end{vmatrix} f = \begin{cases} 0 & \text{if } m > n \\ [\mathbf{x}^m, \mathbf{x}^{m-1}, \ldots, \mathbf{x}^1]\Omega f & \text{if } m = n \end{cases}.$$

CHAPTER III

POLARIZATION PRESERVES INVARIANCE

If $f(\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^m)$ is an invariant under the group of linear transformations Γ then we will show that by the Capelli identity, $\Omega f(\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^m)$ is an absolute invariant provided the elements of Γ are unimodular (i.e., of determinant 1). However, if the elements of Γ are not unimodular, then Ωf is still relatively invariant. For example, let f be invariant under group Γ, then if $\mathbf{A} \in \Gamma$,

$$f(\mathbf{A}\mathbf{x}^1, \ldots, \mathbf{A}\mathbf{x}^m) = f(\mathbf{x}^1, \ldots, \mathbf{x}^m).$$

Therefore,

$$\Omega f(\mathbf{A}\mathbf{x}^1, \ldots, \mathbf{A}\mathbf{x}^m) = \Omega f(\mathbf{x}^1, \ldots, \mathbf{x}^m) / \det(\mathbf{A}).$$

To show this, we first need a lemma.

Lemma 3.1. *If f is an invariant under Γ, then $D_{\mathbf{yx}} f$ is invariant under Γ.*

Proof: Let $f(\mathbf{x}, \mathbf{y})$ be invariant in Γ. Then $f(\mathbf{x}, \mathbf{y}) = f(\mathbf{A}\mathbf{x}, \mathbf{A}\mathbf{y})$ provided $\mathbf{A} \in \Gamma$. Let

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \ \mathbf{u} = \mathbf{A}\mathbf{x}, \text{ and } \mathbf{v} = \mathbf{A}\mathbf{y}.$$

Then $u_j = \sum_{i=1}^{n} a_{ji} x_i$, and $v_j = \sum_{i=1}^{n} a_{ji} y_i$. So, by the chain rule,

$$\frac{\partial f(\mathbf{x}, \mathbf{y})}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial f(\mathbf{u}, \mathbf{v})}{\partial u_j} \cdot \frac{\partial u_j}{\partial x_i} = \sum_{j=1}^{n} a_{ji} \frac{\partial f(\mathbf{u}, \mathbf{v})}{\partial u_j}.$$

So,

$$D_{\mathbf{yx}} f(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{n} y_i \frac{\partial f(\mathbf{x},\mathbf{y})}{\partial x_i} = \sum_{i=1}^{n} y_i \sum_{j=1}^{n} a_{ji} \frac{\partial f(\mathbf{u},\mathbf{v})}{\partial u_j} =$$

$$\sum_{j=1}^{n} \frac{\partial f(\mathbf{u},\mathbf{v})}{\partial u_j} \sum_{i=1}^{n} a_{ji} y_i = \sum_{j=1}^{n} \frac{\partial f(\mathbf{u},\mathbf{v})}{\partial u_j} v_j = D_{\mathbf{vu}} f(\mathbf{u},\mathbf{v}).$$

Q.E.D.

Therefore, if $\mathbf{u}^j = \mathbf{A}\mathbf{x}^j$, and $f(\mathbf{x}^1,\ldots,\mathbf{x}^m) = f(\mathbf{u}^1,\ldots,\mathbf{u}^m)$, then it is obvious that $D_{\mathbf{x}^i\mathbf{x}^j} f(\mathbf{x}^1,\ldots,\mathbf{x}^m) = D_{\mathbf{u}^i\mathbf{u}^j} f(\mathbf{u}^1,\ldots,\mathbf{u}^m)$. So, by induction on the number of polarizations, $D_{\mathbf{x}^r\mathbf{x}^s} \ldots D_{\mathbf{x}^i\mathbf{x}^j} f(\mathbf{x}^1,\ldots,\mathbf{x}^m) = D_{\mathbf{u}^r\mathbf{u}^s} \ldots D_{\mathbf{u}^i\mathbf{u}^j} f(\mathbf{u}^1,\ldots,\mathbf{u}^m)$. Clearly, sum and multiplication by a constant preserves invariance. Therefore, since the determinant on the left side of Capelli's identity is simply the sum and constant multiplication of consecutive polarizations, it is invariant under Γ. So

$$\begin{vmatrix} D_{\mathbf{x}^m\mathbf{x}^m} + m - 1 & D_{\mathbf{x}^m\mathbf{x}^{m-1}} & \cdots & D_{\mathbf{x}^m\mathbf{x}^1} \\ D_{\mathbf{x}^{m-1}\mathbf{x}^m} & D_{\mathbf{x}^{m-1}\mathbf{x}^{m-1}} + m - 2 & \cdots & D_{\mathbf{x}^{m-1}\mathbf{x}^1} \\ \vdots & \vdots & \ddots & \vdots \\ D_{\mathbf{x}^1\mathbf{x}^m} & D_{\mathbf{x}^1\mathbf{x}^{m-1}} & \cdots & D_{\mathbf{x}^1\mathbf{x}^1} \end{vmatrix} f(\mathbf{x}^1,\ldots,\mathbf{x}^m) =$$

$$\begin{vmatrix} D_{\mathbf{u}^m\mathbf{u}^m} + m - 1 & D_{\mathbf{u}^m\mathbf{u}^{m-1}} & \cdots & D_{\mathbf{u}^m\mathbf{u}^1} \\ D_{\mathbf{u}^{m-1}\mathbf{u}^m} & D_{\mathbf{u}^{m-1}\mathbf{u}^{m-1}} + m - 2 & \cdots & D_{\mathbf{u}^{m-1}\mathbf{u}^1} \\ \vdots & \vdots & \ddots & \vdots \\ D_{\mathbf{u}^1\mathbf{u}^m} & D_{\mathbf{u}^1\mathbf{u}^{m-1}} & \cdots & D_{\mathbf{u}^1\mathbf{u}^1} \end{vmatrix} f(\mathbf{u}^1,\ldots,\mathbf{u}^m).$$

Also,

$$[\mathbf{u}^1,\mathbf{u}^2,\ldots,\mathbf{u}^m] = [\mathbf{A}\mathbf{x}^1,\mathbf{A}\mathbf{x}^2,\ldots,\mathbf{A}\mathbf{x}^m] = \left| \mathbf{A} \begin{bmatrix} x_1^1 & \cdots & x_1^m \\ \vdots & \ddots & \vdots \\ x_n^1 & \cdots & x_n^m \end{bmatrix} \right| = (\det \mathbf{A})[\mathbf{x}^1,\mathbf{x}^2,\ldots,\mathbf{x}^m]$$

So,

$$\Omega f(\mathbf{u}^1,\ldots,\mathbf{u}^m) =$$

$$\frac{\begin{vmatrix} D_{\mathbf{u}^m\mathbf{u}^m}+m-1 & D_{\mathbf{u}^m\mathbf{u}^{m-1}} & \cdots & D_{\mathbf{u}^m\mathbf{u}^1} \\ D_{\mathbf{u}^{m-1}\mathbf{u}^m} & D_{\mathbf{u}^{m-1}\mathbf{u}^{m-1}}+m-2 & \cdots & D_{\mathbf{u}^{m-1}\mathbf{u}^1} \\ \vdots & \vdots & \ddots & \vdots \\ D_{\mathbf{u}^1\mathbf{u}^m} & D_{\mathbf{u}^1\mathbf{u}^{m-1}} & \cdots & D_{\mathbf{u}^1\mathbf{u}^1} \end{vmatrix} f(\mathbf{u}^1,\ldots,\mathbf{u}^m)}{[\mathbf{u}^1,\ldots,\mathbf{u}^m]} =$$

$$\frac{\begin{vmatrix} D_{\mathbf{x}^m\mathbf{x}^m}+m-1 & D_{\mathbf{x}^m\mathbf{x}^{m-1}} & \cdots & D_{\mathbf{x}^m\mathbf{x}^1} \\ D_{\mathbf{x}^{m-1}\mathbf{x}^m} & D_{\mathbf{x}^{m-1}\mathbf{x}^{m-1}}+m-2 & \cdots & D_{\mathbf{x}^{m-1}\mathbf{x}^1} \\ \vdots & \vdots & \ddots & \vdots \\ D_{\mathbf{x}^1\mathbf{x}^m} & D_{\mathbf{x}^1\mathbf{x}^{m-1}} & \cdots & D_{\mathbf{x}^1\mathbf{x}^1} \end{vmatrix} f(\mathbf{x}^1,\ldots,\mathbf{x}^m)}{(\det\mathbf{A})[\mathbf{x}^1,\mathbf{x}^2,\ldots,\mathbf{x}^m]}$$

$$= \frac{\Omega f(\mathbf{x}^1,\ldots,\mathbf{x}^m)}{\det\mathbf{A}}.$$

CHAPTER IV

REDUCTION OF BASIC INVARIANTS

A **form** is a polynomial $f(\mathbf{x}^1,\ldots,\mathbf{x}^m)$ such that every monomial has the same degree r_j in the components x_j^i of $\mathbf{x}^i$ for each *j = 1,...,m*. For example,

$$f(\mathbf{x},\mathbf{y}) = (x_1)^3(x_2)^4(y_1)(y_2)^4 - 5(x_1)^2(x_2)^5(y_1)^3(y_2)^2$$

is a form because in each monomial, the degrees of the components of **x** sum to 7 (3+4=7 and 2+5=7), and the degrees of the components of **y** sum to 5 (1+4=5, and 3+2=5). We then say that **x** is of degree 7, and **y** is of degree 5. If $\{r_1, r_2, \ldots, r_m\}$ are the respective degrees of $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$, then the **total degree** r is defined as $r = r_1 + r_2 + \ldots + r_m$. The total degree the above example is $r = 7+5 = 12$. The **rank** of form $f(\mathbf{x}^1,\ldots,\mathbf{x}^m)$is an ordered (*m+1*)-tuple $(r, r_1, r_2, \ldots, r_m)$ composed of the total degree *r* of *f*, and degrees $\{\, r_i \,\}$ of the vectors $\{\mathbf{x}_j\}$.A lexicographical ordering is placed on the rank by making the following definition: if $(r, r_1, r_2, \ldots, r_m)$ is the rank of $f(\mathbf{x}^1,\ldots,\mathbf{x}^m)$ and $(r', r_1', r_2' \ldots, r_m')$ is the rank of $g(\mathbf{x}^1,\ldots,\mathbf{x}^m)$, then $(r, r_1, r_2, \ldots, r_m) < (r', r_1', r_2', \ldots, r_m')$ provided either $r < r'$ or if $r = r'$ and $r_i = r_i'$ for all $i < k$ then $r_k < r'$.If $r = r'$and $r_i = r_i'$ for all $i \in \{1, \ldots, m\}$, then the ranks are said to be equal. For example, let $f(\mathbf{x},\mathbf{y})$ be defined as above, and let

$$g(\mathbf{x},\mathbf{y}) = 3(x_1)^5(x_2)^4(y_1)^2(y_2)^1 - 7(x_1)^7(x_2)^2(y_1)^1(y_2)^2.$$

Let us verify that $rank(f(\mathbf{x},\mathbf{y})) = (12,7,5) < (12,9,3) = rank(g(\mathbf{x},\mathbf{y}))$.Check the first component (total rank): 12=12. Check the second component (rank of **x**):, 7<9. Therefore, *rank(f) < rank(g)*. Let Ωf be defined as previously. Ωf takes the partial derivative of each component of each vector in *f*. Thus, if the rank of *f* is $(r, r_1, r_2, \ldots, r_m)$, then the rank of Ωf is $(r-m, r_1-1, r_2-1, \ldots r_m-1)$. For example, let $f(\mathbf{x}, \mathbf{y})$ again be defined as above. Then

$$\Omega f(\mathbf{x},\mathbf{y}) = \begin{vmatrix} \frac{\partial}{\partial y_1} & \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial y_2} & \frac{\partial}{\partial x_2} \end{vmatrix} f(\mathbf{x},\mathbf{y}) = \frac{\partial^2 f}{\partial y_1 \partial x_2} - \frac{\partial^2 f}{\partial y_2 \partial x_1} =$$

$$4(x_1)^3(x_2)^3(y_2)^4 - 75(x_1)^2(x_2)^4(y_1)^2(y_2)^2 - 12(x_1)^2(x_2)^4(y_1)^1(y_2)^3 + 20(x_1)(x_2)^5(y_1)^3(y_2)$$

So, $rank(\Omega f) = (10,6,4)$.

