# ASYMPTOTIC, SPECTRAL, AND NUMERICAL ANALYSIS OF AN AIRCRAFT WING MODEL IN SUBSONIC AIRFLOW

by

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CONTENTS

# ABSTRACT

In this dissertation, we carry out asymptotic, spectral and numerical analysis of an aircraft wing model in subsonic air flow. The wing is modeled as a finite length beam, which can bend and twist. The model is governed by a system of two coupled partial integro-differential equations and a two parameter family of physically meaningful boundary conditions modeling the action of the self-straining actuators. The unknown functions depend on time and one spatial variable. The differential parts of the above equations form a coupled linear hyperbolic system; the integral parts are of the convolution type and represent the generalized forces and moments exerted on the wing due to the air flow. The system of equations of motion is equivalent to a single operator evolution–convolution equation in the state space of the system, which is a Hilbert space equipped with the so-called energy metric. The Laplace transform of the solution can be represented in terms of the generalized resolvent operator, which is an analytic operator-valued function of the spectral parameter. This operator is a finite–meromorphic function with infinitely many poles, which is defined on the complex plane having the branch cut along the negative real semi–axis. The poles of the generalized resolvent are called the aeroelastic modes and they are precisely the quantities that can be measured experimentally. The residues at these poles are the projectors on the generalized eigenspaces. In this dissertation, our main object of interest is the dynamics generator of the differential part of the system. It is a nonselfadjoint operator in the state space with a pure discrete spectrum. We show that the spectrum consists of two branches, and derive the precise spectral asymptotics of those branches. The importance of the information on the spectrum of the differential part is related to the fact that the two sets of complex points, i.e.,

the set of the aeroelastic modes and the set of the eigenvalues of the differential part of the model, are asymptotically close. This means that we have derived the asymptotic distribution of the aeroelatic modes, which is of importance for aircraft engineers.

# CHAPTER I

# INTRODUCTION

We study the spectral properties and derive the spectral asymptotics of a family of nonselfadjoint operators generated by a coupled Euler–Bernoulli and Timoshenko beam model. Such a model actually occurs in classical aeroelastic textbooks such as [12, 15, 19]. We formulate and prove spectral asymptotics of nonselfadjoint operators that are the dynamics generators for hyperbolic systems which govern the motion of a coupled Euler–Bernoulli and Timoshenko beam model subject to a two–parameter family of nonconservative boundary conditions, which model the action of self–sensing, self–straining actuators. The introduction of these specific boundary parameters [4–10, 33–48] is motivated by the desire to obtain a controller to suppress flutter in aircraft wings and tails.

We recall that flutter is a phenomenon of self–excited oscillations, often destructive, wherein energy is absorbed from the airstream. Flutter analysis is one of most important areas of aeroelasticity. We give now a brief overview of the state–of–art in flutter problem (see [11, 13, 14, 16-18, 20, 25, 27-31, 45,46]).

- In the well–known paper *Renaissance of Aeroelasticity and Its Future* [17], the author, Peretz P. Friedmann, writes,

> "The primary objective of this paper is to demonstrate that the field of aeroelasticity continues to play a *critical role* in the design of modern aerospace vehicles, and several important problems are still far from being well understood. Furthermore, the emergence of new technologies, such as the use of adaptive materials (sometimes denoted as smart structures technology), providing new actuator and sensor capabilities, has invigorated aeroelasticity, and generated a host of new and challenging research topics that can have a major impact on the design of a new generation of aerospace vehicles."

According to the author, aeroelaslicity deals with the behavior of an elastic body or vehicle in an airstream, wherein there is significant reciprocal interaction, or feedback, between deformation and flow. While dramatic instabilities are often a cause for concern, it is important to emphasize that subcritical aeroelastic response problem is equally, if not even more important, and for such classes of vehicles, as helicopters and tilt–rotors. In a modern aerospace vehicle, there is also a strong interaction between aeroelasticity and high–gain control systems leading to *aeroservoelasticity.* Furthermore, in high–speed supersonic or hypersonic vehicles, thermal effects become important, producing an even more complex class of *aerothermo–servoelastic problems.* Paper [17] focuses on the state–of–art in selected important topics, such as (1) experimental aeroelasticity in wind tunnels; (2) aeroservoplasticity; (3) computational and nonlinear aeroelasticity; (4) rotary–wing aeroelasticity; (5) impact of new technologies on aeroelasticity; (6) concise comments on experimental verification of aeroelastic behavior, aeroelastic problems in new configurations, and aeroelasticity and design. *Aeroservoelasticity* is a multidisciplinary technology dealing with the interaction of the aircraft flexible structure, the steady and unsteady aerodynamic forces resulting from the motion of the aircraft, and the flight control system. Its role and importance are increasing in modem aircraft with high–gain digital control systems.

• In other survey–type papers [20,27], the authors give an overview of an aircraft flutter in historical retrospective. When an aircraft is in flight, it necessarily deforms appreciably under load. Such deformations change the distribution of the aerodynamic load, which in turn changes the deformations: the interacting feedback process may lead to *flutter.* The initiation of flutter depends directly on the stiffness, and only indirectly on the strength of an airplane, analogous to depending on the slope of the lift curve rather than on the maximum lift. This implies that the airplane must be treated not as a rigid body but as an elastic structure. Despite the fact that the subject is an old one, this requires for a modern airplane a large effort in many areas, including ground vibration testing, use of dynamically scaled wind–tunnel models, theoretical analysis, and flight flutter testing. As has already been mentioned, from the present perspective, flutter is included in the broader term aeroelasticity, the study of the static and dynamic response of an elastic airplane. Since flutter involves the problems of interaction of aerodynamics and structural deformation, including inertial effects, at subcritical as well as at critical speeds, it really involves all aspects of aeroelasticity. In a broad sense, aeroelasticity is at work in natural phenomena such as in the motion of insects, fish, and birds (biofluid–dynamics).

Aeroelastic model tests in wind tunnel supported by mathematical analysis gave designers that much needed feeling of confidence that neither theory nor experiment alone could provide. These wind–tunnel model investigations ranged from measurements of the oscillating airloads to flutter–proof tests using complete aeroelastic models of *prototype aircraft.* In addition to providing designers with solutions that might not be obtainable by theory in a reasonable length of time, such experiments also are extremely useful tools for evaluating and guiding the development of theory.

The usefulness of wind–tunnel flutter model tests to validate theory, study flutter trends, and determine margins of safety for full scale prototypes had already been well established for low–speed aircrafts. Decades later, with flight speeds approaching that of sound, and aircraft of all–metal construction, new requirements arose for the design and fabrication of aeroelastically scaled flutter models.

The replica model concept was replaced by a much simpler model design approach wherein only those modes of vibration, expected to be significant from the standpoint of flutter, were represented by the model. With this approach, beam–like wings could be simulated by a single metal spar having the proper stiffness distribution.

With the advent of flight at transonic speeds brought about mainly by the jet engine, came a host of new and challenging aeroelastic problems, many of which remain to this day, as the transonic speed range is nearly always the most critical one from the standpoint of flutter.

*Supersonic speeds* also produce a new type of flutter. *The panel flutter*, could occur involving the skin covering wherein standing or travelling ripples in the skin persisted, which could often lead to an abrupt fatigue failure. Panels are natural structural elements of both aircraft and spacecraft so that avoidance of panel flutter is important. Panel flutter depends on many parameters, including the Mach number and the boundary layer, but especially on a compressive or thermal effects that tend to create low buckles in the skin.

• The carriage of external stores affects the aeroelastic stability of an aircraft. The store carriage problem is still significant today [27], particularly when the many store configurations that an aircraft can carry. Certain combinations of external stores (carried, e.g., by F–16, F–18, and F–111 aircrafts) produce an aeroelastic instability known as a *limit cycle oscillations.* Although these oscillations are mostly characterized by sinusoidal oscillations of limited amplitude, flight testing has shown that the amplitudes may either decrease or increase as a function of load factor (angle of attack) and airspeed. Much has been learned about the prevention of flutter through proper aircraft design and testing. Flutter testing, however, is still a hazardous test for a number of reasons [27]. First, it is necessary to fly close to actual flight speeds before imminent instabilities can be detected. Secondly, subcritical damping trends cannot be extrapolated to predict stability at higher speeds. Thirdly, the aeroelastic stability may change abruptly from a stable to the unstable one with only a few knots' change in airspeed [27].

Flutter analysis is an extremely important and active area of research in modern aeronautical studies. In recent years, extensive research has been carried out to develop *active flutter suppression and gust alleviation systems*, in which aerodynamic control surfaces are operated according to a control law, which relates the motion of the controls to some measurements taken on the vehicle. The aerodynamic forces generated by the controls modify the overall forces in such a way as to suppress flutter and alleviate structural turbulence response within the aircraft flight envelope (including safety margins). Some recent analytical developments, wind–tunnel tests, and flight–test demonstrations show the potential and feasibility of active control systems.

• In papers [30, 31], theoretical analysis of the flutter suppression of oscillating thin airfoils using *active acoustic excitations* in incompressible flow is presented. The aerodynamic control surfaces driven by the *hydraulic power* units, such as leading and trailing edge flaps, ailerons, spoilers, tailerons, and additional vanes, are often employed for flutter suppression and gust alleviation. Known difficulties in active control techniques, such as design of the feedback control laws and the implementation of the hydroservo systems, are usually complemented with a specific problem, namely, the hydraulic actuator is usually sluggish in response and sometimes cannot cope with the high–frequency oscillations.

In papers [16, 18, 28, 29], the authors study analytically, numerically, and experimentally the effect of the *trailing–edge flap* on flutter control of incompressible flow, and in addition, an important paper [16] addresses the question of *aeroelastic scaling.*

Recent advances in the area of smart structures have led to the use of such materials as actuators for aeroservoelastic applications in order to introduce continuous structural deformations of the lifting surface that can be exploited to manipulate the unsteady aerodynamic loads and prevent flutter. There exist many papers dealing with analysis and applications of piezoelectric materials to the control of static aeroelastic problems in a composite wing, and also demonstrating flutter suppression by using piezoelectric actuation on small–scale wind–tunnel models in incompressible flow.

• In paper [28], the authors suggest mathematical analysis and then numerical verification of a specific structural nonlinearity. Namely, the effect of a cubic structural restoring force on the flutter characteristics of a two–dimensional airfoil placed in an incompressible flow is investigated.

This dissertation is organized as follows.

• In Chapter I, we give a formulation of the boundary value problem for an aircraft wing. It describes a motion of a wing by taking into account the interaction between mechanically vibrating structure and an air flow around it. The right–hand sides of the system of two equations represent the generalized forces and moments due to an airflow. Detailed derivation of those loading terms can be found in [3, 12, 19] textbooks. As the result, we obtain a coupled system of differential equations, which will be our main system in the present work. We also introduce a two–parameter family of boundary conditions, which contains two control parameters.

• Beginning with Chapter II, we focus our attention on the initial boundary–value problem, which can be obtained from problem (1.1)–(1.7) by assuming that the right–hand sides of Eqs.(1.1) are zeros. At this moment, we notice that the aforementioned reduced problem is well known in mathematical and engineering literature [4, 5, 11–13, 15, 25]. The reason for this fact is that the reduced problem, which is called the *bending–torsion vibration model*, is widely used in structural and civil engineering to study the vibrations of suspended bridges; the same model is used also to study flutter phenomenon in high speed hard disk drives, in helicopter and turbomachinary blades, and in biomathematical analysis of blood/air and vessel and capillary interactions. [45, 46].

• In Chapter II, we justify the new setting of the initial boundary–value problem in the form of the first order in time evolution equation. The asymptotic properties of the spectrum of the dynamics generator is our main interest in the work. It is known that to make a new setting for the problem in the form of an evolution equation, one needs to find correctly the metric of the state space of the system, which is, in our case, a Hilbert space equipped with the so–called energy norm. We will call this Hilbert space an *energy space*. We note here that the metric of the energy space is highly nonstandard (see formula (2.14) below).