When the main diagonal of the polarization determinant of Capelli's identity operates on *f*, we get

$$(D_{\mathbf{x}^m\mathbf{x}^m} + m - 1)(D_{\mathbf{x}^{m-1}\mathbf{x}^{m-1}} + m - 2)\cdots D_{\mathbf{x}^1\mathbf{x}^1} f,$$

which equals by Euler's theorem

$$(r_m + m - 1)(r_{m-1} + m - 2)\cdots r_1 f = \rho f.$$

The ρ multiplying f is simply a constant greater than zero when $r_1 > 0$ since each of the other factors $(r_i + i - 1) \geq i - 1 > 0$ when $i > 1$. Thus ρ is greater than zero when *f* actually contains the vector $\mathbf{x}^1$. When the polarization operator determinant in Capelli's identity is expanded, any term that contains 1 or more of the diagonal factors $D_{\mathbf{x}^i\mathbf{x}^i} + i - 1$ can replace each diagonal factor with $r_i + i - 1$ again by Euler's theorem.

Example 4.1. Consider

$$D_{x^3x^m}(D_{x^{m-1}x^{m-1}}+m-2)D_{x^4x^{m-2}}\cdots D_{x^1x^3}(D_{x^2x^2}+1)D_{x^5x^1}f,$$

then we can replace the two main diagonal factors and rewrite the term as

$$(r_{m-1}+m-2)(r_2+1)D_{x^3x^m}D_{x^4x^{m-2}}\cdots D_{x^1x^3}D_{x^5x^1}f.$$

For each term of the operator determinant let's associate a $\hat{\rho}_\kappa$ to represent the product of the scalar factors described above. ($\kappa \in \{1,\ldots,m!\}$ to distinguish each $\hat{\rho}$ for each term). If there are no factors from the main diagonal in a term then $\hat{\rho}_\kappa = 1$. Then each term will be of the form

$$\hat{\rho}_\kappa D_{x^{\beta_r}x^{\alpha_r}}\cdots D_{x^{\beta_2}x^{\alpha_2}}D_{x^{\beta_1}x^{\alpha_1}}$$

where $\alpha_r > \alpha_{r-1} > \ldots > \alpha_2 > \alpha_1$, $\alpha_i \neq \beta_i$ (otherwise $D_{x^{\beta_i}x^{\alpha_i}}$ would already be included in $\hat{\rho}_\kappa$), and $\{\beta_r,\beta_{r-1},\ldots,\beta_1\}$ is a permutation of $\{\alpha_r,\alpha_{r-1},\ldots,\alpha_1\}$. Notice that since α_1 is the smallest element of $\{\alpha_r,\alpha_{r-1},\ldots,\alpha_1\}$, and $\alpha_1 \neq \beta_1$ therefore, $r \geq 2$. Also, the main diagonal is the only term in which all the factors $D_{x^ix^i}$ have the same indices. Therefore, let

$$\mathrm{P}_\kappa = D_{x^{\beta_2}x^{\alpha_2}}\cdots D_{x^{\beta_2}x^{\alpha_2}}, \text{ and } \hat{f} = -\hat{\rho}_\kappa D_{x^{\beta_1}x^{\alpha_1}}f$$

Then we can write the left side of Capelli's identity as $\rho f - \sum_\kappa \mathrm{P}_\kappa \hat{f}_\kappa$. Since

$$\hat{f}_\kappa = -\hat{\rho}_\kappa D_{x^{\beta_1}x^{\alpha_1}}f = -\hat{\rho}_\kappa \sum_{i=1}^{m} x_i^{\beta_1}\frac{\partial f}{\partial x_i^{\alpha_1}} \tag{10}$$

the degree of $\hat{f}_\kappa$ has degree one less than f in terms of $\mathbf{x}^{\alpha_1}$ and degree one greater in terms of $\mathbf{x}^{\beta_1}$. Since $\alpha_1 < \beta_1$, the difference in rank between the forms is first decided by the degree of $\mathbf{x}^{\alpha_1}$ according to our ordering. Thus, the rank of f is greater than the rank of $\hat{f}_\kappa$.

We can rewrite Capelli's identity in terms of P_κ and ρ:

$$\rho f = \sum_\kappa P_\kappa \hat{f}_\kappa \qquad \text{if } m > n$$

$$\rho f = \sum_\kappa P_\kappa \hat{f}_\kappa + [\mathbf{x}^m, \mathbf{x}^{m-1}, \ldots, \mathbf{x}^1]\Omega f \qquad \text{if } m = n$$

The four following properties of Capelli's identity are now realized:

1) $\hat{f}_\kappa$ and Ωf Ωf are of lower rank than f.

Since

$$\Omega f = \begin{vmatrix} \frac{\partial}{\partial x_1^m} & \cdots & \frac{\partial}{\partial x_1^1} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_n^m} & \cdots & \frac{\partial}{\partial x_n^1} \end{vmatrix} f,$$

the degree of each component of each vector is reduced m times. Therefore, the degree of each vector in f is reduced by m times, thus the rank of Ωf is lower than the rank of f.

2) If f is an invariant, then $\hat{f}_\kappa$ and Ωf are invariant.

Since $\hat{f}_\kappa = -\hat{\rho}_\kappa D_{\mathbf{x}^{\beta_1}\mathbf{x}^{\alpha_1}} f$, $D_{\mathbf{x}^{\beta_1}\mathbf{x}^{\alpha_1}} f$ is invariant and $-\hat{\rho}_\kappa$ is just a constant, $\hat{f}_\kappa$ is invariant.

3) P_κ is a succession of polarizations.

Thus, if $\hat{f}_\kappa$ is invariant, then $P_\kappa \hat{f}_\kappa$ is invariant.

4) If f contains $\mathbf{x}^1$ (i.e. $\mathbf{x}^1$ is not of 0 degree in f), then $\rho > 0$.

Now, let us define $\Lambda_{m,\Gamma(n)}$ to be the set of all invariant forms under subgroup $\Gamma(n)$ of *GLR(n)* depending on *m* vectors $\mathbf{x}^1,\mathbf{x}^2,...,\mathbf{x}^m \in \Re^n$. That is $f(\mathbf{x}^1,\ldots,\mathbf{x}^m) \in \Lambda_{m,\Gamma(n)}$ provided $f(\mathbf{x}^1,\ldots,\mathbf{x}^m) = f(\mathbf{A}\mathbf{x}^1,\ldots,\mathbf{A}\mathbf{x}^m)$ for every $\mathbf{A} \in \Gamma(n)$. .Let us suppose that we can choose a finite subset $\{\varphi_1,\varphi_2,...,\varphi_l\} \subset \Lambda_{m,\Gamma(n)}$ such that $\{\varphi_1,\varphi_2,...,\varphi_l\}$ form an integrity basis for $\Lambda_{m,\Gamma(n)}$. That is each $f \in \Lambda_{m,\Gamma(n)}$ is a polynomial in $\{\varphi_1,\varphi_2,...,\varphi_l\}$. When this occurs we say that *f* is expressible by the elements of $\{\varphi_1,\varphi_2,...,\varphi_l\}$ or more concisely, *f* is expressible in $\{\varphi_1,\varphi_2,...,\varphi_l\}$.

It will be convenient to formulate the notion of a **typical basic invariant**. Let

$$\phi_1^*(\mathbf{u}^1,...,\mathbf{u}^k),\phi_2^*(\mathbf{u}^1,...,\mathbf{u}^l),... \qquad (5.4)$$

be some polynomial functions depending linearly on some vectors $\mathbf{u}^1,\mathbf{u}^2,...$ (not necessarily the same number for each function). These functions $\phi_1^*,\phi_2^*,...$ are called a **complete table of typical basic invariants** for *m* arguements if (5.4) becomes an integrity basis for invariants of *m* arguements $\mathbf{x}^1,...,\mathbf{x}^m$ by substituting for $\mathbf{u}^1,\mathbf{u}^2,...$ these $\mathbf{x}^i$'s in all possible combinations (repetitions included). For example, if $\phi_1^*(\mathbf{u}^1,\mathbf{u}^2)$ is the dot product $\langle \mathbf{u}^1,\mathbf{u}^2 \rangle$, then this function will product the table of invariants for 3 arguements $\mathbf{x}^1,\mathbf{x}^2,\mathbf{x}^3$:

$$\{\langle \mathbf{x}^1,\mathbf{x}^1\rangle,\langle \mathbf{x}^2,\mathbf{x}^2\rangle,\langle \mathbf{x}^3,\mathbf{x}^3\rangle,\langle \mathbf{x}^2,\mathbf{x}^1\rangle,\langle \mathbf{x}^1,\mathbf{x}^2\rangle,\langle \mathbf{x}^3,\mathbf{x}^1\rangle,\langle \mathbf{x}^1,\mathbf{x}^3\rangle,\langle \mathbf{x}^3,\mathbf{x}^2\rangle,\langle \mathbf{x}^2,\mathbf{x}^3\rangle\},$$

which can be reduced since $\langle \mathbf{x}^j,\mathbf{x}^i\rangle = \langle \mathbf{x}^i,\mathbf{x}^j\rangle$.. Denote the set of all such substitutions in (5.4) with Φ.

Since each $\varphi \in \Phi$ is linear in its vector arguments, polariztion $D_{\mathbf{x}^i\mathbf{x}^j}\varphi$ simply replaces the $\mathbf{x}^j$ vector in φ with the $\mathbf{x}^i$ vector. Thus, $D_{\mathbf{x}^i\mathbf{x}^j}\varphi \in \Phi$. (If φ is not a function in terms of $\mathbf{x}^j$, then $D_{\mathbf{x}^i\mathbf{x}^j}\varphi = 0$, which is trivally a polynomial in the elements of Φ.)

Note that in the cases of $Sl(n, \Re)$ and $O(n)$, which will be treated in this thesis, the typical basic invariants would be

$$\phi_1^*(\mathbf{u}^1, \mathbf{u}^2) = u_1^1 u_1^2 + \ldots + u_n^1 u_n^2 \qquad \text{and} \qquad \phi_2^*(\mathbf{u}^1, \ldots, \mathbf{u}^n) = \det\left[\mathbf{u}^1, \ldots, \mathbf{u}^n\right].$$

These are indeed linear in each separate vector argument.

Now can formulate the following powerful theorem:

Theorem 4.1. *If* Φ *is an* ***integrity basis*** *for all invariant forms depending on n vectors in* $\Re^n$ *with the added condition that if* $\varphi \notin \Phi$, *then either* $D_{\mathbf{x}^j\mathbf{x}^i}\varphi \in \Phi$ *or,* $D_{\mathbf{x}^j\mathbf{x}^i}\varphi$ *can be written as a polynomial in the elements of* Φ, *then* Φ *is an* ***integrity basis*** *for all invariant forms depending on m>n vectors in* $\Re^n$. *This is true for any subgroup* $\Gamma(n)$ *of GLR(n) under which the forms are invariant.*

The proof of this theorem is by induction on rank. Let F_{n+1} be an arbitrary invariant form in *n+1* vectors. We assume for induction that every invariant form with rank less than F_{n+1} can be expressed in terms of the elements of Φ. We then show that F_{n+1} is therefore expressible in terms of Φ by Lemma 3.1. Finally, we use Capelli's general identity to show that this can be extended to a form in *m* vectors. The details follow:

Proof: First we must realize that polarization has the same formal properties differentiation:

a) $D_{\mathbf{x}^i\mathbf{x}^j}(f+g) = D_{\mathbf{x}^i\mathbf{x}^j} f + D_{\mathbf{x}^i\mathbf{x}^j} g$

Because $D_{\mathbf{x}^i\mathbf{x}^j}(f+g) = \sum_{k=1}^{n} x_k^i \frac{\partial (f+g)}{\partial x_k^j} = \sum_{k=1}^{n} (x_k^i \frac{\partial f}{\partial x_k^j} + x_k^i \frac{\partial g}{\partial x_k^j}) = D_{\mathbf{x}^i\mathbf{x}^j} f + D_{\mathbf{x}^i\mathbf{x}^j} g$

b) $D_{\mathbf{x}^i\mathbf{x}^j}(\alpha f) = \alpha D_{\mathbf{x}^i\mathbf{x}^j} f \qquad \alpha \in \Re$

Since $D_{\mathbf{x}^i\mathbf{x}^j}(\alpha f) = \sum_{k=1}^{n} x_k^i \frac{\partial (\alpha f)}{\partial x_k^j} = \sum_{k=1}^{n} \alpha x_k^i \frac{\partial f}{\partial x_k^j} = \alpha \sum_{k=1}^{n} x_k^i \frac{\partial f}{\partial x_k^j} = \alpha D_{\mathbf{x}^i\mathbf{x}^j} f$

c) $D_{\mathbf{x}^i\mathbf{x}^j}(fg) = g D_{\mathbf{x}^i\mathbf{x}^j} f + f D_{\mathbf{x}^i\mathbf{x}^j} g$

Because

$$D_{\mathbf{x}^i\mathbf{x}^j}(fg) = \sum_{k=1}^{n} x_k^i \frac{\partial (fg)}{\partial x_k^j} = \sum_{k=1}^{n} x_k^i (g \frac{\partial f}{\partial x_k^j} + f \frac{\partial g}{\partial x_k^i}) = g \sum_{k=1}^{n} x_k^i \frac{\partial f}{\partial x_k^j} + f \sum_{k=1}^{n} x_k^i \frac{\partial g}{\partial x_k^j} = g D_{\mathbf{x}^i\mathbf{x}^j} f + f D_{\mathbf{x}^i\mathbf{x}^j} g$$

The proof of the theorem is by induction on rank. Assume that Φ is an **integrity basis** for all invariant forms of *n* vectors in $\Re^n$. Let F_{n+1} be an invariant form in *n+1* vectors, say $\{\mathbf{x}^1, \ldots, \mathbf{x}^n, \mathbf{x}^{n+1}\}$ in $\Re^n$. Let r_i represent the degree of $\mathbf{x}^i$ in F_{n+1}, and let $r_{i,\kappa}^*$ represent the degree of $\mathbf{x}^i$ in F_κ^*, where $F_\kappa^* = -\hat{\rho}_\kappa D_{\mathbf{x}^{\beta_1}\mathbf{x}^{\alpha_1}} F_{n+1}$. By the fact that P_κ is a successions of polarizations, the above formal properties of polar operators, and the first hypothesis of the theorem, if F_κ^*, is expressible in Φ, then $\rho F_{n+1} = \sum_\kappa P_\kappa F_\kappa^*$ is expressible in Φ provided $\rho > 0$ (i.e. $r_1 > 0$).

Base case: The lowest rank that a form *f* can have in *n*+1 vectors is (*n*+1,1,1,...,1). Thus, the rank of $\hat{f}_\kappa = -\hat{\rho}_\kappa D_{\mathbf{x}^{\beta_1}\mathbf{x}^{\alpha_1}} f$ is (*n*+1,1,..,0,...,2,...,1). So $\hat{f}_\kappa$ is a polynomial in *n* vectors and thus by hypothesis can be written as a polynomial in Φ. Also, the rank of $P_\kappa \hat{f}_\kappa$ is less than $\hat{f}_\kappa$ thus $\mathrm{P}_\kappa \hat{f}_\kappa$ is a polynomial in n vectors and can be expressed by Φ. So by the Capelli identity, f can be written as a polynomial in Φ.

Induction step:

Suppose all invariant forms with rank less than F_{n+1} can be expressed in terms of elements of Φ. Each $F_\kappa^* = -\hat{\rho}_\kappa D_{\mathbf{x}^{\beta_1}\mathbf{x}^{\alpha_1}} F_{n+1}$ is an invariant form with rank less than F_{n+1}. So, each F_κ^* can be written in terms of the elements of Φ. Therefore, as stated earlier, this implies that F_{n+1} can be written in terms of. Thus, this is true for any invariant form with finite rank by induction.

This proof can be extended easily to m vectors by induction again. The fact that Φ is an integrity basis for F_{n+k} implies that Φ is an integrity basis for F_{n+k+1} since $n+k+1>n$ and therefore Capelli's general identity can still be used. **Q.E.D.**

It will now be shown that we can do one better provided the determinant of the matrix of the vectors (i.e., $[\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^n]$) is either an element or expressible by the elements of Φ.

Theorem 4.2. *If the determinant $[\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^n] \in \Phi'$ or at is a polynomial in Φ', and if Φ' is an integrity basis for all invariant forms in terms of n-1 vectors in $\Re^n$, then Φ' is an integrity basis for all invariant form of n vectors. This also true for any subgroup $\Gamma(n)$ of GLR(n) under which the forms are invariant.*

The proof is similar to the proof of Theorem 4.1 in that it is by induction on rank. Let F_n be an invariant form in n vectors. Assume for induction that every invariant form with rank less than F_n can be expressed in Φ'. Then by polarization property 1, ΩF_n has rank less than F_n. Also, by polarization property 2, ΩF_n is an invariant form. Therefore, ΩF_n is expressible in Φ'. However, this time since *n=m*, Capelli's special identity will be used. That is $\rho F_n = \sum_{\kappa} \mathrm{P}_\kappa \hat{F}_\kappa + [\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^n]\Omega F_n$. Therefore, F_n is expressible in Φ'. The details of the proof are given below.

Proof: Again, we need $\rho > 0$. However, if $\rho = 0$, then $r_1 = 0$, and thus, the degree of $\mathbf{x}^1$ is zero. Thus, F_n is a form in the vectors $\{\mathbf{x}^2, \ldots, \mathbf{x}^{n-1}, \mathbf{x}^n\}$. So, by hypothesis, F_n is an invariant form expressible in Φ'. So we can assume that $\rho > 0$. It will suffice to show that each $\hat{F}_\kappa$ is expressible in Φ' as before and to show ΩF_n is expressible in Φ'. By hypothesis $[\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^n] \in \Phi'$, therefore $[\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^n]\Omega F_n$ is expressible in Φ'. The theorem will now be proved by induction on rank.

Base case: Suppose f has rank $(r, r_1, \ldots, 1, \ldots, r_n)$. Then Ωf has rank $(r-n, r_1-1, \ldots, 0, \ldots, r_n-1)$. Thus Ωf is a polynomial in *n-1* vectors and can be expressed as a polynomial in Φ'. The rest follows from the proof of theorem 3.

Induction step: Suppose that every invariant form that has rank less than F_n is expressible in Φ'. By property **1**, each $\hat{F}_\kappa$ has rank less than F_n, and ΩF_n has rank less than F_n. Therefore, ΩF_n and each $\hat{F}_\kappa$ is expressible in Φ'. Thus, by Capelli's identity, F_n is expressible in Φ'. So by induction, all invariant forms in *n* vectors in $\Re^n$ can be expressed by the elements of Φ'. **Q.E.D.**

CHAPTER V

THE UNIMODULAR GROUP *SL(n)*

SL(n) is the collection of all $n \times n$ linear transformations with determinant 1. *SL(n)* is easily verified as a group with matrix multiplication as its group operation. For if $\mathbf{A}, \mathbf{B} \in SL(n)$, then $det(\mathbf{AB}) = (det(\mathbf{A}))(det(\mathbf{B})) = (1)(1) = 1$. Thus, $\mathbf{AB} \in SL(n)$. Also, the identity matrix, $\mathbf{I}_n$, has determinant 1. Since $det(\mathbf{A}) = 1$, then $det(\mathbf{A}^{-1}) = 1$. So $\mathbf{A}^{-1} \in SL(n)$. Therefore, *SL(n)* is a group.

Our goal is to find a finite table of invariant forms (basic invariants) under *SL(n)* such that every invariant form (under *SL(n)*) can be generated as a polynomial in the entries of the table. In order to do this, we need two types of vectors: 1) the **covariant** or column vector and 2) the **contravariant** or row vector.

Let **x** be a covariant vector, and let **y** be a contravariant vector. If $f(\mathbf{x},\mathbf{y})$ is invariant under *SL(n)*, then $f(\mathbf{x},\mathbf{y}) = f(\mathbf{Ax},\mathbf{yA}^{-1})$ given $\mathbf{A} \in SL(n)$. The importance of these two types of vectors is shown through example:

Define $h(\mathbf{x},\mathbf{y}) = x_1y_1 + x_2y_2$ (i.e. dot product). We will show later that $h(\mathbf{x},\mathbf{y})$ is invariant under *SL(2)*. $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \in SL(2)$, and thus, $\mathbf{A}^{-1} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}$. So

$$\mathbf{Ax} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 3x_2 \\ x_1 + 4x_2 \end{bmatrix}, \text{ and}$$

$$\mathbf{yA}^{-1} = \begin{bmatrix} y_1 & y_2 \end{bmatrix}\begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} (4y_1 - y_2) & (-3y_1 + y_2) \end{bmatrix}$$

So, we have

$$h(\mathbf{Ax},\mathbf{yA}^{-1}) = (x_1+3x_2)(4y_1-y_2)+(x_1+4x_2)(-3y_1+y_2) =$$

$$4x_1y_1+12x_2y_1-x_1y_2-3x_2y_2-3x_1y_1-12x_2y_1+x_1y_2+4x_2y_2 =$$

$$x_1y_1+x_2y_2 = h(\mathbf{x},\mathbf{y}).$$

However, if we let **z** be a covariant vector and define $\tilde{h}(\mathbf{x},\mathbf{z}) = x_1z_1+x_2z_2$, then

$$\tilde{h}(\mathbf{Ax},\mathbf{Az}) = (x_1+3x_2)(z_1+3z_2)+(x_1+4x_2)(z_1+4z_2) =$$

$$x_1z_1+3x_2z_1+3x_1z_2+9x_2z_2+x_1z_1+4x_2z_1+4x_1z_2+x_2z_2 =$$

$$x_1z_1+7x_2z_1+7x_1z_2+10x_2z_2 \neq x_1z_1+x_2z_2 = \tilde{h}(\mathbf{x},\mathbf{z}).$$

Thus, $\tilde{h}(\mathbf{x},\mathbf{z})$ is not an invariant form under *SL(2)*.