• In Chapter III, we initiate methodical study of the spectrum of the dynamics generator. As the first step, we reduce the problem for the spectrum of the generator to the problem of finding spectral asymptotics for the corresponding operator pencil. We recall the definitions related to operator pencils and provide the explanation why the pencil considered in the present work is highly nonstandard and extremely complicated. We show that the first necessary step is to analyze the asymptotic behavior of the fundamental system of solutions of the sixth order ordinary differential equation involving the spectral parameter $\lambda$ (see Eq.(3.16) below). The latter analysis requires us to derive asymptotics for the roots of the sixth order polynomial of a special type.

• In Chapter IV, we derive asymptotic approximations for the roots of the characteristic polynomial, which is associated to the aforementioned sixth–order ordinary differential equation. Namely, we obtain the approximations for six roots of the sixth order polynomial, approximations when the spectral parameter $|\lambda| \to \infty$ and those approximations are uniform with respect to the spatial variable $x$. Technically, this Chapter is very complicated and the results obtained in it are of crucial importance for the remaining Chapters V–VIII.

• In Chapter V, we introduce a special method to solve the boundary problem, i.e., we use the so–called *two–step procedure.* This is a relatively new method that has been introduced in papers [5, 33, 40, 44] to solve the boundary–value problem for a spatially nonhomogeneous Timoshenko beam model. By using the aforementioned procedure, we examine the effect of applying the boundary conditions from the left and from the right separately. Namely, in Chapter V, we look for a solution of the spectral equation for the operator pencil, a solution which satisfies only three left–hand side boundary conditions without any restrictions on the behavior of such a solution at the right–hand side of the flexible structure. In this Section, we introduce an important notion, which we call the *left–reflection matrix* (denoted as $\mathbb{R}_l$).

• In Chapter VI, we derive an asymptotic approximation for the *right–reflection matrix*, (denoted as $\mathbb{R}_r$), which is similar to the left–reflection matrix of Chapter V. The right–reflection matrix is useful to describe the solution of the main differential equation, which satisfies only the right–hand side boundary conditions without imposing any restrictions on the solution at the left end.

• In Chapter VII, we incorporate all information obtained in the previous two Chapters in order to derive the spectral asymptotics. The main tool in proving asymptotic formulae (3.2)–(3.3) with the necessary accuracy is the well–known Rouche's Theorem.

## 1.1 Precise Formulation of the Dissertation Problem.

We start with the description of the system of coupled integro–differential equation supplied with a nonstandard set of the boundary conditions. As was already mentioned, in the present dissertation, we will carry out investigation of the differential part of the system, which is a challenging problem in itself, and the necessary step in investigation of the entire integro–differential problem.

The boundary–value problem consists of a system of two coupled partial integro–differential equations in two unknown functions $h$ and $\alpha$, with $h(t,x)$ being a deflection at a point $x$ at time $t$ and $\alpha(t,x)$ being a torsion angle at a point $x$ and time $t$. We assume that the spatial extent of the flexible structure is $L$ and $t > 0$. Technically, it is convenient to assume that the spatial variable $x$ belongs to the interval $[-l, 0]$. We consider the following system:

$$\begin{cases} \tilde{m}h_{tt}(t,x) + \tilde{S}\alpha_{tt}(t,x) + EIh''''(t,x) + \pi\rho u\alpha_t(t,x) = f_1(t,x), \\ \\ \tilde{S}h_{tt}(t,x) + \tilde{I}_\alpha\alpha_{tt}(t,x) - GJ\alpha''(t,x) - \pi\rho u h_{tt}(t,x) - \pi\rho u^2\alpha = f_2(t,x), \\ \\ -L < x < 0; \quad 0 < t, \end{cases} \tag{1.1}$$

where

$$\tilde{m} = m + \pi\rho, \quad \tilde{S} = S - a\pi\rho, \quad \tilde{I}_\alpha = I_\alpha + \pi\rho(a^2 + 1/8). \tag{1.2}$$

In (1.1) and (1.2), the following notation have been introduced: $m$ is a mass per unit length, $S$ is a mass moment per unit length, $EI$ is a bending stiffness, $GJ$ is a torsion stiffness, $I_\alpha$ is a moment of inertia, $\rho$ is the density of air, and $u$ is the speed of the aircraft. By prime, we have denoted the spatial derivative and by the sub t, the time derivative.

Now we describe the right–hand side functionals $f_1$ and $f_2$. These functionals are nonzeros if and only if an aircraft is in flight, i.e., $u \neq 0$. We mentioned here that the same problem with $u = 0$ (and thus $f_1 = f_2 = 0$) describes the so–called ground vibrations of a wing. The latter problem is also known to be difficult and important since any test of an aircraft wing performance starts with the identification of the ground vibration frequencies.

Now we turn to the description of the integral convolution–type functionals $f_1$ and $f_2$. The right hand side of system (1.1) can be represented as the following system of two convolution–type integral operations:

$$f_1(x,t) = -2\pi\rho \int_0^t \left[uC_2(t-\sigma) - (C_3)_t(t-\sigma)\right] g(x,\sigma)d\sigma, \tag{1.3}$$

$$\begin{aligned} f_2(x,t) = -2\pi\rho \int_0^t & \left[1/2C_1(t-\sigma) - auC_2(t-\sigma) + a(C_3)_t(t-\sigma)\right. \\ & \left. +uC_4(t-\sigma) + 1/2(C_5)_t(t-\sigma)\right] g(x,\sigma)d\sigma, \end{aligned} \tag{1.4}$$

$$g(x,t) = u\alpha_t(x,t) + h_{tt}(x,t) + (1/2 - a)\alpha_{tt}(x,t). \tag{1.5}$$

The aerodynamical functions $C_i$, $i = 1 \ldots 5$, are defined in the following ways:

$$\begin{aligned} \hat{C}_1(\lambda) &= \int_0^\infty e^{-\lambda t} C_1(t)dt = \frac{u}{\lambda} \frac{e^{-\lambda/u}}{K_0(\lambda/u) + K_1(\lambda/u)}, \quad Re\lambda > 0, \\ C_2(t) &= \int_0^t C_1(\sigma)d\sigma, \\ C_3(t) &= \int_0^t C_1(t-\sigma)(u\sigma - \sqrt{u^2\sigma^2 + 2u\sigma})d\sigma, \\ C_4(t) &= C_2(t) + C_3(t), \\ C_5(t) &= \int_0^t C_1(t-\sigma)((1+u\sigma)\sqrt{u^2\sigma^2 + 2u\sigma} - (1+u\sigma)^2)d\sigma, \end{aligned} \tag{1.6}$$

where $K_0$ and $K_1$ are the modified Bessel functions of the zero and first orders, respectively [12, 19]. The derivation of the formulae for the aerodynamical functions can be found in [19]. The system is supplied with a two–parameter family of boundary conditions

$$
\begin{aligned}
h(t,-L) = h'(t,-L) = \alpha(t,-L) = 0,& \\
h'''(t,0) = 0, \quad EIh''(t,0) + g_h h_t'(t,0) = 0,& \qquad (1.7)\\
GJ\alpha'(t,0) + g_\alpha \alpha_t(t,0) = 0.&
\end{aligned}
$$

We notice now that the set of boundary conditions at the left end is standard. However, the right–hand side boundary conditions are highly nonstandard, i.e., they contain two arbitrary parameters $g_h$ and $g_\alpha$. These parameters are used in current mathematical and engineering literature in order to model the action of "smart materials" [4–10, 33, 35, 37, 39, 42, 47, 48]. Technically, the analysis can be carried out for any complex values of the boundary parameters. An important case for practical applications is the one when $g_h$ and $g_\alpha$ are positive numbers. In that case, $g_h$ is called *a bending control gain* and $g_\alpha$ is called *a torsion control gain* [9]. The idea for introducing the control gains in the boundary conditions is to suppress flutter in a wing by using the properties of smart materials. However, the reality is that original expectations, concerning power of piezo–electric inclusions into a wing structure, were impossibly high. Namely, there are two different types of actions of smart materials, which are characterized by two different control gains, i.e., $g_h$ and $g_\alpha$. Mathematical analysis (see formulae (3.2) and (3.3) of Chapter III) shows that controlling the torsion motion with the control gain $g_\alpha$ is highly promising; at the same time control with the control gain $g_h$ is not efficient at all. This important result is in a good agreement with experimental data.

In addition to the boundary conditions, we introduce a set of initial conditions in a standard manner

$$h(0,x) = h_0(x),\ h_t(t,x)\,\big|_{t=0} = h_1(x),\ \alpha(0,t) = \alpha_0(x),\ \alpha_t(t,x)\,\big|_{t=0} = \alpha_1(x). \qquad (1.8)$$

To conclude the Introduction, we note that we consider a wing that is perfectly elastic, and rigid in cross sections perpendicular to the lengthwise direction. It has an elastic axis, implying that the wing is unswept and without structural discontinuities, so that elastic coupling between bending and twisting is eliminated. The elastic axis is straight, and rotary inertia and shear deformation are neglected.

CHAPTER II

# DYNAMICS GENERATOR AND ITS GENERAL SPECTRAL PROPERTIES

## 2.1 Dynamics Generator

As has been already mentioned, our goal is to analyze the properties of the differential part (the left–hand side) of the system (1.1). More precisely, we consider the problem, given by (1.1)–(1.8) when $f_1 = f_2 = 0$. First, we would like to present it as a first order in time evolution equation. The dynamics generator will be our main object of interest. We will show that the dynamics generator is a $4 \times 4$ matrix differential operator, which acts in the energy space. To introduce the formula for the dynamics generator and to describe its domain, we carry out the following steps. First, we can verify that our system of equations (1.1) (if we set $f_1 = f_2 = 0$)

$$\begin{cases} \tilde{m}h_{tt}(t,x) + \tilde{S}\alpha_{tt}(t,x) + \pi\rho u\alpha_t(t,x) = -EIh''''(t,x), \\ \tilde{S}h_{tt}(t,x) + \tilde{I}_\alpha\alpha_{tt}(t,x) - \pi\rho u h_t(t,x) - \pi\rho u^2\alpha(t,x) = GJ\alpha''(t,x) \end{cases} \tag{2.1}$$

can be represented in the following form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \tilde{m} & \pi\rho u & \tilde{S} \\ 0 & 0 & 1 & 0 \\ -\pi\rho u & \tilde{S} & 0 & \tilde{I}_\alpha \end{bmatrix} \begin{bmatrix} h_t(t,x) \\ h_{tt}(t,x) \\ \alpha_t(t,x) \\ \alpha_{tt}(t,x) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -EI\dfrac{\partial^4}{\partial x^4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & GJ\dfrac{\partial^2}{\partial x^2} + \pi\rho u^2 & 0 \end{bmatrix} \begin{bmatrix} h(t,x) \\ h_t(t,x) \\ \alpha(t,x) \\ \alpha_t(t,x) \end{bmatrix}. \tag{2.2}$$

If we introduce a 4–component vector $Y$ by the formula $Y = (h, h_t, \alpha, \alpha_t)^T$ (the superscript "T" means the transposition), and denote the matrices in (2.2) by $\mathbf{M}$ and $\mathbf{A}$, then Eq.(2.2) can be written in the form

$$\mathbf{M}Y_t = \mathbf{A}Y. \tag{2.3}$$

Now we introduce two important assumptions

$$\Delta = mI_\alpha - S^2 > 0, \qquad 0 < u \le \frac{\sqrt{2GJ}}{L\sqrt{\pi\rho}}. \tag{2.4}$$

It can be easily shown that from the first estimate, the following estimate is also valid: $\tilde{m}\tilde{I}_\alpha - \tilde{S}^2 > 0$. Note that by the first assumption, the value of a coupling constant $\tilde{S}$ is not necessarily small. In a subsonic air flow, the second assumption is always satisfied. Due to (2.4), we can rewrite Eq.(2.3) as

$$Y_t = \mathbf{M}^{-1}\mathbf{A}Y = i[-i\mathbf{M}^{-1}\mathbf{A}]Y. \tag{2.5}$$

Denoting $\mathcal{L}_{g_h g_\alpha} = -i\mathbf{M}^{-1}\mathbf{A}$, we finally rewrite Eq.(2.5) in the desired form

$$Y_t = i\mathcal{L}_{g_h g_\alpha}Y. \tag{2.6}$$

If $\Phi^T(t,x) = \{\phi_0(t,x),\ \phi_1(t,x),\ \phi_2(t,x),\ \phi_3(t,x)\} = \{h,\ h_t,\ \alpha,\ \alpha_t\}$, $-L \le x \le 0$, $t \ge 0$, then the initial–boundary value problem (1.1)–(1.8) can be rewritten in the form of the first–order in time evolution equation

$$\Phi_t = i\mathcal{L}_{g_h g_\alpha}\Phi, \qquad \Phi\,\big|_{t=0} = \Phi_0, \tag{2.7}$$

with the dynamics generator $\mathcal{L}_{g_h g_\alpha}$ being defined on smooth functions $\Phi = (\phi_0, \phi_1, \phi_2, \phi_3)^T$ by the formula

$$\mathcal{L}_{g_h g_\alpha} = -i \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{EI\tilde{I}_\alpha}{\Delta}\frac{d^4}{dx^4} & -\frac{\pi\rho u\tilde{S}}{\Delta} & -\frac{\tilde{S}}{\Delta}\left(GJ\frac{d^2}{dx^2} + \pi\rho u^2\right) & -\frac{\pi\rho u\tilde{I}_\alpha}{\Delta} \\ 0 & 0 & 0 & 1 \\ \frac{EI\tilde{S}}{\Delta}\frac{d^4}{dx^4} & \frac{\pi\rho u\tilde{m}}{\Delta} & \frac{\tilde{m}}{\Delta}\left(GJ\frac{d^2}{dx^2} + \pi\rho u^2\right) & \frac{\pi\rho u\tilde{S}}{\Delta} \end{bmatrix}. \tag{2.8}$$

The functions from the domain of $\mathcal{L}_{g_h g_\alpha}$ are subject to the following boundary conditions:

$$\begin{aligned} \phi_1(-L) = \phi_1'(-L) = \phi_3(-L) &= 0, \\ \phi_0'''(0) &= 0, \\ EI\phi_0''(0) + g_h\phi_1'(0) &= 0, \\ GJ\phi_2'(0) + g_\alpha\phi_3(0) &= 0. \end{aligned} \tag{2.9}$$

## 2.2 Operator Setting

As has already been mentioned, to have rigorous operator setting of the problem, we need to find a metric for the state space. As the first step, we will derive the energy functional. Let us derive the formula the energy of the system by carrying out the following steps. Let us multiply the first of Eq.(2.1) by $\overline{h}_t$ and the second of Eq.(2.1) by $\bar{\alpha}_t$ and then add the resulting two equations. If we denote the resulting sum as EQ1, we have

$$\begin{aligned} EQ1 = &\tilde{m}h_{tt}(t,x)\bar{h}_t(t,x) + \tilde{S}\alpha_{tt}(t,x)\bar{h}_t(t,x) + EI\ h''''(t,x)\bar{h}_t(t,x) + \\ &\pi\rho u\alpha_t(t,x)\bar{h}_t(t,x) + \quad \tilde{S}h_{tt}(t,x)\bar{\alpha}_t(t,x) + \tilde{I}_\alpha\alpha_{tt}(t,x)\bar{\alpha}_t(t,x) - \\ &\pi\rho u h_t(t,x)\bar{\alpha}_t(t,x) - \pi\rho u^2\alpha(t,x)\bar{\alpha}_t(t,x) - GJ\ \alpha''(t,x)\bar{\alpha}_t(t,x) = 0. \end{aligned} \tag{2.10}$$

Let us take complex conjugate of Eq.(2.10) and denote this new equation by EQ2. It can be verified by a direct calculation that we have the following result:

$$
\begin{aligned}
EQ1 + EQ2 = \tilde{m}\frac{d}{dt}\left|h_t(t,x)\right|^2 + \tilde{S}\frac{d}{dt}\left(\alpha_t(t,x)\bar{h}_t(t,x) + \bar{\alpha}_t(t,x)h_t(t,x)\right) + \\
EI\left(h''''(t,x)\bar{h}_t(t,x) + \overline{h}''''(t,x)h_t(t,x)\right) + \tilde{I}_\alpha\frac{d}{dt}\left|\alpha_t(t,x)\right|^2 - \\
GJ\left(\alpha''(t,x)\bar{\alpha}_t(t,x) + \overline{\alpha}''(t,x)\alpha_t(t,x)\right) + \pi\rho u\left(\alpha_t(t,x)\bar{h}_t(t,x) + \right. \\
\bar{\alpha}_t(t,x)h_t(t,x) - h_t(t,x)\bar{\alpha}_t(t,x) - h_t(t,x)\alpha_t(t,x) - \\
\left. u\alpha(t,x)\bar{\alpha}_t(t,x) - u\bar{\alpha}(t,x)\alpha_t(t,x)\right) = 0.
\end{aligned}
\tag{2.11}
$$

Eq.(2.11) suggests that a convenient expression for the energy of the system can be taken in the form of the following functional:

$$
\begin{aligned}
\mathcal{E}(t) = \frac{1}{2}\int_{-L}^{0}\Big[EI\left|h''(t,x)\right|^2 + GJ\left|\alpha'(t,x)\right|^2 + \tilde{m}\left|h_t(t,x)\right|^2 + \\
\tilde{I}_\alpha\left|\alpha_t(t,x)\right|^2 + \tilde{S}\left(\alpha_t(t,x)\bar{h}_t(t,x) + \bar{\alpha}_t(t,x)h_t(t,x)\right) - \pi\rho u^2|\alpha(t,x)|^2\Big]dx.
\end{aligned}
\tag{2.12}
$$

Integrals involving the first two terms in (2.12) represent the potential energy of vibrations; the third and fourth terms represent the kinetic energy of vibrations; the fifth and sixth terms represent the kinetic energy due to the coupling between different types of motion; the last term is nontrivial as long as $u \neq 0$. The following Lemma regarding this energy has been proved in our paper [5].

**Lemma 2.1.** *Under conditions (2.4), the energy of vibrations, given by formula (2.12), is nonnegative and is equal to zero if and only if* $h(t,x) = \alpha(t,x) = 0$, $x \in [-L, 0]$, $t \geq 0$.

With this energy of vibrations, we can define the operator setting of the problem. First we describe the state space of the system, which will be denoted by $\mathcal{H}$. Let $\mathcal{H}$ be a set of 4-component vector-valued functions $\Phi = (\phi_0, \phi_1, \phi_2, \phi_3)^T$ obtained as a closure of smooth functions satisfying the conditions

$$
\phi_0(-L) = \phi_0'(-L) = \phi_2(-L) = 0 \tag{2.13}
$$

in the following energy norm:

$$
\begin{aligned}
\|\Phi\|_{\mathcal{H}}^2 = \frac{1}{2}\int_{-L}^{0} \Big[ & EI\,|\phi_0''(x)|^2 + GJ\,|\phi_2'(x)|^2 + \tilde{m}\,|\phi_2(x)|^2 + \tilde{I}_\alpha\,|\phi_3(x)|^2 + \\
& \tilde{S}\left(\bar{\phi}_1(x)\phi_3(x) + \phi_1(x)\bar{\phi}_3(x)\right) - \pi\rho u^2|\phi_2|^2 \Big]\,dx.
\end{aligned} \tag{2.14}
$$

This is shown to be a norm in our paper [35].

The operator $\mathcal{L}_{g_h g_\alpha}$ is given by formula (2.8) and is defined on the domain

$$
\begin{aligned}
D\left(\mathcal{L}_{g_h g_\alpha}\right) = \Big\{ \Phi \in \mathcal{H} : \ & \phi_0 \in H^4(-L,0),\ \phi_1 \in H^2(-L,0), \\
& \phi_2 \in H^2(-L,0),\ \phi_3 \in H^1(-L,0); \\
& \phi_1(-L) = \phi_1'(-L) = \phi_3(-L) = 0,\ \phi_0'''(0) = 0; \\
& EI\ \phi_0''(0) + g_h\phi_1'(0) = 0,\ GJ\ \phi_2'(0) + g_\alpha\phi_3(0) = 0 \Big\},
\end{aligned} \tag{2.15}
$$

where $H^i$ , $i = 1, 2, 4$, are the standard Sobolev spaces [1].

## 2.3 Properties of the Dynamics Generator

The following theorem is proved in our paper [5].

**Theorem 2.1.** *The operator $\mathcal{L}_{g_h g_\alpha}$ has the following properties.*

*(i) $\mathcal{L}_{g_h g_\alpha}$ is an unbounded closed nonselfadjoint (unless $\Re\, g_h = \Re\, g_\alpha = 0$) operator in $\mathcal{H}$.*

*(ii) If $\Re\, g_h \geq 0$ and $\Re\, g_\alpha \geq 0$, then $\mathcal{L}_{g_h g_\alpha}$ is a dissipative operator in $\mathcal{H}$ (i.e., if $\Phi \in D(\mathcal{L}_{g_h g_\alpha})$, then $\Im\,(\mathcal{L}_{g_h g_\alpha}\Phi, \Phi)_{\mathcal{H}} \geq 0$, [26] ).*

*(iii) The inverse operator $\mathcal{L}_{g_h g_\alpha}^{-1}$ exists and it is a compact operator in $\mathcal{H}$. Therefore, $\mathcal{L}_{g_h g_\alpha}^{-1}$ has a purely discrete spectrum consisting of normal eigenvalues. (Recall that $\lambda$ is a normal eigenvalue of a bounded operator $A$ in the space $H$ if a) $\lambda$ is an isolated point of the spectrum of $A$, b) the algebraic multiplicity of $\lambda$ is finite, c) the range $(A - \lambda I)\,H$ of the operator $(A - \lambda I)$ is closed [24, 32] ).*

We emphasize that from Theorem 2.1, the following two important results can be seen immediately:

a. the operator $\mathcal{L}_{g_h g_\alpha}$ has a purely discrete spectrum, which can accumulate only at infinity;

b. when $\Re g_h \geq 0$ and $\Re g_\alpha \geq 0$, the spectrum is located in the closed upper half plane.

**Remark 2.1.** As is accustomed in the engineering literature, we reformulate the conclusions of Theorem 2.1 for the operator $\mathfrak{L}_{g_\alpha g_h} = i\mathcal{L}_{g_h g_\alpha}$. For $\mathfrak{L}_{g_\alpha g_h}$, we obtain that when $\Re\, g_h \geq 0$ and $\Re\, g_\alpha \geq 0$, then the spectrum of this operator is located in the closed left half–plane, and consists of, at most, countable number of eigenvalues that can accumulate only at infinity.

# CHAPTER III

# GENERAL SOLUTION OF SPECTRAL EQUATION

## 3.1 Precise Statement of the Asymptotics of the Spectrum

We formulate now a precise statement of the spectral results which will be proven in the rest of the paper. In fact, Theorem 3.1 below is the main analytical result of the dissertation research.