For a second type of example define $g(\mathbf{x},\mathbf{z}) = x_1z_2-x_2z_1$ (i.e., determinant). With **Ax**, and **Az** the same as above, we have

$$g(\mathbf{Ax},\mathbf{Az}) = (x_1+3x_2)(z_1+4z_2)-(x_1+4x_2)(z_1+3z_2) =$$

$$x_1z_1+3x_2z_1+4x_1z_2+12x_2z_2-x_1z_1-4x_2z_1-3x_1z_2-12x_2z_1 =$$

$$x_1z_2-x_2z_1 = g(\mathbf{x},\mathbf{z}).$$

If we let **w** be a contravariant vector and define $\tilde{g}(\mathbf{y},\mathbf{w}) = y_1w_2-y_2w_1$, we get a similar result:

$$\tilde{g}(\mathbf{yA}^{-1},\mathbf{wA}^{-1}) = (4y_1-y_2)(-3w_1+w_2)-(-3y_1+y_2)(4w_1-w_2) =$$

$$-12y_1w_1+3y_2w_1+4y_1w_2-y_2w_2+12y_1w_1-4y_2w_1-3y_1w_2+y_2w_2 =$$

$$y_1w_2-y_2w_1 = \tilde{g}(\mathbf{y},\mathbf{w})$$

We will see later that $g(\mathbf{x},\mathbf{z})$ and $\tilde{g}(\mathbf{y},\mathbf{w})$ are both in fact invariant under *SL(2)*. Let the letter **x** represent any covariant vector and the letter **y** represent any contravariant vector.

Therefore, the dot product **xy** is clearly invariant under any group of nonsingular linear transformations since $(\mathbf{yA}^{-1})(\mathbf{Ax}) = \mathbf{yA}^{-1}\mathbf{Ax} = \mathbf{yIx} = \mathbf{yx}$.

Let the symbol

$$\Theta = \begin{vmatrix} \mathbf{y}^1\mathbf{x}^1 & \cdots & \mathbf{y}^1\mathbf{x}^{n-1} \\ \vdots & \ddots & \vdots \\ \mathbf{y}^{n-1}\mathbf{x}^1 & \cdots & \mathbf{y}^{n-1}\mathbf{x}^{n-1} \end{vmatrix}.$$

Lemma 5.1. *If $\Theta \neq 0$, we can introduce a new coordinate system by a unimodular transformation such that the covariant vectors $\mathbf{x}^1,\ldots,\mathbf{x}^{n-1}$ are replaced by the first n-1 basic vectors $\mathbf{e}^1,\ldots,\mathbf{e}^{n-1}$, and the n^{th} component of each of the n-1 contravariant vectors $\mathbf{y}^1,\ldots,\mathbf{y}^{n-1}$ is zero.*

That is if before the transformation we have

$$\mathbf{x}^1 = \begin{bmatrix} x_1^1 \\ \vdots \\ x_{n-1}^1 \\ x_n^1 \end{bmatrix},\ldots,\mathbf{x}^{n-1} = \begin{bmatrix} x_1^{n-1} \\ \vdots \\ x_{n-1}^{n-1} \\ x_n^{n-1} \end{bmatrix}, \text{ and}$$

$$\mathbf{y}^1 = \begin{bmatrix} y_1^1 & \cdots & y_{n-1}^1 & y_n^1 \end{bmatrix},\ldots,\mathbf{y}^{n-1} = \begin{bmatrix} y_1^{n-1} & \cdots & y_{n-1}^{n-1} & y_n^{n-1} \end{bmatrix},$$

then after the unimodular transformation we have

$$\mathbf{x}^1 = \begin{bmatrix} 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix},\ldots,\mathbf{x}^{n-1} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}, \text{ and}$$

$$\mathbf{y}^1 = \begin{bmatrix} \bar{y}_1^1 & \cdots & \bar{y}_{n-1}^1 & 0 \end{bmatrix},\ldots,\mathbf{y}^{n-1} = \begin{bmatrix} \bar{y}_1^{n-1} & \cdots & \bar{y}_{n-1}^{n-1} & 0 \end{bmatrix},$$

where $\bar{y}_j^i$ is the new value for y_j^i from the transformation.

Proof. First, since $\Theta \neq 0$,

$$\Theta = \det\begin{bmatrix} \mathbf{y}^1\mathbf{x}^1 & \cdots & \mathbf{y}^1\mathbf{x}^{n-1} \\ \vdots & \ddots & \vdots \\ \mathbf{y}^{n-1}\mathbf{x}^1 & \cdots & \mathbf{y}^{n-1}\mathbf{x}^{n-1} \end{bmatrix} = \det\left(\begin{bmatrix} y_1^1 & \cdots & y_{n-1}^1 \\ \vdots & \ddots & \vdots \\ y_1^{n-1} & \cdots & y_{n-1}^{n-1} \end{bmatrix}\begin{bmatrix} x_1^1 & \cdots & x_1^{n-1} \\ \vdots & \ddots & \vdots \\ x_{n-1}^1 & \cdots & x_{n-1}^{n-1} \end{bmatrix}\right) =$$

$$\det\begin{bmatrix} y_1^1 & \cdots & y_{n-1}^1 \\ \vdots & \ddots & \vdots \\ y_1^{n-1} & \cdots & y_{n-1}^{n-1} \end{bmatrix}\det\begin{bmatrix} x_1^1 & \cdots & x_1^{n-1} \\ \vdots & \ddots & \vdots \\ x_{n-1}^1 & \cdots & x_{n-1}^{n-1} \end{bmatrix} \neq 0.$$

Thus,

$$\det\begin{bmatrix} x_1^1 & \cdots & x_1^{n-1} \\ \vdots & \ddots & \vdots \\ x_{n-1}^1 & \cdots & x_{n-1}^{n-1} \end{bmatrix} \neq 0 \quad \text{and} \quad \det\begin{bmatrix} y_1^1 & \cdots & y_{n-1}^1 \\ \vdots & \ddots & \vdots \\ y_1^{n-1} & \cdots & y_{n-1}^{n-1} \end{bmatrix} \neq 0.$$

Therefore, both { $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^{n-1}$ } and { $\mathbf{y}^1, \mathbf{y},^2 \ldots, \mathbf{y}^{n-1}$ } are linearly independent. Now a new coordinate system, $\{\overline{\mathbf{e}}_1, \overline{\mathbf{e}}_2, \ldots, \overline{\mathbf{e}}_n\}$ can be created by:

$$\begin{aligned} \overline{\mathbf{e}}^1 &= x_1^1\mathbf{e}^1 + \ldots + x_n^1\mathbf{e}^n \\ \vdots \quad & \quad \vdots \qquad\quad \vdots \\ \overline{\mathbf{e}}^{n-1} &= x_1^{n-1}\mathbf{e}^1 + \ldots + x_n^{n-1}\mathbf{e}^n \\ \overline{\mathbf{e}}^n &= z_1\mathbf{e}^1 + \ldots + z_n\mathbf{e}^n \end{aligned}$$

Thus each $\overline{\mathbf{e}}^i$ is exactly $\mathbf{x}^i$ for $i \in \{1, \ldots, n-1\}$ and $\overline{\mathbf{e}}^n$ is exactly some vector $\mathbf{z}$ that must be found.We want our transformation $\mathbf{A}$ to be the matrix

$$\begin{bmatrix} x_1^1 & \cdots & x_1^{n-1} & z_1 \\ x_2^1 & \cdots & x_2^{n-1} & z_2 \\ \vdots & \ddots & \vdots & \vdots \\ x_n^1 & \cdots & x_n^{n-1} & z_n \end{bmatrix}.$$

In order to be unimodular we need the condition that $det(\mathbf{A}) = 1$.

Let

$$M_i = \begin{vmatrix} x_1^1 & \cdots & x_1^{n-1} \\ \vdots & \ddots & \vdots \\ x_{i-1}^1 & \cdots & x_{i-1}^{n-1} \\ x_{i+1}^1 & \cdots & x_{i+1}^{n-1} \\ \vdots & \ddots & \vdots \\ x_n^1 & \cdots & x_n^{n-1} \end{vmatrix},$$

then we need $det(\mathbf{A}) = \sum_{k=1}^{n}(-1)^{k+1}M_k z_k = 1$. The contravariant vectors transform

$\overline{\mathbf{y}}^i = \mathbf{y}^i\mathbf{A}$. So to find each component:

$$\begin{aligned} \overline{y}_1^i &= x_1^1 y_1^i + \ldots + x_n^1 y_n^i \\ &\vdots \\ \overline{y}_{n-1}^i &= x_1^{n-1} y_1^i + \ldots + x_n^{n-1} y_n^i \\ \overline{y}_n^i &= z_1 y_1^i + \ldots + z_n y_n^i \end{aligned}$$

Now to satisfy the lemma, we need the n^{th} component of the vectors $\{\overline{\mathbf{y}}^1, \overline{\mathbf{y}}^2, \ldots, \overline{\mathbf{y}}^{n-1}\}$ to be zero. In other words we need $\overline{y}_n^i = z_1 y_1^i + \ldots + z_n y_n^i = 0$ for $i \in \{1, \ldots, n-1\}$. Thus, we have system of n equations for the n variables $\{z_1, z_2, \ldots, z_n\}$:

$$\mathbf{Bz} = \begin{bmatrix} y_1^1 & y_2^1 & \cdots & y_n^1 \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{n-1} & y_2^{n-1} & \cdots & y_n^{n-1} \\ M_1 & -M_2 & \cdots & (-1)^{n+1}M_n \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_{n-1} \\ z_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Thus, we can find a unique $\mathbf{z}$ if $det(\mathbf{B}) \neq 0$. Since it was shown earlier that $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^{n-1}\}$ are linearly independent in $\Re^n$, we can find a vector $\mathbf{z}'$ in $\Re^n$ that is linearly independent of $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^{n-1}\}$. Thus,

$$\begin{vmatrix} x_1^1 & \cdots & x_1^{n-1} & z_1' \\ x_2^1 & \cdots & x_2^{n-1} & z_2' \\ \vdots & \ddots & \vdots & \vdots \\ x_n^1 & \cdots & x_n^{n-1} & z_n' \end{vmatrix} \neq 0 .$$

Notice that

$$x_1^i M_1 - x_2^i M_2 + \ldots + (-1)^{n+1} x_n^i M_n = \begin{vmatrix} x_1^i & x_1^1 & \cdots & x_1^i & \cdots & x_1^{n-1} \\ x_2^i & x_2^1 & \cdots & x_2^i & \cdots & x_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ x_n^i & x_n^1 & \cdots & x_n^i & \cdots & x_n^{n-1} \end{vmatrix} = 0 .$$