**Theorem 3.1.** *(a) The operator $\mathcal{L}_{g_h g_\alpha}$ has a countable set of complex eigenvalues. Under the assumption*

$$g_\alpha \neq \sqrt{\tilde{I}_\alpha \, GJ}, \tag{3.1}$$

*the set of eigenvalues is located in a strip parallel to the real axis.*

*(b) The entire set of the eigenvalues asymptotically splits into two disjoint subsets. We call them the h–branch and the $\alpha$–branch and denote these branches by $\{\lambda_n^h\}_{n\in\mathbb{Z}}$ and $\{\lambda_n^\alpha\}_{n\in\mathbb{Z}}$ respectively. If $\Re\, g_\alpha \geq 0$ and $\Re\, g_h > 0$, then the $\alpha$–branch is asymptotically close to some horizontal line in the upper half–plane. If $\Re\, g_h = \Re\, g_\alpha = 0$, then the operator $\mathcal{L}_{g_h g_\alpha}$ is selfadjoint and thus its spectrum is real. The entire set of eigenvalues may have only two points of accumulation: $+\infty$ and $-\infty$ in the sense that $\Re\, \lambda_n^{h(\alpha)} \to \pm\infty$ and $\Im\, \lambda_n^{h(\alpha)} < const$ as $n \to \pm\infty$ (see formulae (3.2) and (3.3) below).*

*(c) The following asymptotic formula is valid for the h–branch of the spectrum:*

$$\lambda_n^h = \pm\pi^2/L^2\sqrt{\tilde{I}EI/\Delta}(|n| - 1/4)^2 + O(1), \quad |n| \to \infty. \tag{3.2}$$

*In formula (3.2), the sign "+" should be taken for $n > 0$ and "$-$" for $n < 0$.*

*(d) The following asymptotic formula is valid for the $\alpha$–branch of the spectrum:*

$$\lambda_n^\alpha = \frac{\pi n}{L\sqrt{\tilde{I}_\alpha/GJ}} + \frac{i}{2L\sqrt{\tilde{I}_\alpha/GJ}} \ln \frac{g_\alpha + \sqrt{\tilde{I}_\alpha GJ}}{g_\alpha - \sqrt{\tilde{I}_\alpha GJ}} + O(|n|^{-1/2}), \quad |n| \to \infty. \tag{3.3}$$

*In (3.3), the principle value of the logarithm is understood.*

**Remark 3.1.** One can see that the leading term of the asymptotics (3.2) does not contain control parameters $g_h$ or $g_\alpha$. At the same time, the leading term of the asymptotics (3.3) does contain one control parameter, $g_\alpha$, which is associated with controlling the torsion motion. Based on the behavior of the leading terms of the asymptotics, we can suggest that there is a difference in the performance of two types of controls, i.e., it is unlikely to control bending displacement by using $g_h$ parameter as a controller.

## 3.2 Operator Pencil

In this subsection, we introduce an operator–valued polynomial function, which we call an *operator pencil.* To introduce this operator pencil, which is associated to the dynamics generator $\mathcal{L}_{g_h g_\alpha}$, we start with the spectral equation for this operator

$$\mathcal{L}_{g_h g_\alpha}\Phi = \lambda\Phi, \qquad \Phi \in \mathcal{D}(\mathcal{L}_{g_h g_\alpha}). \tag{3.4}$$

Using the explicit formula for $\mathcal{L}_{g_h g_\alpha}$, we obtain

$$-i\begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{EI\tilde{I}_\alpha}{\Delta}\frac{d^4}{dx^4} & -\frac{\pi\rho u\tilde{S}}{\Delta} & -\frac{\tilde{S}}{\Delta}\left(GJ\frac{d^2}{dx^2}+\pi\rho u^2\right) & -\frac{\pi\rho u\tilde{I}_\alpha}{\Delta} \\ 0 & 0 & 0 & 1 \\ \frac{EI\tilde{S}}{\Delta}\frac{d^4}{dx^4} & \frac{\pi\rho u\tilde{m}}{\Delta} & \frac{\tilde{m}}{\Delta}\left(GJ\frac{d^2}{dx^2}+\pi\rho u^2\right) & \frac{\pi\rho u\tilde{S}}{\Delta} \end{bmatrix}\begin{bmatrix}\phi_0\\ \phi_1\\ \phi_2\\ \phi_3\end{bmatrix} = \lambda\begin{bmatrix}\phi_0\\ \phi_1\\ \phi_2\\ \phi_3\end{bmatrix}. \tag{3.5}$$

Rewriting component–wise, yields the following four equations:

$$\phi_1 = i\lambda\phi_0, \tag{3.6}$$

$$\phi_3 = i\lambda\phi_1, \tag{3.7}$$

$$\frac{\tilde{I}_\alpha EI}{\Delta}\phi_0^{IV} + \frac{\tilde{S}GJ}{\Delta}\phi_2'' + \frac{\pi\rho u\tilde{S}}{\Delta}\phi_1 + \frac{\pi\rho u^2\tilde{S}}{\Delta}\phi_2 + \frac{\pi\rho u\tilde{I}_\alpha}{\Delta}\phi_3 = -i\lambda\phi_2, \tag{3.8}$$

$$\frac{\tilde{S}EI}{\Delta}\phi_0^{IV} + \frac{\tilde{m}GJ}{\Delta}\phi_2'' + \frac{\pi\rho u\tilde{m}}{\Delta}\phi_1 + \frac{\pi\rho u^2\tilde{m}}{\Delta}\phi_2 + \frac{\pi\rho u\tilde{S}}{\Delta}\phi_3 = i\lambda\phi_3. \tag{3.9}$$

Our goal is to eliminate the three components $\phi_1$, $\phi_2$, and $\phi_3$ from system (3.6)–(3.9) and to derive a single equation with respect to the one component $\phi_0$. Substituting Eqs.(3.6) and (3.7) into Eqs.(3.8) and (3.9) to eliminate $\phi_2$ and $\phi_3$, we obtain

$$\frac{EI\tilde{I}_\alpha}{\Delta}\phi_0^{IV} + i\lambda\frac{\pi\rho u\tilde{S}}{\Delta}\phi_0 + \frac{GJ\tilde{S}}{\Delta}\phi_2'' + \frac{\pi\rho u^2\tilde{S}}{\Delta}\phi_2 + i\lambda\frac{\pi\rho u\tilde{I}_\alpha}{\Delta}\phi_2 = \lambda^2\phi_0, \tag{3.10}$$

$$\frac{EI\tilde{S}}{\Delta}\phi_0^{IV} + i\lambda\frac{\pi\rho u\tilde{m}}{\Delta}\phi_0 + \frac{GJ\tilde{m}}{\Delta}\phi_2'' + \frac{\pi\rho u^2\tilde{m}}{\Delta}\phi_2 + i\lambda\frac{\pi\rho u\tilde{S}}{\Delta}\phi_2 = -\lambda^2\phi_0 \tag{3.11}$$

where $\Delta$ is defined in (2.4). Solving both Eqs. (3.10) and (3.11), we have the following system of equations:

$$\begin{aligned}\frac{EI\tilde{I}_\alpha}{\Delta}\phi_0^{IV} &= -i\lambda\frac{\pi\rho u\tilde{S}}{\Delta}\phi_0 - \frac{GJ\tilde{S}}{\Delta}\phi_2'' - \frac{\pi\rho u^2\tilde{S}}{\Delta}\phi_2 - i\lambda\frac{\pi\rho u\tilde{I}_\alpha}{\Delta}\phi_2 + \lambda^2\phi_0\\ \frac{EI\tilde{S}}{\Delta}\phi_0^{IV} &= -i\lambda\frac{\pi\rho u\tilde{m}}{\Delta}\phi_0 - \frac{GJ\tilde{m}}{\Delta}\phi_2'' - \frac{\pi\rho u^2\tilde{m}}{\Delta}\phi_2 - i\lambda\frac{\pi\rho u\tilde{S}}{\Delta}\phi_2 - \lambda^2\phi_0.\end{aligned} \tag{3.12}$$

Excluding $\phi_0^{IV}$ from the system of Eqs. (3.12) and simplifing we obtain the one following equation:

$$\begin{gathered}\left[i\lambda\left(\frac{\pi\rho u\Delta}{EI\tilde{I}_\alpha\tilde{S}}\right) - \lambda^2\left(\frac{\Delta}{EI\tilde{I}_\alpha}\right)\right]\phi_0 - \left[\frac{GJ\Delta}{EI\tilde{I}_\alpha\tilde{S}}\right]\phi_2'' + \\ \left[\frac{\pi\rho u^2\Delta}{EI\tilde{I}_\alpha\tilde{S}} - \lambda^2\left(\frac{\Delta}{EI\tilde{S}}\right)\right]\phi_2 = 0.\end{gathered} \tag{3.13}$$

Solving (3.13) for $\phi_0$ gives:

$$\phi_0 = -\frac{GJ}{\lambda^2\tilde{S} + i\lambda\pi\rho u}\phi_2'' - \frac{\lambda^2\tilde{I}_\alpha + \pi\rho u^2}{\lambda^2\tilde{S} + i\lambda\pi\rho u}\phi_2. \tag{3.14}$$

Substituting (3.14) into Eq.(3.11), we obtain

$$\begin{aligned} EI\tilde{S}GJ\phi_2^{VI} + EI\tilde{S}(\lambda^2\tilde{I}_\alpha + \pi\rho u^2)\phi_2^{IV} + i\lambda\pi\rho u\tilde{m}GJ\phi_2'' + i\lambda\pi\rho u\tilde{m}(\lambda^2\tilde{I}_\alpha \\ +\pi\rho u^2)\phi_2 - \tilde{m}GJ(\lambda^2\tilde{S} + i\lambda\pi\rho u)\phi_2'' - \pi\rho u^2\tilde{m}(\lambda^2\tilde{S} + i\lambda\pi\rho u)\phi_2 \\ -i\lambda\pi\rho u\tilde{S}(\lambda^2\tilde{S} + i\lambda\pi\rho u)\phi_2 - \lambda^2\Delta(\lambda^2\tilde{S} + i\lambda\pi\rho u)\phi_2 = 0, \end{aligned} \tag{3.15}$$

where $\phi_2^{VI}$ means the sixth–order derivative with respect to $x$. Collecting like terms in Eq.(3.15) and then simplifying, we arrive at the following final form of the equation for the component $\phi_2$:

$$EI\,GJ\phi_2^{VI} + EI(\lambda^2\tilde{I}_\alpha + \pi\rho u^2)\phi_2^{IV} - \lambda^2\tilde{m}GJ\phi_2'' - [\lambda^2\pi\rho u^2\tilde{m} - \lambda^2(\pi\rho u)^2 + \lambda^4\Delta]\phi_2 = 0. \tag{3.16}$$

Thus, we have an equation in which $\phi_2$ and its derivatives are the only unknown functions. We notice that the left–hand side of Eq.(3.16) is a fourth–order polynomial with respect to $\lambda$. However, coefficients in that polynomial are high–order differential operations. It is convenient to introduce special notation for this polynomial operation.

Let $\mathcal{P}(\cdot)$ be an operation defined by the formula

$$\mathcal{P}(\lambda)\phi_2 = EI\,GJ\phi_2^{VI}+EI(\lambda^2\tilde{I}_\alpha+\pi\rho u^2)\phi_2^{IV}-\lambda^2\tilde{m}GJ\phi_2''-[\lambda^2\pi\rho u^2\tilde{m}-\lambda^2(\pi\rho u)^2+\lambda^4\Delta]\phi_2, \tag{3.17}$$

where $\phi_2$ is a smooth function for $x \in [-L, 0]$. In order to determine the domain of $\mathcal{P}(\cdot)$, we have to calculate the conditions, which $\phi_2$ inherits from the domain of the operator $\mathcal{L}_{g_h g_\alpha}$. To do this, we will take the boundary conditions from the domain of the dynamics generator $\mathcal{L}_{g_h g_\alpha}$ (see (2.9)) and rewrite them in terms of $\phi_2$ and its derivatives. First of all, we need to write $\phi_1$ in terms of $\phi_2$. To carry out this elimination, we substitute Eq.(3.14) into Eq.(3.6)

$$\phi_1 = i\lambda - \frac{GJ}{\lambda^2 \tilde{S} + i\lambda\pi\rho u}\phi_2'' - \frac{\lambda^2 \tilde{I}_\alpha + \pi\rho u^2}{\lambda^2 \tilde{S} + i\lambda\pi\rho u}\phi_2. \tag{3.18}$$

Using Eq.(3.18) and the first and third condition in Eq.(2.9), we arrive at the following condition:

$$\phi_2''(-L) = 0. \tag{3.19}$$

Using Eq.(3.18) along with the second condition of Eq.(2.9), we have another condition

$$GJ\phi_2'''(-L) + \kappa(\lambda)\phi_2'(-L) = 0, \tag{3.20}$$

where

$$\kappa(\lambda) = \lambda^2 \tilde{I}_\alpha + \pi\rho u^2. \tag{3.21}$$

The third condition from Eq.(2.9), being only in terms of $\phi_2$, remains unchanged, i.e., $\phi_2(-L) = 0$. Substituting Eq.(3.14) into the fourth boundary condition in Eq.(2.9) yields

$$\kappa(\lambda)\phi_2'''(0) + GJ\phi_2^{v}(0) = 0. \tag{3.22}$$

Substituting the appropriate expressions Eq.(3.7), Eq.(3.14) and Eq.(3.18) into the fifth and sixth conditions of Eq.(2.9) yields the following two boundary conditions

$$g_h i\lambda\kappa(\lambda)\phi_2'(0) + EI\kappa(\lambda)\phi_2''(0) + g_h i\lambda GJ\phi_2'''(0) + EI\,GJ\phi_2^{iv}(0) = 0, \tag{3.23}$$

$$GJ\phi_2'(0) + i\lambda g_\alpha\phi_2(0) = 0. \tag{3.24}$$

Therefore, the problem of finding the eigenvalues and eigenfunctions of the operator $\mathcal{L}_{g_h g_\alpha}$ (see Eq.(3.5)) has been reduced to the problem of finding those values of the parameter $\lambda$ for which the sixth–order ordinary differential equation (3.21) has nontrivial solutions satisfying six boundary conditions.