Let

$$a = z_1' M_1 - z_2' M_2 + \ldots + (-1)^{n+1} z_n' M_n = \begin{vmatrix} z_1' & x_1^1 & \cdots & x_1^{n-1} \\ z_2' & x_2^1 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ z_n' & x_n^1 & \cdots & x_n^{n-1} \end{vmatrix} \neq 0 .$$

So

$$\begin{vmatrix} y_1^1 & y_2^1 & \cdots & y_n^1 \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{n-1} & y_2^{n-1} & \cdots & y_n^{n-1} \\ M_1 & -M_2 & \cdots & (-1)^{n+1} M_n \end{vmatrix} \begin{vmatrix} x_1^1 & \cdots & x_1^{n-1} & z_1' \\ x_2^1 & \cdots & x_2^{n-1} & z_2' \\ \vdots & \ddots & \vdots & \vdots \\ x_n^1 & \cdots & x_n^{n-1} & z_n' \end{vmatrix}$$

$$= \begin{vmatrix} \mathbf{y}^1\mathbf{x}^1 & \cdots & \mathbf{y}^1\mathbf{x}^{n-1} & \mathbf{y}^1\mathbf{z}' \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{y}^{n-1}\mathbf{x}^1 & \cdots & \mathbf{y}^{n-1}\mathbf{x}^{n-1} & \mathbf{y}^{n-1}\mathbf{z}' \\ 0 & \cdots & 0 & a \end{vmatrix} = a\Theta \neq 0.$$

Therefore, $det(\mathbf{B}) \neq 0$, so $\mathbf{z}$ is uniquely determined. **Q.E.D.**

If $\mathbf{z}'$ is replaced by $\mathbf{z}$ then, by the construction, we get $\mathbf{y}^i\mathbf{z} = 0$ for all $i \in (1,\ldots,n-1\}$ and $a = 1$. So $det(\mathbf{B}) = \Theta$, and if we let

$$N_i = \begin{vmatrix} y_1^1 & \cdots & y_{i-1}^1 & y_{i+1}^1 & \cdots & y_n^1 \\ y_1^2 & \cdots & y_{i-1}^2 & y_{i+1}^2 & \cdots & y_n^2 \\ \vdots & & \vdots & \vdots & & \vdots \\ y_1^{n-1} & \cdots & y_{i-1}^{n-1} & y_{i+1}^{n-1} & \cdots & y_n^{n-1} \end{vmatrix},$$

then

$$z_i = \frac{(-1)^{n+1} N_i}{\Theta}.$$

Example 5.1. Let

$$\mathbf{y}^1 = [2 \quad 0 \quad 1],\ \mathbf{y}^2 = [1 \quad 0 \quad 2],\ \mathbf{x}^1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \text{ and } \mathbf{x}^2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

Then

$$\Theta = \begin{vmatrix} \mathbf{y}^1\mathbf{x}^1 & \mathbf{y}^1\mathbf{x}^2 \\ \mathbf{y}^2\mathbf{x}^1 & \mathbf{y}^2\mathbf{x}^2 \end{vmatrix} = \begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix} = 8 - 5 = 3.\ N_1 = \begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} = 0,\ N_2 = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3, \text{ and}$$

$$N_3 = \begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix} = 0. \text{ So let } z_1 = \frac{(-1)^2 N_1}{\Theta} = \frac{0}{3} = 0,\ z_2 = \frac{(-1)^3 N_2}{\Theta} = \frac{-3}{3} = -1, \text{ and}$$

$$z_3 = \frac{(-1)^4 N_3}{\Theta} = \frac{0}{3} = 0.$$

Thus,

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \text{ and } det(\mathbf{A}) = 1.$$

So, we get

$$\bar{\mathbf{e}}^1 = \mathbf{A}\mathbf{e}^1 = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \mathbf{x}^1$$

$$\overline{\mathbf{e}}^2 = \mathbf{A}\mathbf{e}^2 = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \mathbf{x}^2$$

$$\overline{\mathbf{y}}^1 = \mathbf{y}^1\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 0 \end{bmatrix}$$

$$\overline{\mathbf{y}}^2 = \mathbf{y}^2\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 \end{bmatrix}.$$

Dot product is an obvious invariant, since by definition:

$$(\mathbf{y}\mathbf{A}^{-1})(\mathbf{A}\mathbf{x}) = \mathbf{y}\mathbf{A}^{-1}\mathbf{A}\mathbf{x} = \mathbf{y}\mathbf{I}\mathbf{x} = \mathbf{y}\mathbf{x}.$$

So the only nontrivial invariant under *SL(n)* is the determinant. Since the determinant of some $n \times n$ matrix **B** is the *n* dimensional volume of the parallelpiped formed by the vectors in the rows of **B**, the determinant has geometric meaning. Therefore, it should be invariant under *Sl(n)*. Thus, we are ready to state our main theorem about the group *SL(n)*.

Theorem 5.2.

$$\begin{vmatrix} x_1^1 & \cdots & x_1^n \\ \vdots & \ddots & \vdots \\ x_n^1 & \cdots & x_n^n \end{vmatrix}, \quad \begin{vmatrix} y_1^1 & \cdots & y_n^1 \\ \vdots & \ddots & \vdots \\ y_1^n & \cdots & y_n^n \end{vmatrix}, \quad \textit{and } \mathbf{y}^i\mathbf{x}^j$$

is the complete table of basic invariants for the SL(n) group.

Proof. By theorems 4.1 and 4.2, it suffices to show that these form a complete table for *n*-1 covariant and *n*-1 contravariant vectors in $\Re^n$. First, we need to show that

$$D_{\mathbf{x}^i\mathbf{x}^j}\begin{vmatrix} x_1^1 & \cdots & x_1^n \\ \vdots & \ddots & \vdots \\ x_n^1 & \cdots & x_n^n \end{vmatrix}, \; D_{\mathbf{y}^i\mathbf{y}^j}\begin{vmatrix} y_1^1 & \cdots & y_n^1 \\ \vdots & \ddots & \vdots \\ y_1^n & \cdots & y_n^n \end{vmatrix}, \; D_{\mathbf{x}^i\mathbf{x}^j}\mathbf{y}^\alpha\mathbf{x}^\beta, \text{ and } D_{\mathbf{y}^i\mathbf{y}^j}\mathbf{y}^\alpha\mathbf{x}^\beta$$

are all in the table. First for $i \neq j$,

$$D_{\mathbf{x}^i\mathbf{x}^j}\begin{vmatrix} x_1^1 & \cdots & x_1^n \\ \vdots & \ddots & \vdots \\ x_n^1 & \cdots & x_n^n \end{vmatrix} = D_{\mathbf{x}^i\mathbf{x}^j}\sum_{\Psi}\lambda x_{\sigma(1)}^1 \cdot\cdot x_{\sigma(i)}^i \cdot x_{\sigma(j)}^j \cdot\cdot x_{\sigma(n)}^n =$$

$$\sum_{k=1}^{n} x_k^i \frac{\partial}{\partial x_k^j}\sum_{\Psi}\lambda x_{\sigma(1)}^1 \cdot\cdot x_{\sigma(i)}^i \cdot x_{\sigma(j)}^j \cdot\cdot x_{\sigma(n)}^n =$$

$$\sum_{\Psi}\lambda x_{\sigma(1)}^1 \cdot\cdot x_{\sigma(i)}^i \cdot x_{\sigma(j)}^i \cdot\cdot x_{\sigma(n)}^n = \begin{vmatrix} x_1^1 & \cdots & x_1^i & \cdots & x_1^i & \cdots & x_1^n \\ \vdots & & \vdots & & \vdots & & \vdots \\ x_n^1 & \cdots & x_n^i & \cdots & x_n^i & \cdots & x_n^n \end{vmatrix} = 0$$

which is trivially in the table. ,If $i = j$, then by Euler's theorem,

$$D_{\mathbf{x}^i\mathbf{x}^i}\begin{vmatrix} x_1^1 & \cdots & x_1^n \\ \vdots & \ddots & \vdots \\ x_n^1 & \cdots & x_n^n \end{vmatrix} = \begin{vmatrix} x_1^1 & \cdots & x_1^n \\ \vdots & \ddots & \vdots \\ x_n^1 & \cdots & x_n^n \end{vmatrix}.$$

The result is entirely similar for

$$D_{\mathbf{y}^i\mathbf{y}^j}\begin{vmatrix} y_1^1 & \cdots & y_n^1 \\ \vdots & \ddots & \vdots \\ y_1^n & \cdots & y_n^n \end{vmatrix}.$$

Again, for $i \neq j$, and $\beta = j$

$$D_{\mathbf{x}^i\mathbf{x}^j}\mathbf{y}^\alpha\mathbf{x}^j = \sum_{k=1}^{n} x_k^i \frac{\partial}{\partial x_k^j}(y_1^\alpha x_1^j + .. + y_k^\alpha x_k^j + .. + y_n^\alpha x_n^j) = \sum_{k=1}^{n} y_k^\alpha x_k^i = \mathbf{y}^\alpha\mathbf{x}^i$$

which is clearly in the table. If $i = j$, and $\beta = j$, then $D_{\mathbf{x}^j\mathbf{x}^j}\mathbf{y}^\alpha\mathbf{x}^j = \mathbf{y}^\alpha\mathbf{x}^j$, by Euler's theorem. If $\beta \neq j$, then $D_{\mathbf{x}^i\mathbf{x}^j}\mathbf{y}^\alpha\mathbf{x}^\beta = 0$. Again, the result is similar for $D_{\mathbf{y}^i\mathbf{y}^j}\mathbf{y}^\alpha\mathbf{x}^\beta$.

Since

$$\begin{vmatrix} x_1^1 & \cdots & x_1^n \\ \vdots & \ddots & \vdots \\ x_n^1 & \cdots & x_n^n \end{vmatrix}$$

is in our table, by Theorem 4.2, we have a complete table for n covariant vectors, and since

$$\begin{vmatrix} y_1^1 & \cdots & y_n^1 \\ \vdots & \ddots & \vdots \\ y_1^n & \cdots & y_n^n \end{vmatrix}$$

is in our table, again by Theorem 4.2, we have a complete table for n contravariant vectors. Thus, by Theorem 4.1, we have a complete table for any $m>n$ covariant and contravariant vectors. Since

$$\begin{vmatrix} x_1^1 & \cdots & x_1^{n-1} \\ \vdots & \ddots & \vdots \\ x_n^1 & \cdots & x_n^{n-1} \end{vmatrix}, \text{ and } \begin{vmatrix} y_1^1 & \cdots & y_n^1 \\ \vdots & \ddots & \vdots \\ y_1^{n-1} & \cdots & y_n^{n-1} \end{vmatrix}$$

do not make sense (determinants of non-square matrices), we need to show only that each invariant form dependent on those n-1 covariant and contravariant vectors can be written in terms of the (n-1) dot products $\mathbf{y}^i\mathbf{x}^j$. From the Lemma 5.1, and the fact that dot product is invariant, we have

$$\mathbf{y}^i\mathbf{x}^j = \overline{\mathbf{y}}^i\mathbf{e}^j = \overline{y}_j^i. \tag{10}$$

Let $f(\mathbf{x}^1,...,\mathbf{x}^{n-1},\mathbf{y}^1,...,\mathbf{y}^{n-1})$ be an invariant form under *SL(n)* such that

$\{\mathbf{x}^1,...,\mathbf{x}^{n-1},\mathbf{y}^1,...,\mathbf{y}^{n-1}\} = \{\mathbf{X},\mathbf{Y}\}$ satisfy the hypothesis of the Lemma 5.1.