Now we are in a position to introduce a pencil associated with the operator $\mathcal{L}_{g_h g_\alpha}$. We recall [32] that a *polynomial operator pencil* $A(\lambda)$ is an operator–valued function defined by the formula $A(\lambda) = \lambda^n + \lambda^{n-1}A_{n-1} + ... + A_0$ in which $A_k$ are linear operators. Those operators may be either bounded or unbounded and either selfadjoint or nonselfadjoint. The degree $n$ of this polynomial is called the order of the pencil. Let $\mathcal{P}(\cdot)$ be the fourth–order operator pencil that acts on a function $\phi \in H^6(-L,0)$ by the formula

$$\mathcal{P}(\lambda)\phi = EI\,GJ\phi^{VI} + EI(\lambda^2\tilde{I}_\alpha + \pi\rho u^2)\phi^{IV} - \lambda^2\tilde{m}GJ\phi'' - [\lambda^2\pi\rho u^2\tilde{m} - \lambda^2(\pi\rho u)^2 + \lambda^4\Delta]\phi \tag{3.25}$$

and is defined on the domain

$$\begin{aligned} D(\mathcal{P}) = \{\phi \in H^6(-L,0) : \ & \phi(-L) = \phi''(-L) = 0; \\ & GJ\psi'''(-L) + \kappa(\lambda)\psi'(-l) = 0; \\ & g_h i\lambda\kappa(\lambda)\psi'(0) + EI\kappa(\lambda)\psi''(0) + g_h i\lambda GJ\psi'''(0) + EIGJ\psi''''(0) = 0 \\ & \kappa(\lambda)\psi'''(0) + GJ\psi^{vi}(0) = 0; \\ & GJ\psi'(0) + i\lambda g_\alpha\psi(0) = 0.\} \end{aligned} \tag{3.26}$$

We note that $H^6$ is the standard Sobolev space [1]. We call a nontrivial solution $\phi \in D(\mathcal{P})$ of the pencil equation $\mathcal{P}(\lambda)\phi = 0$ an *eigenfunction* of the pencil $\mathcal{P}(\cdot)$ and the corresponding value of $\lambda$ an *eigenvalue* of $\mathcal{P}(\cdot)$. It is clear that having an eigenfunction of the pencil and using (3.19), we can find $\phi_1$ and then find all four components of the eigenvector of the operator $\mathcal{L}$.

We mention that $\mathcal{P}(\cdot)$ is a nonstandard pencil due to the fact that the spectral parameter $\lambda$ enters the domain explicitly. This type of pencil has not been considered in the monograph [32]. However, it is convenient to keep the terminology because there exists an extensive literature in which the pencils with the parameter dependent boundary conditions appear naturally.

### 3.3 An Alternative Derivation of the Pencil Equation

In this subsection, we present the derivation of the pencil problem by using slightly less rigorous approach, which is actively used in engineering literature. Let us look for a solution to our system in the form

$$\begin{pmatrix} h(t,x) \\ \alpha(t,x) \end{pmatrix} = e^{i\lambda t} \begin{pmatrix} \hat{h}(x) \\ \hat{\alpha}(x) \end{pmatrix}. \tag{3.27}$$

where $\left(\hat{h}(x), \hat{\alpha}(x)\right)^T$ does not depend on time. Substituting (3.27) into the system (2.1), we obtain the following equations for $\hat{h}$ and $\hat{\alpha}$

$$-\tilde{m}\hat{h}(x)\lambda^2 e^{i\lambda t} - \tilde{S}\hat{\alpha}(x)\lambda^2 e^{i\lambda t} + EIe^{i\lambda t}\hat{h}''''(x) + i\lambda\pi\rho u e^{i\lambda t}\hat{\alpha}(x) = 0, \tag{3.28}$$

$$-\tilde{S}e^{i\lambda t}\hat{h}(x)\lambda^2 - \tilde{I}_\alpha e^{i\lambda t}\hat{\alpha}(x)\lambda^2 - i\lambda\pi\rho u e^{i\lambda t}\hat{h}(x) - GJ\, e^{i\lambda t}\hat{\alpha}''(x) - \pi\rho u^2 e^{i\lambda t}\hat{\alpha}(x) = 0. \tag{3.29}$$

Eliminating $e^{i\lambda t}$ from (3.28) and (3.29), we obtain a new system for $\hat{h}$ and $\hat{\alpha}$

$$-\tilde{m}\hat{h}(x)\lambda^2 - \tilde{S}\hat{\alpha}(x)\lambda^2 + EI\hat{h}^{IV}(x) + i\lambda\pi\rho u\hat{\alpha}(x) = 0, \tag{3.30}$$

$$\tilde{S}\hat{h}(x)\lambda^2 + \tilde{I}_\alpha\hat{\alpha}(x)\lambda^2 + i\lambda\pi\rho u\hat{h}(x) + GJ\,\hat{\alpha}''(x) + \pi\rho u^2\hat{\alpha}(x) = 0. \tag{3.31}$$

Our next goal is to eliminate $\hat{h}$ from system (3.30), (3.31). To this end, we start with Eq.(3.31) and rewrite it as

$$\left(-\lambda^2\tilde{S} - i\lambda\pi\rho u\right)\hat{h}(x) = \left(\lambda^2\tilde{I}_\alpha + \pi\rho u^2\right)\hat{\alpha}(x) + GJ\hat{\alpha}''(x). \tag{3.32}$$

From Eq.(3.32), we obtain that

$$\hat{h}(x) = -\frac{(\lambda^2 \tilde{I}_\alpha + \pi\rho u^2)\hat{\alpha}(x) + GJ\hat{\alpha}''(x)}{\lambda^2 \tilde{S} + i\lambda\pi\rho u}. \tag{3.33}$$

Differentiating $\hat{h}$ four times, we have

$$\hat{h}^{IV}(x) = -\frac{(\lambda^2 \tilde{I}_\alpha + \pi\rho u^2)\hat{\alpha}^{IV}(x) + GJ\hat{\alpha}^{VI}(x)}{\lambda^2 \tilde{S} + i\lambda\pi\rho u}. \tag{3.34}$$

Substituting (3.33), (3.34) into Eq.(3.30), we obtain the following result

$$\begin{aligned} &\tilde{m}\lambda^2 \frac{(\lambda^2 \tilde{I}_\alpha + \pi\rho u^2)\hat{\alpha}(x) + GJ\hat{\alpha}''(x)}{\lambda^2 \tilde{S} + i\lambda\pi\rho u} - \tilde{S}\hat{\alpha}(x)\lambda^2 - \\ &EI\frac{(\lambda^2 \tilde{I}_\alpha + \pi\rho u^2)\hat{\alpha}''''(x) + GJ\hat{\alpha}^{VI}(x)}{\lambda^2 \tilde{S} + i\lambda\pi\rho u} + i\lambda\pi\rho u\hat{\alpha}(x) = 0. \end{aligned} \tag{3.35}$$

Simplifying all coefficients in Eq.(3.35), we arrive at the following final form of the equation with respect to $\hat{\alpha}$:

$$\begin{aligned} &\hat{\alpha}^{VI}(x) + (\lambda^2 \tilde{I}_\alpha + \pi\rho u^2)(GJ)^{-1}\hat{\alpha}^{IV}(x) - \tilde{m}\lambda^2(EI)^{-1}\hat{\alpha}''(x) + \\ &\left[(\tilde{S}^2 - \tilde{m}\tilde{I}_\alpha)\lambda^4 + (\rho^2 u^2 \pi^2 - \tilde{m}\pi\rho u^2)\lambda^2\right](GJ\,EI)^{-1}\hat{\alpha}(x) = 0. \end{aligned} \tag{3.36}$$

Eq. (3.36) is a sixth–order ordinary differential equation with respect to $\hat{\alpha}$ where $\hat{\alpha} = e^{-i\lambda t}\alpha$ being from our original system (2.1).

Now we have to rewrite six boundary conditions in terms of $\hat{\alpha}$ only, i.e., we will eliminate $\hat{h}$ from all conditions (2.10). We note that our first boundary condition for $\hat{\alpha}$ remains unchanged, i.e.,

$$\hat{\alpha}(-L) = 0. \tag{3.37}$$

We have to find two more left–end boundary conditions in terms of $\hat{\alpha}$. Recall that those conditions have the following forms:

$$\hat{h}(-L) = \hat{h}'(-L) = 0. \tag{3.38}$$

Evaluating Eq.(3.32) at $x = -L$, we obtain

$$-\tilde{S}\lambda^2\hat{h}(-L) - \tilde{I}_\alpha\lambda^2\hat{\alpha}(-L) - \pi\rho u i\lambda\hat{h}(-L) - GJ\hat{\alpha}''(-L) - \pi\rho u^2\hat{\alpha}(-L) = 0. \quad (3.39)$$

Using (3.37) and (3.8), we have the second boundary condition at $x = -L$

$$\hat{a}''(-L) = 0. \quad (3.40)$$

To find the third boundary condition at $x = -L$, let us differentiate Eq.(3.31) with respect to the spatial variable and have

$$\lambda^2 S\hat{h}'(x) + \lambda^2\tilde{I}_\alpha\hat{\alpha}'(x) + i\lambda\pi\rho u\hat{h}'(x) + GJ\hat{\alpha}'''(x) + \pi\rho u^2\hat{\alpha}'(x) = 0. \quad (3.41)$$

Taking into account that $\hat{h}'(-L) = 0$, we obtain for $x = -L$

$$GJ\hat{\alpha}'''(-L) + (\lambda^2\tilde{I}_\alpha + \pi\rho u^2)\hat{\alpha}'(-L) = 0. \quad (3.42)$$

Now we will derive the right boundary conditions. The first boundary condition is already given in terms of $\alpha$, i.e., $GJ\alpha'(t,0) + g_\alpha\alpha_t(t,0) = 0$ (see formulae (1.7)). Let

$$\kappa(\lambda) = \lambda^2\tilde{I}_\alpha + \pi\rho u^2, \quad (3.43)$$

then from Eq.(3.34), we have

$$\hat{h}(x) = -\frac{\kappa(\lambda)\hat{\alpha}(x) + GJ\hat{\alpha}''(x)}{\lambda^2\tilde{S} + i\lambda\pi\rho u}. \quad (3.44)$$

Substituting (3.44) into the boundary condition $EI\hat{h}''(0) + i\lambda g_h\hat{h}'(0) = 0$, we obtain

$$EI\frac{\kappa(\lambda)\hat{\alpha}''(0) + GJ\hat{\alpha}^{IV}(0)}{\lambda^2\tilde{S} + i\lambda\pi\rho u} + g_h i\lambda\frac{\kappa(\lambda)\hat{\alpha}'(0) + GJ\hat{\alpha}'''(0)}{\lambda^2\tilde{S} + i\lambda\pi\rho u} = 0. \quad (3.45)$$

Simplifying (3.45), we arrive at the second boundary condition

$$i\lambda g_h\kappa(\lambda)\hat{\alpha}'(0) + EI\kappa(\lambda)\hat{\alpha}''(0) + i\lambda g_h GJ\hat{\alpha}'''(0) + EI\,GJ\hat{\alpha}^{IV}(0) = 0. \quad (3.46)$$

Finally, we rewrite the remaining boundary condition $\hat{h}'''(0) = 0$ in terms of $\hat{\alpha}$. Differentiating twice Eq.(3.31), and evaluating at $x = 0$, we get

$$\lambda^2 \tilde{S}\hat{h}'''(0) + \lambda^2 \tilde{I}_\alpha \hat{\alpha}'''(0) + i\lambda\pi\rho u\hat{h}'''(0) + GJ\hat{\alpha}^V(0) + \pi\rho u^2\hat{\alpha}'''(0) = 0. \tag{3.47}$$

Due to the fact that $\hat{h}'''(0) = 0$, we obtain

$$-(\lambda^2 \tilde{I}_\alpha + \pi\rho u^2)\hat{\alpha}'''(0) - GJ\hat{\alpha}^V(0) = 0. \tag{3.48}$$

Thus, we have obtained the same boundary problem by using an alternative method.

## 3.4 Characteristic Equation

In this subsection, we initiate the analysis of the pencil equation $\mathcal{P}(\lambda)\phi = 0$. In particular, we will focus on the differential equation (3.16), which is a sixth-order ordinary differential equation with constant coefficients containing the complex parameter $\lambda$. We are looking for the asymptotic representation for the fundamental system of its solutions. It is important to mention that we are looking for the asymptotics with respect to $\lambda$ when $|\lambda| \to \infty$ and those asymptotics must be uniform with respect to the spatial variable $x \in [-L, 0]$.

Thus, from now on our main object of interest is the following equation:

$$\begin{gathered}\Psi^{VI}(x) + \lambda^2(1 + \pi\rho u I_\alpha^{-1}\lambda^{-2})(GJ)^{-1}\tilde{I}_\alpha\Psi^{IV}(x) - \lambda^2(EI)^{-1}\tilde{m}\Psi''(x) - \\ \lambda^4\left[1 - \Delta^{-1}(\rho^2u^2\pi^2 - \tilde{m}\pi\rho u^2)\lambda^{-2}\right](GJ)^{-1}(EI)^{-1}\Delta\Psi(x) = 0,\end{gathered} \tag{3.49}$$

with $\Delta$ being defined in formula (2.4). Let us introduce the following notation:

$$\begin{gathered}A = (GJ)^{-1}\tilde{I}_\alpha, \quad B(u) = \tilde{I}_\alpha^{-1}\pi\rho u^2, \quad C = (EI)^{-1}\tilde{m}, \\ D = \Delta(GJ)^{-1}(EI)^{-1}, \quad F(u) = \Delta^{-1}(\rho^2u^2\pi^2 - \tilde{m}\pi\rho u^2).\end{gathered} \tag{3.50}$$

In terms of (3.51), Eq.(3.50) has the form

$$\Psi^{VI}(x)+\lambda^2(1+\lambda^{-2}B(u))A\Psi^{IV}(x)-C\lambda^2\Psi''(x)-\left[1-F(u)\lambda^{-2}\right]D\lambda^4\Psi(x)=0. \quad (3.51)$$

As is well-known, to find the fundamental system of solutions of the sixth-order ordinary differential equation, we have to find approximations for the roots of the sixth order polynomial, which is the characteristic polynomial for the differential equation. The characteristic equation for Eq.(3.51) has the following form:

$$x^6+A\lambda^2(1+\lambda^{-2}B(u))x^4-C\lambda^2x^2-D\lambda^4[1-\lambda^{-2}F(u)]=0. \quad (3.52)$$

Through the rest of this section, we analyze Eq. (3.52).

Clearly, Eq.(3.52) is of a sixth order, but if we change an independent variable, we can reduce it to a cubic equation. Namely, let

$$y=x^2, \quad (3.53)$$

and then rewrite Eq.(3.52) in terms of $y$, as

$$y^3+\tilde{A}\lambda^2y^2-C\lambda^2y-\tilde{D}\lambda^4=0, \quad (3.54)$$

where

$$\tilde{A}=A(1+B(u)\lambda^{-2}), \quad \tilde{D}=D(1-F(u)\lambda^{-2}). \quad (3.55)$$

Using the Cardano's Formulae [2], one can obtain the roots of a cubic polynomial. However, in order to apply those formulae, we need the cubic polynomial to be monic with no quadratic term. We already have a monic-polynomial. Next we need to make the quadratic term in (3.54) vanish. To do so, we notice that for a polynomial

$$f(y)=y^3+a_2y^2+a_1y+a_0, \quad (3.56)$$

the substitution

$$y=z-a_2/3 \quad (3.57)$$

will result in a polynomial with respect to $z$, with no quadratic term. So if we substitute

$$y = z - \frac{\tilde{A}\lambda^2}{3} \tag{3.58}$$

into (3.54), then we have an equation, for which we can use the Cardano's Formulae. Taking into account that

$$\begin{aligned} y^3 = y(y^2) &= \left(z - \frac{\tilde{A}\lambda^2}{3}\right)\left(z^2 - \frac{2\tilde{A}\lambda^2}{3}z + \left(\frac{\tilde{A}\lambda^2}{3}\right)^2\right) \\ &= z^3 - \tilde{A}\lambda^2 z^2 + \frac{1}{3}\left(\tilde{A}\lambda^2\right)^2 z - \left(\frac{\tilde{A}\lambda^2}{3}\right)^3, \end{aligned} \tag{3.59}$$

and substituting (3.58) and (3.59) into Eq.(3.54), we obtain

$$\begin{aligned} &\left[z^3 - \tilde{A}\lambda^2 z^2 + \frac{1}{3}\left(\tilde{A}\lambda^2\right)^2 z - \left(\frac{\tilde{A}\lambda^2}{3}\right)^3\right] \\ &+\tilde{A}\lambda^2\left[z^2 - \frac{2\tilde{A}\lambda^2}{3}z + \left(\frac{\tilde{A}\lambda^2}{3}\right)^2\right] - C\lambda^2\left[z - \frac{\tilde{A}\lambda^2}{3}\right] - \tilde{D}\lambda^4 = 0, \end{aligned} \tag{3.60}$$

and cancel the two quadratic terms as expected. Then we combine same powers of $z$ and have the following cubic equation:

$$z^3 + \left[-\frac{\tilde{A}^2}{3}\lambda^4 - C\lambda^2\right] z + \frac{2\tilde{A}^3}{3^3}\lambda^6 + \left(\frac{C\tilde{A}}{3} - \tilde{D}\right)\lambda^4 = 0. \tag{3.61}$$

It is this equation that we will apply Cardano's Formulae to in the next section.

• We recall that our goal is to find the asymptotic approximations for six roots of Eq.(3.54) when $|\lambda| \to \infty$. However, before proceeding to find these six roots, we will see what knowledge may be gained by investigating the asymptotic behavior of the solutions of a simpler equation than Eq.(3.61). We will call this simpler equation the *model equation.* Let us rewrite Eq.(3.61) in the asymptotical form as $|\lambda| \to \infty$. We have

$$z^3 - \frac{\tilde{A}^2}{3}\lambda^4(1 + O(\lambda^{-2}))z + \lambda^6\frac{2\tilde{A}^3}{3^3}(1 + O(\lambda^{-2}) = 0. \tag{3.62}$$

By omitting $O(\lambda^{-2})$ in Eq.(3.62), we obtain the model equation of the form

$$z^3 - \frac{\tilde{A}^2}{3}\lambda^4 z + \lambda^6\frac{2\tilde{A}^3}{3^3} = 0. \tag{3.63}$$

Assuming that a solution $z_1$ will be a multiple of $\lambda^2$, we substitute

$$z_1 = a\lambda^2 \tag{3.64}$$

into Eq.(3.63) and divide by $\lambda^6$ to obtain a cubic equation for the multiple $a$

$$a^3 - \tilde{A}^2\frac{1}{3}a + \frac{2}{3^3}\tilde{A}^3 = 0, \tag{3.65}$$

from which we guess that a solution $a$ will be a multiple of $\tilde{A}$. So, now making the substitution $a = b\tilde{A}$ and then dividing by $\tilde{A}^3$ yields

$$b^3 - \frac{1}{3}b + \frac{2}{3^3} = 0. \tag{3.66}$$

One can check directly that a solution of this equation is $b = 1/3$. Thus we find that one solution $z_1$ of the model equation (3.63) can be found by successive substitutions into (3.66) as follows:

$$z_1 = a\lambda^2 = \frac{\tilde{A}}{3}\lambda^2 = \frac{\tilde{I}_\alpha}{3GJ}(1 + \lambda^{-2}B(u))\lambda^2 \equiv \lambda^2\frac{R}{3}, \tag{3.67}$$

where

$$R = \frac{\tilde{I}_\alpha}{GJ}(1 + \lambda^{-2}B(u)). \tag{3.68}$$

Factoring out the model equation (3.62), we obtain

$$\left(z - \frac{\tilde{A}\lambda^2}{3}\right)\left(z^2 + \frac{\tilde{A}\lambda^2}{3}z - \frac{2}{9}\tilde{A}^2\lambda^4\right) = 0. \tag{3.69}$$

The following expressions for the roots of the quadratic polynomial from (3.69) are valid:

$$z_2 = \frac{1}{3}\tilde{A}\lambda^2, \qquad z_3 = -\frac{2}{3}\tilde{A}\lambda^2. \tag{3.70}$$

Thus we have the following three solutions to the model equation:

$$z_1 = z_2 = \frac{\tilde{A}\lambda^2}{3}, \qquad z_3 = -\frac{2}{3}\tilde{A}\lambda^2. \tag{3.71}$$

Note that $z_1 = z_2$. From the latter fact we can expect that two roots of Eq.(3.62) will have a similar behavior in nature, while the third solution will behave differently. This exact difference in behavior remains to be seen.

# CHAPTER IV

# ASYMPTOTIC ANALYSIS OF THE ROOTS OF THE CHARACTERISTIC POLYNOMIAL

## 4.1 Cardano's Formulae

It is well–known that Cardano's Formulae [2] gives a solution for a monic cubic equation with no quadratic term such as

$$z^3 + pz + q = 0, \tag{4.1}$$

where $p$ and $q$ are constants. We will use the following version of Cardano's Formulae:

$$z_1 = \left[-\frac{q}{2} + \sqrt{\frac{4p^3+27q^2}{108}}\right]^{1/3} + \left[-\frac{q}{2} - \sqrt{\frac{4p^3+27q^2}{108}}\right]^{1/3}, \tag{4.2}$$

$$z_2 = \left[-\frac{q}{2} + \sqrt{\frac{4p^3+27q^2}{108}}\right]^{1/3} + w^2\left[-\frac{q}{2} - \sqrt{\frac{4p^3+27q^2}{108}}\right]^{1/3}, \tag{4.3}$$

$$z_3 = \left[-\frac{q}{2} + \sqrt{\frac{4p^3+27q^2}{108}}\right]^{1/3} + w\left[-\frac{q}{2} - \sqrt{\frac{4p^3+27q^2}{108}}\right]^{1/3}, \tag{4.4}$$

where $w$ is a cube root of unity, but $w \neq 1$. Noticing that $\frac{4p^3+27q^2}{108} = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2$, we start with calculations of the asymptotic approximations for the quantities $(p/3)^3$ and $(q/2)^2$.