Then

$$f(\mathbf{X},\mathbf{Y}) = f(\mathbf{A}^{-1}\mathbf{X},\mathbf{YA}) = f(\mathbf{e}^1,...,\mathbf{e}^{n-1},\overline{\mathbf{y}}^1,...,\overline{\mathbf{y}}^{n-1}).$$

Since each $\mathbf{e}^i$ is fixed, the last *f* actually only depends on the vectors $\{\overline{\mathbf{y}}^1,...,\overline{\mathbf{y}}^{n-1}\}$. Thus.

$$f(\mathbf{x}^1,...,\mathbf{x}^{n-1},\mathbf{y}^1,...,\mathbf{y}^{n-1}) = \bar{f}(\overline{\mathbf{y}}^1,...,\overline{\mathbf{y}}^{n-1}) \qquad (11)$$

Let $\mathcal{P}(\{\overline{y}_j^i\})$ be the polynomial in the $(n-1)^2$ variables $\{\overline{y}_j^i\}$ such that

$$\mathcal{P}(\{\overline{y}_j^i\}) = \bar{f}(\overline{\mathbf{y}}^1,...,\overline{\mathbf{y}}^{n-1}).$$

It follows from (10) and (11) that

$$\mathcal{P}(\{\mathbf{y}^i\mathbf{x}^j\}) = f(\mathbf{x}^1,...,\mathbf{x}^{n-1},\mathbf{y}^1,...,\mathbf{y}^{n-1}),$$

which holds whenever $\Theta \neq 0$, and thus is an identity by the principle of algebraic irrelevance. So, every invariant form under *SL(n)* that depends on *n*-1 covariant vectors and *n*-1 contravariant vectors can be written as a polynomial of the dot product of those vectors. **Q.E.D.**

CHAPTER VI

THE ORTHOGONAL GROUP *O(n)*

A linear transformation **A** is **orthogonal** if and only if $\mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{I}$. An orthogonal transformation **A** is **proper** if *det*(**A**) = 1, and **improper** if *det*(**A**) = -1. Let *O(n)* be the collection of all proper and improper $n \times n$ orthogonal transformations. It is easy to verify that *O(n)* is a group with matrix multiplication as its group operator. Let $\mathbf{I}_n$ be the identity matrix. Obviously, $\mathbf{I}_n(\mathbf{I}_n)^{\mathrm{T}} = \mathbf{I}_n\mathbf{I}_n = \mathbf{I}_n$, so $\mathbf{I}_n \in O(n)$. Let **A, B**$\in$ *O(n)*, then $\mathbf{AB}(\mathbf{AB})^{\mathrm{T}} = \mathbf{ABB}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} = \mathbf{AI}_n\mathbf{A}^{\mathrm{T}} = \mathbf{AA}^{\mathrm{T}} = \mathbf{I}_n$, so **AB**$\in$ *O(n)*. Clearly, $\mathbf{A}^{\mathrm{T}} = \mathbf{A}^{-1}$, thus $\mathbf{A}^{-1}(\mathbf{A}^{-1})^{\mathrm{T}} = \mathbf{A}^{-1}(\mathbf{A}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n$, so $\mathbf{A}^{-1} \in O(n)$. Notice that this holds for both proper and improper orthogonal transformation since *det*(**A**) = *det*($\mathbf{A}^{\mathrm{T}}$). Thus, even if **A** is improper, *det*($\mathbf{AA}^{\mathrm{T}}$) = *det*(**A**)*det*($\mathbf{A}^{\mathrm{T}}$) = (-1)(-1) = 1= *det*($\mathbf{I}_n$)

We are going to define an invariant form as **even** if $f(\mathbf{x}) = f(\mathbf{Ax})$, and **odd** if $f(\mathbf{x}) = \det(\mathbf{A})f(\mathbf{Ax})$, where $\det(\mathbf{A}) = \pm 1$. The determinant $\left[\mathbf{x}^1 \ \cdots \ \mathbf{x}^n\right]$ is an odd invariant.

$$\left[\mathbf{A}\mathbf{x}_1 \ \cdots \ \mathbf{A}\mathbf{x}_n\right] = \det\left(\mathbf{A}\begin{bmatrix} x_1^1 & \cdots & x_1^n \\ \vdots & \ddots & \vdots \\ x_n^1 & \cdots & x_n^n \end{bmatrix}\right) =$$

$$\det(\mathbf{A})\det\left(\begin{bmatrix} x_1^1 & \cdots & x_1^n \\ \vdots & \ddots & \vdots \\ x_n^1 & \cdots & x_n^n \end{bmatrix}\right) = \det(\mathbf{A})\left[\mathbf{x}^1 \ \cdots \ \mathbf{x}^n\right].$$

Since $\det(\mathbf{A}) = \pm 1$, $\det(\mathbf{A}) = (\det(\mathbf{A}))^{-1}$. Thus

$$\begin{bmatrix}\mathbf{x}^1 & \cdots & \mathbf{x}^n\end{bmatrix} = \det(\mathbf{A})\begin{bmatrix}\mathbf{A}\mathbf{x}^1 & \cdots & \mathbf{A}\mathbf{x}^n\end{bmatrix}.$$

By the Capelli special identity, if $f(\mathbf{x}^1,\ldots,\mathbf{x}^n)$ is an odd invariant then

$$\Omega f(\mathbf{A}\mathbf{x}^1,\ldots,\mathbf{A}\mathbf{x}^n) = \left(\begin{bmatrix}\mathbf{A}\mathbf{x}^1 & \cdots & \mathbf{A}\mathbf{x}^n\end{bmatrix}\right)^{-1}\begin{vmatrix} D_{\mathbf{A}\mathbf{x}^n\mathbf{A}\mathbf{x}^n}+n-1 & \cdots & D_{\mathbf{A}\mathbf{x}^n\mathbf{A}\mathbf{x}^1} \\ \vdots & \ddots & \vdots \\ D_{\mathbf{A}\mathbf{x}^1\mathbf{A}\mathbf{x}^n} & \cdots & D_{\mathbf{A}\mathbf{x}^1\mathbf{A}\mathbf{x}^1}\end{vmatrix} f(\mathbf{A}\mathbf{x}^1,\ldots,\mathbf{A}\mathbf{x}^n) =$$

$$\det(\mathbf{A})\left(\begin{bmatrix}\mathbf{x}^1 & \cdots & \mathbf{x}^1\end{bmatrix}\right)^{-1}\begin{vmatrix} D_{\mathbf{x}^n\mathbf{x}^n}+n-1 & \cdots & D_{\mathbf{x}^n\mathbf{x}^1} \\ \vdots & \ddots & \vdots \\ D_{\mathbf{x}^1\mathbf{x}^n} & \cdots & D_{\mathbf{x}^1\mathbf{x}^1}\end{vmatrix}(\det(\mathbf{A}))f(\mathbf{x}^1,\ldots,\mathbf{x}^n) =$$

$$(\det(\mathbf{A}))^2\,\Omega f(\mathbf{x}^1,\ldots,\mathbf{x}^n) = \Omega f(\mathbf{x}^1,\ldots,\mathbf{x}^n),$$

thus Ωf is an even invariant. However, if $h(\mathbf{x}^1,\ldots,\mathbf{x}^n)$ is an even invariant the second $\det(\mathbf{A})$ will not appear in above and thus, $\Omega h(\mathbf{A}\mathbf{x}^1,\ldots,\mathbf{A}\mathbf{x}^n) = \det(\mathbf{A})\Omega h(\mathbf{x}^1,\ldots,\mathbf{x}^n)$. Since $\det(\mathbf{A}) = (\det(\mathbf{A}))^{-1}$, we have $\Omega h(\mathbf{x}^1,\ldots,\mathbf{x}^n) = (\det(\mathbf{A}))h(\mathbf{A}\mathbf{x}^1,\ldots,\mathbf{A}\mathbf{x}^n)$. So Ωh is an odd invariant. If $\mathbf{y}$ is an contravariant vector, then $\mathbf{y}^{\mathrm{T}}$ is a covariant vector. Since $(\mathbf{y}\mathbf{A}^{-1})^{\mathrm{T}} = (\mathbf{y}\mathbf{A}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{A}\mathbf{y}^{\mathrm{T}}$, we do not have to worry about the distinction between covariant and contravariant vectors.

Both the odd and even invariants are actually absolute invariants for the group of proper orthogonal transformations $O^+(n)$ since if $\mathbf{A}$ is proper then $\det(\mathbf{A}) = 1$. On the other hand, if f is an absolute invariant of $O^+(n)$, then

$$f(\mathbf{x}^1,\ldots,\mathbf{x}^\alpha) = f(\mathbf{A}\mathbf{x}^1,\ldots,\mathbf{A}\mathbf{x}^\alpha)$$

provided $\mathbf{A}$ is proper. Let

$$f'(\mathbf{Ax}^1,\ldots,\mathbf{Ax}^\alpha) = \det(\mathbf{A}) f(\mathbf{Ax},\ldots,\mathbf{Ax}).$$

Then, if f is even then f' is odd, and if f is odd, then f' is even. Suppose f is even. If **A** is a proper transformation then

$$f + f' = f(\mathbf{Ax}^1,\ldots,\mathbf{Ax}^\alpha) + f'(\mathbf{Ax}^1,\ldots,\mathbf{Ax}^\alpha) = 2f \text{ and}$$

$$f - f' = f(\mathbf{Ax}^1,\ldots,\mathbf{Ax}^\alpha) - f'(\mathbf{Ax}^1,\ldots,\mathbf{Ax}^\alpha) = 0.$$

If **A** is an improper transformation, then

$$f + f' = f(\mathbf{Ax}^1,\ldots,\mathbf{Ax}^\alpha) + f'(\mathbf{Ax}^1,\ldots,\mathbf{Ax}^\alpha) =$$

$$f(\mathbf{Ax}^1,\ldots,\mathbf{Ax}^\alpha) - f(\mathbf{Ax}^1,\ldots,\mathbf{Ax}^\alpha) = 0,$$

and

$$f - f' = f(\mathbf{Ax}^1,\ldots,\mathbf{Ax}^\alpha) - f'(\mathbf{Ax}^1,\ldots,\mathbf{Ax}^\alpha) =$$

$$f(\mathbf{Ax}^1,\ldots,\mathbf{Ax}^\alpha) + f(\mathbf{Ax}^1,\ldots,\mathbf{Ax}^\alpha) = 2f.$$

Thus, $f + f'$ is even and $f - f'$ is odd. If f is odd, then by a similar argument, $f + f'$ is odd and $f - f'$ is even. So, f can be written as the sum of an odd and even invariant:

$$f = \frac{1}{2}(f + f') + \frac{1}{2}(f - f').$$

Since, intuitively, the only functions in terms of vectors that have geometric meaning are length of the vector, and volume of the parallelepiped formed from the vectors (volume being related to the length of the vectors and the angles between the vectors), these are clearly invariant under rigid rotations (orthogonal transformations). Thus, it makes sense that dot product (square of length) and the determinant (the polynomial that gives volume of the parallelepiped) are the only invariants. We now can formulate our main theorem for orthogonal transformations.

Theorem 6.1. *A complete table of typical basic invariants of the orthogonal group consist of* (1) *the scalar product* $\langle \mathbf{y},\mathbf{x}\rangle$ *and* (2) *the determinant* $[\mathbf{x}^1 \ \cdots \ \mathbf{x}^n]$.

The product of two determinant factors can be expressed as the determinant of their scalar products:

$$[\mathbf{x}^1 \ \cdots \ \mathbf{x}^n][\mathbf{y}^1 \ \cdots \ \mathbf{y}^n] = \det\left(\begin{bmatrix} x_1^1 & \cdots & x_1^n \\ \vdots & \ddots & \vdots \\ x_n^1 & \cdots & x_n^n \end{bmatrix}\right)\det\left(\begin{bmatrix} y_1^1 & \cdots & y_1^n \\ \vdots & \ddots & \vdots \\ y_n^1 & \cdots & y_n^n \end{bmatrix}\right) =$$

$$\det\left(\begin{bmatrix} x_1^1 & \cdots & x_n^1 \\ \vdots & \ddots & \vdots \\ x_1^n & \cdots & x_n^n \end{bmatrix}\right)\det\left(\begin{bmatrix} y_1^1 & \cdots & y_1^n \\ \vdots & \ddots & \vdots \\ y_n^1 & \cdots & y_n^n \end{bmatrix}\right) =$$

$$\det\left(\begin{bmatrix} x_1^1 & \cdots & x_n^1 \\ \vdots & \ddots & \vdots \\ x_1^n & \cdots & x_n^n \end{bmatrix}\begin{bmatrix} y_1^1 & \cdots & y_1^n \\ \vdots & \ddots & \vdots \\ y_n^1 & \cdots & y_n^n \end{bmatrix}\right) = \begin{vmatrix} \langle \mathbf{x}^1,\mathbf{y}^1\rangle & \cdots & \langle \mathbf{x}^1,\mathbf{y}^n\rangle \\ \vdots & \ddots & \vdots \\ \langle \mathbf{x}^1,\mathbf{y}^1\rangle & \cdots & \langle \mathbf{x}^n,\mathbf{y}^n\rangle \end{vmatrix}..$$

We may rewrite the theorem as follows:

Statement $\mathbf{T}_n^m$) a) *Every even orthogonal invariant depending on m vectors* $\{\mathbf{x}^1,\ldots,\mathbf{x}^m\}$ *in* $\mathfrak{R}^n$ *can be expressed in terms of the* m^2 *scalar products* $\langle \mathbf{x}^i,\mathbf{x}^j\rangle$.

b) *Every odd invariant is a sum of terms* $[\mathbf{u}^1 \ \cdots \ \mathbf{u}^n]f^*(\mathbf{x}^1,\ldots,\mathbf{x}^m)$ *where* $\{\mathbf{u}^1,\ldots,\mathbf{u}^n\} \subset \{\mathbf{x}^1,\ldots,\mathbf{x}^m\}$ *and* f^* *is an even invariant.*

Proof of this theorem is by using Theorems 4.1 and 4.2 to show 3 inductive steps:

1) $\mathbf{T}_{n-1}^{n-1} \Rightarrow \mathbf{T}_n^{n-1}$

$$2)\ \mathbf{T}_n^{n-1} \Rightarrow \mathbf{T}_n^n$$

$$3)\ \mathbf{T}_n^n \Rightarrow \mathbf{T}_n^m \quad (m > n).$$

The details follow.

Proof: Proof of this theorem is by induction on rank and number of vectors.

Base case: $\mathbf{T}_1^1$.. Let *f* be an even invariant under *O(1)*. If $\mathbf{A} \in O(1)$, then $\mathbf{A} = \pm 1$. Let **A**=-1, and $f(x) = f(\mathbf{x}) = f(\mathbf{Ax}) = f(-x)$. This occurs when $f(x)$ is an even polynomial,

$f(x) = a_k x^{2k} + a_{k-1}x^{2k-2} + \ldots + a_1 x^2 + a_0 = a_k(x^2)^k + a_{k-1}(x^2)^{k-1} + \ldots + a_1(x^2)^1 + a_0(x^2)^0$, thus

$f(x)$ is expressible in the 1^2 scalar products $\langle \mathbf{x}, \mathbf{x} \rangle = x^2$. It's trivial when **A**=1.

Now let *f* be an odd invariant. So, if **A**=1, then

$$f(x) = f(\mathbf{x}) = \det(\mathbf{A}) f(\mathbf{Ax}) = f(x).$$

Thus, we need to look the case **A**=-1. Then

$$f(x) = f(\mathbf{x}) = \det(\mathbf{A}) f(\mathbf{Ax}) = -f(-x).$$

This occurs when $f(x)$ is an odd polynomial. Thus,

$$f(x) = a_k x^{2k+1} + a_{k-1}x^{2k-1} + \ldots + a_1 x^3 + a_0 x = x(a_k x^{2k} + a_{k-1}x^{2k-2} + \ldots + a_1 x^2 + a_0) =$$

$$xf^*(x) = [\mathbf{x}] f^*(\mathbf{x}), \qquad \text{and}$$

$$f^*(\mathbf{x}) = a_k x^{2k} + a_{k-1}x^{2k-2} + \ldots + a_1 x^2 + a_0$$

is the prescribed even invariant.

Induction step $\mathbf{T}_{n-1}^{n-1} \Rightarrow \mathbf{T}_n^{n-1}$.

Let $f(\mathbf{x}^1, \ldots, \mathbf{x}^{n-1})$ be an even invariant depending on *n*-1 vectors in $\Re^n$ such that

$x_n^i = 0$ for all $i \in \{1, \ldots, n-1\}$.

Define

$$f_0(\overline{\mathbf{x}}_1,\ldots,\overline{\mathbf{x}}_{n-1}) = f_0\begin{pmatrix} x_1^1 & \cdots & x_1^{n-1} \\ \vdots & ,\cdots, & \vdots \\ x_{n-1}^1 & \cdots & x_{n-1}^{n-1} \end{pmatrix} = f\begin{pmatrix} x_1^1 & \cdots & x_1^{n-1} \\ \vdots & ,\cdots, & \vdots \\ x_{n-1}^1 & \cdots & x_{n-1}^{n-1} \\ 0 & \cdots & 0 \end{pmatrix}.$$

Assume f_0 to be an odd invariant. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & 0 \\ 0 & \cdots & 0 & -1 \end{bmatrix}.$$

Then since f is an even invariant,

$$f_0 = f(\mathbf{x}_1,\ldots,\mathbf{x}_{n-1}) = f(\mathbf{A}\mathbf{x}_1,\ldots,\mathbf{A}\mathbf{x}_{n-1}) = \det(\mathbf{A}) f_0 = -f_0.$$

Thus, $f_0 = 0$. So, f_0 is an even invariant. Notice that f_0 depends on n-1 vectors in $\Re^{n-1}$, thus, by the $\mathbf{T}_{n-1}^{n-1}$ hypothesis, f_0 can be written as a polynomial $\mathcal{P}$ in terms of the $(n-1)^2$ scalar products

$$\begin{matrix} \langle \overline{\mathbf{x}}^1, \overline{\mathbf{x}}^1 \rangle, & \cdots & \langle \overline{\mathbf{x}}^1, \overline{\mathbf{x}}^{n-1} \rangle, \\ \vdots & & \vdots \\ \langle \overline{\mathbf{x}}^{n-1}, \overline{\mathbf{x}}^1 \rangle, & \cdots & \langle \overline{\mathbf{x}}^{n-1}, \overline{\mathbf{x}}^{n-1} \rangle \end{matrix}$$

where $\overline{\mathbf{x}}^i$ is a vector in $\Re^{n-1}$ and $\overline{x}_j^i = x_j^i$ for all $i, j \in \{1,\ldots,n-1\}$. Thus, it suffices to show that there exist an orthogonal transformation $\mathbf{A}$ such that

$$\mathbf{A}\mathbf{x}^i = \begin{bmatrix} \overline{x}_1^i \\ \vdots \\ \overline{x}_{n-1}^i \\ 0 \end{bmatrix}.$$

Since $\mathbf{x}^1,\ldots,\mathbf{x}^{n-1} \in \Re^n$, there exist $\mathbf{z} \in \Re^n$ such that $\mathbf{z}$ is perpendicular to each $\mathbf{x}^i$, and $\sqrt{\langle \mathbf{z},\mathbf{z} \rangle} = 1$. Now, by the Gram-Schimdt method, we can construct a new orthonormal coordinate system $\{\hat{\mathbf{e}}^1,\hat{\mathbf{e}}^2,\ldots,\hat{\mathbf{e}}^n\}$ such that $\mathbf{z} = \hat{\mathbf{e}}^n$. Let $\mathbf{A}$ be the orthogonal transformation that takes the standard coordinate system $\{\mathbf{e}^1,\mathbf{e}^2,\ldots,\mathbf{e}^n\}$ to $\{\hat{\mathbf{e}}^1,\hat{\mathbf{e}}^2,\ldots,\hat{\mathbf{e}}^n\}$. That is let

$$\mathbf{A} = \begin{bmatrix} \hat{e}_1^1 & \cdots & \hat{e}_1^n \\ \vdots & \ddots & \vdots \\ \hat{e}_n^1 & \cdots & \hat{e}_n^n \end{bmatrix}.$$

By definition of orthonomal basis, $\langle \mathbf{x}^i,\mathbf{x}^k \rangle = 0$ if $i \neq k$, and $\langle \mathbf{x}^i,\mathbf{x}^k \rangle = 1$ if $i=k$. Thus,

$$\mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{bmatrix} \hat{e}_1^1 & \cdots & \hat{e}_n^1 \\ \vdots & \ddots & \vdots \\ \hat{e}_1^n & \cdots & \hat{e}_n^n \end{bmatrix} \begin{bmatrix} \hat{e}_1^1 & \cdots & \hat{e}_1^n \\ \vdots & \ddots & \vdots \\ \hat{e}_n^1 & \cdots & \hat{e}_n^n \end{bmatrix} =$$

$$\begin{bmatrix} \langle \hat{\mathbf{e}}^1,\hat{\mathbf{e}}^1 \rangle & \langle \hat{\mathbf{e}}^1,\hat{\mathbf{e}}^2 \rangle & \cdots & \langle \hat{\mathbf{e}}^1,\hat{\mathbf{e}}^n \rangle \\ \langle \hat{\mathbf{e}}^2,\hat{\mathbf{e}}^1 \rangle & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \langle \hat{\mathbf{e}}^{n-1},\hat{\mathbf{e}}^n \rangle \\ \langle \hat{\mathbf{e}}^n,\hat{\mathbf{e}}^1 \rangle & \cdots & \langle \hat{\mathbf{e}}^n,\hat{\mathbf{e}}^{n-1} \rangle & \langle \hat{\mathbf{e}}^n,\hat{\mathbf{e}}^n \rangle \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Therefore, $\mathrm{A} \in O(n)$. Since each $\mathbf{x}^i$ is orthogonal to $\hat{\mathbf{e}}^n$, the n^{th} component in each $\mathbf{Ax}^i$ will be zero. Thus, since f is an even invariant,

$$f(\mathbf{x}^1,\ldots,\mathbf{x}^{n-1}) = f(\mathbf{Ax}^1,\ldots,\mathbf{Ax}^{n-1}) = f_0(\overline{\mathbf{x}}^1,\ldots,\overline{\mathbf{x}}^{n-1})$$

where $\overline{\mathbf{x}}^i \in \Re^{n-1}$, and $\overline{x}_j^i = (\mathbf{Ax}^i)_j$. Since $\mathrm{A} \in SL(n)$,

$$\langle \mathbf{x}^i,\mathbf{x}^k \rangle = \langle \mathbf{Ax}^i,\mathbf{Ax}^k \rangle = \langle \overline{\mathbf{x}}^i,\overline{\mathbf{x}}^k \rangle.$$

Therefore,

$$f(\mathbf{x}^1,\ldots,\mathbf{x}^{n-1}) = \mathcal{P}\left(\begin{pmatrix}\langle \mathbf{x}^1,\mathbf{x}^1\rangle \\ \vdots \\ \langle \mathbf{x}^{n-1},\mathbf{x}^1\rangle\end{pmatrix}, \quad \cdots, \quad \begin{pmatrix}\langle \mathbf{x}^1,\mathbf{x}^{n-1}\rangle \\ \vdots \\ \langle \mathbf{x}^{n-1},\mathbf{x}^{n-1}\rangle\end{pmatrix}\right).$$

So *f* is expressed as a polynomial of the dot products of *n-1* vectors in $\Re^n$.