- From formula (3.62), we have

$$\begin{aligned} p^3(\lambda) &\equiv \left(-\frac{\tilde{A}^2}{3}\lambda^4 - C\lambda^2\right)^3 = -\frac{\tilde{A}^6}{3^3}\lambda^{12} - 3\frac{\tilde{A}^4 C}{3^2}\lambda^{10} - 3\frac{\tilde{A}^2 C^2}{3}\lambda^8 - C^3\lambda^6 \\ &= -\left(\frac{\tilde{A}^6}{3^3}\lambda^{12} + \frac{\tilde{A}^4 C}{3}\lambda^{10}\right)\left(1 + O(\lambda^{-4})\right). \end{aligned} \tag{4.5}$$

Using formulae (3.50) and (3.55) for $\tilde{A}$, we rewrite (4.5) and have

$$p^3(\lambda) = -\left(\frac{\left((GJ)^{-1}\tilde{I}_\alpha\right)^6 (1+O(\lambda^{-2}))^6}{3^3}\lambda^{12} + \frac{\left((GJ)^{-1}\tilde{I}_\alpha\right)^4 (EI)^{-1}\tilde{m}}{3}(1+O(\lambda^{-2}))^4\lambda^{10}\right)(1+O(\lambda^{-4})) \tag{4.6}$$

Simplifying the latter expression, we finally obtain

$$p^3(\lambda) = -\left[\frac{((GJ)^{-1}\tilde{I}_\alpha)^6}{3^3}\lambda^{12}(1+O(\lambda^{-2})) + \frac{((GJ)^{-1}\tilde{I}_\alpha)^4(EI)^{-1}\tilde{m}}{3}\lambda^{10}(1+O(\lambda^{-2}))\right]. \tag{4.7}$$

- Asymptotic approximation for $q^2(\lambda)$ has the form

$$\begin{aligned} q^2(\lambda) &= \frac{4\tilde{A}^6}{3^6}\lambda^{12} + \frac{4\tilde{A}^3}{3^3}\left(\frac{C\tilde{A}}{3} - \tilde{D}\right)\lambda^{10} + \left(\frac{C^2\tilde{A}^2}{3^2} - \frac{2C\tilde{A}\tilde{D}}{3} + \tilde{D}^2\right) \\ &= \left(\frac{4\tilde{A}^6}{3^6}\lambda^{12} + \frac{4\tilde{A}^3}{3^3}\left(\frac{C\tilde{A}}{3} - \tilde{D}\right)\lambda^{10}\right)(1+O(\lambda^{-4})) \end{aligned} \tag{4.8}$$

Using formulae (3.50) and (3.55), we obtain

$$\begin{aligned} q^2(\lambda) = \Bigg[&\frac{4}{3^6}\left((GJ)^{-1}\tilde{I}_\alpha\right)^6(1+O(\lambda^{-2}))^6\lambda^{12} + \frac{4}{3^3}\left((GJ)^{-1}\tilde{I}_\alpha\right)^3(1+O(\lambda^{-2}))^3 \\ &\times\left[\frac{(EI)^{-1}\tilde{m}(GJ)^{-1}\tilde{I}_\alpha}{3} - \frac{\tilde{m}\tilde{I}_\alpha - \tilde{S}^2}{GJ\,EI}\right](1+O(\lambda^{-2}))\Bigg](1+O(\lambda^{-4})). \end{aligned} \tag{4.9}$$

Simplifying (4.9), we finally obtain

$$\begin{aligned} q^2(\lambda) = \frac{4}{3^3}\left((GJ)^{-1}\tilde{I}_\alpha\right)^3\Bigg[&\left(\frac{(GJ)^{-1}\tilde{I}_\alpha}{3}\right)^3\lambda^{12}(1+O(\lambda^{-2})) \\ &+\left(\tilde{S}^2(GJ\,EI)^{-1} - \frac{2}{3}\tilde{m}\tilde{I}_\alpha(GJ)^{-1}(EI)^{-1}\right)\lambda^{10}(1+O(\lambda^{-2}))\Bigg]. \end{aligned} \tag{4.10}$$

• Let us denote by $Q$, the following quantity:

$$Q = \left| \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 \right|^{1/2}. \tag{4.11}$$

Performing detailed asymptotic analysis, we obtain that $Q$ can be approximated as follows:

$$\begin{aligned}
Q^2 &= \left| \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 \right| \\
&= \left| -\frac{\left((GJ)^{-1}\tilde{I}_\alpha\right)^6}{3^6}\lambda^{12} - \frac{\left((GJ)^{-1}\tilde{I}_\alpha\right)^4 (EI)^{-1}\tilde{m}}{3^4}\lambda^{10} + \frac{\left((GJ)^{-1}\tilde{I}_\alpha\right)^6}{3^6}\lambda^{12} \right. \\
&\quad \left. + \frac{\left((GJ)^{-1}\tilde{I}_\alpha\right)^3 \tilde{S}^2 (GJ)^{-1}(EI)^{-1}}{3^3}\lambda^{10} - \frac{2\left((GJ)^{-1}\tilde{I}_\alpha\right)^4}{3^4}(EI)^{-1}\tilde{m}\lambda^{10} \right| \left(1 + O(\lambda^{-2})\right) \\
&= \left| \frac{\left((GJ)^{-1}\tilde{I}_\alpha\right)^3}{3^3} \left[\tilde{S}^2 (GJ)^{-1}(EI)^{-1} - (GJ)^{-1}\tilde{I}_\alpha (EI)^{-1}\tilde{m}\right] \lambda^{10} \right| \left(1 + O(\lambda^{-2})\right).
\end{aligned} \tag{4.12}$$

Therefore, we have

$$\begin{aligned}
Q &= \frac{\left((GJ)^{-1}\tilde{I}_\alpha\right)^{3/2}}{3\sqrt{3}} \left[\tilde{S}^2 (GJ)^{-1}(EI)^{-1} - (GJ)^{-1}\tilde{I}_\alpha (EI)^{-1}\tilde{m}\right]^{1/2} \lambda^5 \left(1 + O(\lambda^{-2})\right) \\
&= \frac{(GJ)^{-3/2}\tilde{I}_\alpha^{3/2}(GJ)^{-1/2}(EI)^{-1/2}}{3\sqrt{3}} \Delta^{1/2}\lambda^5 \left(1 + O(\lambda^{-2})\right) \\
&= \frac{(GJ)^{-2}\tilde{I}_\alpha^{3/2}(EI)^{-1/2}}{3\sqrt{3}} \Delta^{1/2}\lambda^5 \left(1 + O(\lambda^{-2})\right).
\end{aligned} \tag{4.13}$$

• Now we notice that representations (4.2)–(4.4) can be written in the forms

$$z_1 = -\left[\frac{q}{2} - iQ\right]^{1/3} - \left[\frac{q}{2} + iQ\right]^{1/3},$$

$$z_2 = \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\left[\frac{q}{2} - iQ\right]^{1/3} + \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\left[\frac{q}{2} + iQ\right]^{1/3}, \tag{4.14}$$

$$z_3 = \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\left[\frac{q}{2} - iQ\right]^{1/3} + \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\left[\frac{q}{2} + iQ\right]^{1/3}.$$

To find asymptotic approximations for the roots $z_i$, $i = 1, 2, 3$, we first calculate asymptotic approximation for $q/2 - iQ$ and have

$$\frac{q}{2} - iQ = \left[\frac{\left((GJ)^{-1}\tilde{I}_\alpha\right)^3}{27}\lambda^6 + \left(\frac{(GJ)^{-1}\tilde{I}_\alpha(EI)^{-1}\tilde{m}}{6} - \frac{(\tilde{m}\tilde{I}_\alpha - \tilde{S}^2)(GJ)^{-1}(EI)^{-1}}{2}\right)\lambda^4\right.$$

$$\left. - \frac{i\left((GJ)^{-1}\tilde{I}_\alpha\right)^{3/2}}{3\sqrt{3}}\left[\tilde{S}^2(GJ\,EI)^{-1} - (GJ\,EI)^{-1}\tilde{I}_\alpha\tilde{m}\right]^{1/2}\lambda^5\right]\left[1 + O(\lambda^{-2})\right]$$

$$= \left(\frac{(GJ)^{-1}\tilde{I}_\alpha}{3}\right)^3\lambda^6\left[1 - i\frac{3^2}{\sqrt{3}}\left((GJ)^{-1}\tilde{I}_\alpha\right)^{-3/2}\left[\tilde{S}^2(GJ)^{-1}(EI)^{-1}\right.\right.$$

$$\left.\left. -(GJ)^{-1}\tilde{I}_\alpha(EI)^{-1}\tilde{m}\right]^{1/2}\lambda^{-1}\right]\left[1 + O(\lambda^{-2})\right]. \tag{4.15}$$

Using the Binomial Theorem, we evaluate the cube root and have from (4.15)

$$\left[\frac{q}{2}-iQ\right]^{1/3}=\frac{(GJ)^{-1}\tilde{I}_\alpha}{3}\lambda^2\left[1-3^{3/2}i\left((GJ)^{-1}\tilde{I}_\alpha\right)^{-3/2}\times\right.$$

$$\left.(GJ\,EI)^{-1/2}[\tilde{S}^2-\tilde{I}_\alpha\tilde{m}]^{1/2}\lambda^{-1}\right]^{1/3}\left[1+O(\lambda^{-2})\right]$$

$$=\frac{(GJ)^{-1}\tilde{I}_\alpha}{3}\lambda^2\left[1-\frac{3^{3/2}i\left((GJ)^{-1}\tilde{I}_\alpha\right)^{-3/2}(GJ\,EI)^{-1/2}\Delta^{1/2}}{3\lambda}+\right. \tag{4.16}$$

$$\left.O(\lambda^{-2})\right]\left[1+O(\lambda^{-2})\right]$$

$$=\frac{(GJ)^{-1}\tilde{I}_\alpha}{3}\lambda^2\left[1-\sqrt{3}i\left((GJ)^{-1}E_\alpha\right)^{-3/2}(GJ\,EI)^{-1/2}\Delta^{1/2}\lambda^{-1}+\right.$$

$$\left.O(\lambda^{-2})\right]\left[1+O(\lambda^{-2})\right].$$

Asymptotic approximation for the expression $(q/2+iQ)^{1/3}$ has the form

$$\left[\frac{q}{2}+iQ\right]^{1/3}=\frac{(GJ)^{-1}\tilde{I}_\alpha}{3}\lambda^2\left[1+\sqrt{3}i\left((GJ)^{-1}\tilde{I}_\alpha\right)^{-3/2}(GJ\,EI)^{-1/2}\Delta^{1/2}\lambda^{-1}\right]\left[1+O(\lambda^{-2})\right]$$

$$=\frac{(GJ)^{-1}\tilde{I}_\alpha}{3}\lambda^2\left[1+i\sqrt{3}GJ\tilde{I}_\alpha^{-3/2}(EI)^{-1/2}\Delta^{1/2}\lambda^{-1}\right]\left(1+O(\lambda^{-2})\right)$$

$$\equiv\frac{(GJ)^{-1}\tilde{I}_\alpha}{3}\lambda^2\left[1+C_1\lambda^{-1}\right]\left(1+O(\lambda^{-2})\right). \tag{4.17}$$

Combining (4.16) and (4.17), we can find the approximations for the roots. Namely, for $z_1$ we have

$$z_1 = -\left[\frac{q}{2} - iQ\right]^{1/3} - \left[\frac{q}{2} + iQ\right]^{1/3}$$

$$= -\frac{(GJ)^{-1}\tilde{I}_\alpha}{3}\lambda^2 \Big[1 + \sqrt{3}i\left((GJ)^{-1}EI\right)^{-3/2}(GJ\,EI)^{-1/2}\Delta^{1/2}\lambda^{-1} \tag{4.18}$$

$$+1 - \sqrt{3}i\left((GJ)^{-1}EI\right)^{-3/2}(GJ\,EI)^{-1/2}\Delta^{1/2}\lambda^{-1}\Big]\left[1 + O(\lambda^{-2})\right].$$

By simplifying (4.18), we obtain that

$$z_1 = -\frac{2(GJ)^{-1}\tilde{I}_\alpha}{3}\lambda^2\left[1 + O(\lambda^{-2})\right]. \tag{4.19}$$