Example 6.1. Let

$$\mathbf{x}^1 = \begin{bmatrix}1\\2\\1\end{bmatrix} \text{ and } \mathbf{x}^2 = \begin{bmatrix}1\\1\\1\end{bmatrix}.$$

$\mathbf{x}^1$ and $\mathbf{x}^2$ are clearly linearly independent. Let

$$\mathbf{z}' = \mathbf{x}^1 \times \mathbf{x}^2 = \begin{bmatrix}1\\0\\-1\end{bmatrix} \text{ and } \mathbf{z} = \frac{\mathbf{z}'}{\langle \mathbf{z}',\mathbf{z}'\rangle} = \begin{bmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{bmatrix}.$$

Then $\mathbf{z}$ is clearly orthogonal to $\mathbf{x}^1$ and $\mathbf{x}^2$. By the Gram- Schmidt method, we are going to find an orthonormal basis $\{\hat{\mathbf{e}}^1, \hat{\mathbf{e}}^2, \hat{\mathbf{e}}^3\}$ such that $\mathbf{z} = \hat{\mathbf{e}}^3$. So let

$$\hat{\mathbf{e}}^3 = \mathbf{z},$$

$$\tilde{\mathbf{e}}^1 = \mathbf{x}^1 - \frac{\langle \mathbf{z},\mathbf{x}^1\rangle}{\langle \mathbf{z},\mathbf{z}\rangle}\mathbf{z} = \begin{bmatrix}1\\2\\1\end{bmatrix},$$

$$\hat{\mathbf{e}}^1 = \frac{\tilde{\mathbf{e}}^1}{\langle \tilde{\mathbf{e}}^1,\tilde{\mathbf{e}}^1\rangle} = \begin{bmatrix}1/\sqrt{6}\\2/\sqrt{6}\\1/\sqrt{6}\end{bmatrix},$$

$$\tilde{\mathbf{e}}^2 = \mathbf{x}^2 - \frac{\langle \mathbf{z}, \mathbf{x}^2 \rangle}{\langle \mathbf{z}, \mathbf{z} \rangle}\mathbf{z} - \frac{\langle \hat{\mathbf{e}}^1, \mathbf{x} \rangle}{\langle \hat{\mathbf{e}}^1, \hat{\mathbf{e}}^1 \rangle}\hat{\mathbf{e}}^1 = \begin{bmatrix} 1/3 \\ -1/3 \\ 1/3 \end{bmatrix}, \text{ and}$$

$$\hat{\mathbf{e}}^2 = \frac{\tilde{\mathbf{e}}^2}{(\tilde{\mathbf{e}}\ )^T \tilde{\mathbf{e}}^2} = \begin{bmatrix} \sqrt{3}/3 \\ -\sqrt{3}/3 \\ \sqrt{3}/3 \end{bmatrix}.$$

Then

$$\mathbf{A} = \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ \sqrt{3}/3 & -\sqrt{3}/3 & \sqrt{3}/3 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}.$$

Thus

$$\overline{\mathbf{x}}^1 = \mathbf{A}\mathbf{x}^1 = \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ \sqrt{3}/3 & -\sqrt{3}/3 & \sqrt{3}/3 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

and

$$\overline{\mathbf{x}}^2 = \mathbf{A}\mathbf{x}^2 = \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ \sqrt{3}/3 & -\sqrt{3}/3 & \sqrt{3}/3 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/\sqrt{6} \\ \sqrt{3}/3 \\ 0 \end{bmatrix}.$$

Induction step $\mathbf{T}_n^{n-1} \Rightarrow \mathbf{T}_n^n$:

Proof of this is very similar to the proof of Theorem 4.2. The proof is by induction on rank. Again, we may assume $\rho > 0$.

Base case: Suppose f has rank $(n,1,1,...,1)$. Then, $\hat{f}_\kappa = -\hat{\rho}_\kappa D_{\mathbf{x}^{\beta_1}\mathbf{x}^{\alpha_1}} f$ has rank $(n,1,...,0,...,2,...,1)$ and is expressible in n-1 vectors. Also, the rank of Ωf is $(0,0,...,0)$. Therefore, Ωf is just a constant. Thus, by the Capelli special identity, and the $\mathbf{T}_n^{n-1}$ hypothesis, the conclusion holds.

Induction step: Assume that every even invariant with rank less than f can be expressed in terms of the dot products of its vectors, and every odd invariant with rank less than f can be expressed by

$$\begin{vmatrix} x_1^1 & \cdots & x_1^n \\ \vdots & \ddots & \vdots \\ x_n^1 & \cdots & x_n^n \end{vmatrix} \tilde{f},$$

where $\tilde{f}$ is an even invariant. First suppose that f is even. Then by Capelli's special identity we have

$$\rho f = \sum_\kappa \mathrm{P}_\kappa \hat{f}_\kappa + [\mathbf{x}^1,\ldots,\mathbf{x}^n]\Omega f, \text{ where } \hat{f}_\kappa = -\hat{\rho}_\kappa D_{\mathbf{x}^{\beta_1}\mathbf{x}^{\alpha_1}} f.$$

$\hat{f}_\kappa$ is an even form with rank less than f. Thus, $\hat{f}_\kappa$ can be expressed in terms of the dot products. P_κ is just succession of polarizations which, as shown in the unimodular group section, take a polynomial in terms of dot product to another polynomial in terms of dot product. Thus, $\sum_\kappa \mathrm{P}_\kappa \hat{f}_\kappa$ is expressible in terms of the dot products $\{\langle \mathbf{x}^i, \mathbf{x}^k \rangle\}$. Since f is even, Ωf is odd, and the rank of Ωf is less than the rank of f. Therefore, by the induction hypothesis, Ωf can be expressed as $[\mathbf{x}^n,\ldots,\mathbf{x}^1]\tilde{f}$, where $\tilde{f}$ is expressible in terms of the dot products. So,

$$[\mathbf{x}^n,\ldots,\mathbf{x}^1]\Omega\tilde{f} = [\mathbf{x}^n,\ldots,\mathbf{x}^1][\mathbf{x}^n,\ldots,\mathbf{x}^1]\tilde{f} =$$

$$\begin{vmatrix} x_1^n & \cdots & x_n^n \\ \vdots & \ddots & \vdots \\ x_1^1 & \cdots & x_n^1 \end{vmatrix}\begin{vmatrix} x_1^n & \cdots & x_1^1 \\ \vdots & \ddots & \vdots \\ x_n^n & \cdots & x_n^1 \end{vmatrix}\tilde{f} = \begin{vmatrix} \langle \mathbf{x}^n,\mathbf{x}^n\rangle & \cdots & \langle \mathbf{x}^n,\mathbf{x}^1\rangle \\ \vdots & \ddots & \vdots \\ \langle \mathbf{x}^1,\mathbf{x}^n\rangle & \cdots & \langle \mathbf{x}^1,\mathbf{x}^1\rangle \end{vmatrix}\tilde{f}.$$

Thus, $[\mathbf{x}^n,\ldots,\mathbf{x}^1]\Omega\tilde{f}$ is expressible in terms of the dot products $\{\langle \mathbf{x}^i,\mathbf{x}^k\rangle\}$. Therefore, $f(\mathbf{x}^1,\ldots,\mathbf{x}^n)$ is expressible by the dot products $\{\langle \mathbf{x}^i,\mathbf{x}^k\rangle\}$.. Now suppose *f* is odd. Then Ωf is even and *f* is of the form we want. $\hat{f}_\kappa = -\hat{\rho}_\kappa D_{\mathbf{x}^{\beta_1}\mathbf{x}^{\alpha_1}} f$ is odd and has less rank than *f*. So $\hat{f}_\kappa$ can be expressed by $[\mathbf{x}^n,\ldots,\mathbf{x}^1]\tilde{f}$ where $\tilde{f}$ is some even invariant, and, by the same argument as for the even invariants, $\sum_\kappa \mathrm{P}_\kappa \hat{f}_\kappa$ can be expressed by $[\mathbf{x}^n,\ldots,\mathbf{x}^1]\tilde{f}$.

Thus, by Capelli's general identity, *f* is equal to an even invariant times the determinant $[\mathbf{x}^n,\ldots,\mathbf{x}^1]$. Thus, by induction, $\mathbf{T}_n^{n-1} \Rightarrow \mathbf{T}_n^n$.

Induction step $\mathbf{T}_n^n \Rightarrow \mathbf{T}_n^m$ $(m > n)$:

Since it was shown earlier (Chapter V) that

$$D_{\mathbf{x}^i\mathbf{x}^j}\begin{vmatrix} x_1^1 & \cdots & x_1^n \\ \vdots & \ddots & \vdots \\ x_n^1 & \cdots & x_n^n \end{vmatrix} \text{ and } D_{\mathbf{x}^i\mathbf{x}^j}\mathbf{x}^\alpha\mathbf{x}^\beta$$

can be expressed by $[\mathbf{x}^n,\ldots,\mathbf{x}^1]$ and $\langle \mathbf{x}^i,\mathbf{x}^k\rangle$, this induction step follows directly from Theorem 4.1. **Q.E.D.**

REFERENCES

Weyl, Hermann. *TheClassical Groups: Their Invariants and Representations.* 2nd ed.Princeton, NJ: Princeton University Press, 1946

Taylor, Angus and Mann, Robert. *Advanced Calculus.* 3rd ed. New York, NY: John Wiley and Sons, Inc., 1983

PERMISSION TO COPY

In presenting this thesis in partial fulfillment of the requirements for a master's degree at Texas Tech University or Texas Tech University Health Science Center, I agree that the Library and my major department shall make it freely available for research purposes. Permission to copy this thesis for scholarly purposes may be granted by the Director of the Library or my major professor. It is understood that any copying or publication of this thesis for financial gain shall not be allowed without my further written permission and that any user may be liable for copyright infringement.

Agree (Permission is granted.)

Christopher G. Mouran — 11/7/96

Student's Signature — Date

Disagree (Permission is not granted.)

Student's Signature — Date