- Substituting (4.16), and (4.17) into (4.14), we obtain for the second root

$$z_2 = \frac{(GJ)^{-1}I_\alpha}{3}\lambda^2\left[\left(\frac{1}{2} - i\frac{\sqrt{3}}{2} + \frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{2} - i\frac{\sqrt{3}}{2} - \frac{1}{2} - i\frac{\sqrt{3}}{2}\right)C_1\lambda^{-1}\right]$$

$$\times\left[1 + O(\lambda^{-2})\right] = \frac{(GJ)^{-1}I_\alpha}{3}\lambda^2\left[1 - \sqrt{3}iC_1\lambda^{-1} + O(\lambda^{-2})\right]\left[1 + O(\lambda^{-2})\right], \tag{4.20}$$

where $C_1$ is defined in (4.17). By simplifying the latter expression, we obtain that

$$z_2 = \frac{(GJ)^{-1}\tilde{I}_\alpha}{3}\lambda^2\left[1 + 3GJ\tilde{I}_\alpha^{-3/2}(EI)^{-1/2}\Delta^{1/2}\lambda^{-1}\right]\left[1 + O(\lambda^{-2})\right]. \tag{4.21}$$

Completing similar calculations, we obtain

$$z_3 = \frac{(GJ)^{-1}\tilde{I}_\alpha}{3}\lambda^2\left[\left(\frac{1}{2} + i\frac{\sqrt{3}}{2} + \frac{1}{2} - i\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{2} + i\frac{\sqrt{3}}{2} - \frac{1}{2} + i\frac{\sqrt{3}}{2}\right)C_1\lambda^{-1}\right]$$

$$\times\left[1 + O(\lambda^{-2})\right] = \frac{(GJ)^{-1}\tilde{I}_\alpha}{3}\lambda^2\left[1 + \sqrt{3}iC_1\lambda^{-1} + O(\lambda^{-2})\right]\left[1 + O(\lambda^{-2})\right]. \tag{4.22}$$

By simplifying (4.22), we obtain that

$$z_3 = \frac{(GJ)^{-1}\tilde{I}_\alpha}{3}\lambda^2\left[1 - 3GJ\tilde{I}_\alpha^{-3/2}(EI)^{-1/2}\Delta^{1/2}\lambda^{-1}\right]\left[1 + O(\lambda^{-2})\right]. \tag{4.23}$$

Now we return to the original variable $y$. Recall that we made substitution (3.53)

$$x^2 = y, \tag{4.24}$$

to obtain a cubic polynomial. Next we substituted the shift of (3.57) to obtain a cubic equation with no quadratic term. Making this substitution into (4.24) gave us

$$x^2 = z - \frac{\tilde{A}\lambda^2}{3}. \tag{4.25}$$

Now we have approximations for three roots $z_1$, $z_2$, and $z_3$ of the reduced equation.

• Now we are in a position to find the asymptotic approximations for the roots of the characteristic polynomial (3.52). As the first step, we can write the formulae for $y_i = z_i - \frac{(GJ)^{-1}\tilde{I}_\alpha}{3}\lambda^2\left(1 + O(\lambda^{-2})\right)$, $i = 1, 2, 3$ as follows:

$$\begin{aligned} y_1 &= \left[-\frac{2(GJ)^{-1}\tilde{I}_\alpha}{3}\lambda^2\left[1 + O(\lambda^{-2})\right] - \frac{(GJ)^{-1}\tilde{I}_\alpha}{3}\lambda^2\left[1 + O(\lambda^{-2})\right]\right] \\ &= -\lambda^2(GJ)^{-1}\tilde{I}_\alpha\left[1 + O(\lambda^{-2})\right], \end{aligned} \tag{4.26}$$

$$\begin{aligned} y_2 &= \left[\left((GJ)^{-1}\tilde{I}_\alpha\right)^{-1/2}(GJ\,EI)^{-1/2}\Delta^{1/2}\lambda + O(1)\right]\left[1 + O(\lambda^{-2})\right] \\ &= \lambda(EI\,\tilde{I}_\alpha)^{-1/2}\Delta^{1/2}\left[1 + O(\lambda^{-1})\right], \end{aligned} \tag{4.27}$$

$$\begin{aligned} y_3 &= \left[-\left((GJ)^{-1}\tilde{I}_\alpha\right)^{-1/2}(GJ\,EI)^{-1/2}\Delta^{1/2}\lambda + O(1)\right]\left[1 + O(\lambda^{-2})\right] \\ &= -\lambda(EI\,\tilde{I}_\alpha)^{-1/2}\Delta^{1/2}\left[1 + O(\lambda^{-1})\right]. \end{aligned} \tag{4.28}$$

Notice that $y_3 = -y_2$. Taking into account that $y = x^2$, we can finally write the asymptotic approximations for all six roots of the main characteristic equation (3.53).

$$x_{1,2} = \pm i \left(\frac{\tilde{I}_\alpha}{GJ}\right)^{1/2} \lambda \left[1 + O(\lambda^{-2})\right], \tag{4.29}$$

$$x_{3,4} = \pm \left(\frac{\Delta}{EI\,\tilde{I}_\alpha}\right)^{1/4} \lambda^{1/2} \left[1 + O(\lambda^{-2})\right], \tag{4.30}$$

$$x_{5,6} = \pm i \left(\frac{\Delta}{EI\,\tilde{I}_\alpha}\right)^{1/4} \lambda^{1/2} \left[1 + O(\lambda^{-2})\right]. \tag{4.31}$$

**Remark 4.1.** At this moment, we point out that, as follows form formulae (4.29)–(4.31), there is obvious difference in the formulae (4.29) and (4.30), (4.31). Namely, the parameter $\lambda$ enters formulae (4.29) as $\lambda$, while the same parameter enters formula (4.30) and (4.31) as $\sqrt{\lambda}$. Due to this difference, the asymptotic analysis in the remaining part of the proof will be significantly complicated.

## 4.2 General Solution to the Spectral Equation for the Operator Pencil

In this subsection, we bring the results of the last section together in order to write the general solution of the differential equation $\mathcal{P}(\lambda)\phi = 0$.

The general solution to the sixth–order ordinary differential equation can be represented as a linear combination of exponential–like functions. To simplify subsequent calculations, we introduce the notation

$$x_{1,2} = \pm i\gamma(\lambda), \qquad x_{3,4} = \pm i\hat{\gamma}(\lambda), \qquad x_{5,6} = \pm\Gamma(\lambda), \tag{4.32}$$

where complex–valued functions $\Gamma$, $\gamma$, and $\hat{\gamma}$ are defined by the following formulae:

$$\gamma(\lambda) = Q\lambda + O(\lambda^{-1}) = Q\lambda(1 + O(\lambda^{-1})), \tag{4.33}$$

$$\hat{\gamma}(\lambda) = P\lambda^{1/2} + O(\lambda^{-1/2}) = P\lambda^{1/2}(1 + O(\lambda^{-1})), \tag{4.34}$$

$$\Gamma(\lambda) = P\lambda^{1/2} + O(\lambda^{-1/2}) = P\lambda^{1/2}(1 + O(\lambda^{-2})), \tag{4.35}$$

$$\text{where } P = \left(\frac{\Delta}{\tilde{I}_\alpha EI}\right)^{1/4}, \quad Q \equiv R^{1/2} = \left(\frac{\tilde{I}_\alpha}{GJ}\right)^{1/2}. \tag{4.36}$$

Using notations (4.33)–(4.36), we may write the general solution $\Psi$ of the equation $\mathcal{P}(\lambda)\phi = 0$ in the following form:

$$\begin{aligned}\Psi(x,\lambda) =& \mathcal{A}(\lambda)e^{\gamma(\lambda)(x+L)} + \mathcal{B}(\lambda)e^{i\hat{\gamma}(\lambda)(x+L)} + \mathcal{C}(\lambda)e^{i\Gamma(\lambda)(x+L)} + \\ & \mathcal{D}(\lambda)e^{-\gamma(\lambda)(x+L)} + \mathcal{E}(\lambda)e^{-i\hat{\gamma}(\lambda)(x+L)} + \mathcal{F}(\lambda)e^{-i\Gamma(\lambda)(x+L)}.\end{aligned} \tag{4.37}$$

In the representation (4.37), there are six arbitrary functions of $\lambda$. By imposing six boundary conditions, which have to be satisfied by the solution $\Psi(x,\lambda)$, we will specify both the spectral points and the approximations for the aforementioned six functions of $\lambda$ from (4.37).

# CHAPTER V

# THE LEFT-REFLECTION MATRIX

## 5.1 Two-step Procedure for Applying Boundary Conditions

As stated in Section 3.2, our ultimate goal is to find such values of the complex parameter $\lambda$, for which the equation $\mathcal{L}_{g_h g_\alpha}\Psi = \lambda\Phi$ has nontrivial solutions, i.e., to find the eigenvalues and eigenvectors of the operator $\mathcal{L}_{g_h g_\alpha}$. We have shown that the aforementioned problem is equivalent to the problem of finding eigenvalues and eigenfunctions of the pencil $\mathcal{P}(\cdot)$, i.e., to the problem of finding the values of $\lambda$ for which the equation $\mathcal{P}(\lambda)\phi = 0$ has nontrivial solutions. This is exactly the problem we will focus on in Chapters V–VII. Thus, we are looking for a solution of the equation

$$\mathcal{P}(\lambda)\Psi = 0, \tag{5.1}$$

which can be represented in the form (4.37). More precisely, we are looking for those values of $\lambda$, $\lambda \in \mathbb{C}$, for which there exist coefficients $\mathcal{A}(\lambda)$, $\mathcal{B}(\lambda)$, $\mathcal{C}(\lambda)$, $\mathcal{D}(\lambda)$, $\mathcal{E}(\lambda)$, and $\mathcal{F}(\lambda)$ such that $\Psi(x,\lambda)$ satisfies the boundary conditions given in (3.26).

Substituting $\Psi$ into those boundary conditions gives us a linear system of six equations in six unknowns $\mathcal{A}(\lambda)$, $\mathcal{B}(\lambda)$, $\mathcal{C}(\lambda)$, $\mathcal{D}(\lambda)$, $\mathcal{E}(\lambda)$, and $\mathcal{F}(\lambda)$. Let $\mathbb{M}$ be the $6\times 6$ matrix of coefficients from the aforementioned system. Since our system is homogeneous, it can be written in the form $\mathbb{M}\mathbb{Z} = 0$, where $\mathbb{Z}^T = \{\mathcal{A}(\lambda), \mathcal{B}(\lambda), \mathcal{C}(\lambda), \mathcal{D}(\lambda), \mathcal{E}(\lambda), \mathcal{F}(\lambda)\}$. Thus, we have to find approximations for the solutions of the equation $det\,\mathbb{M}(\lambda) = 0$. It turns out that directly finding approximations for the roots of this determinant is an extremely difficult problem. So, we suggest an alternative approach.

• **Description of a two–step procedure for finding the spectral asymptotics.**

Let us introduce two 3–component vectors

$$X(\lambda) = (\mathcal{A}(\lambda), \mathcal{B}(\lambda), \mathcal{C}(\lambda))^T, \quad Y(\lambda) = (\mathcal{D}(\lambda), \mathcal{E}(\lambda), \mathcal{F}(\lambda))^T, \tag{5.2}$$

and first select only three boundary conditions, the conditions which have to be imposed on the solution $\Psi$ to satisfy the boundary conditions at the left end of the beam model. As a result, we obtain the relation between the vectors $X(\cdot)$ and $Y(\cdot)$, which can be written in the form

$$X(\lambda) = \mathbb{R}_l(\lambda)Y(\lambda). \tag{5.3}$$

We will call a corresponding $3 \times 3$ matrix $\mathbb{R}_l(\cdot)$ in (5.3) *the left–reflection matrix.* Therefore, if the two vectors $X(\cdot)$ and $Y(\cdot)$ are connected through the left–reflection matrix, the corresponding function (4.37) satisfies equation (5.1) and only the left–end boundary conditions. Secondly, let us select only the right–end boundary conditions. We obtain from three right–end boundary conditions, that the following relation between $X(\cdot)$ and $Y(\cdot)$ holds:

$$X(\lambda) = \mathbb{R}_r(\lambda)Y(\lambda), \tag{5.4}$$

where the $3 \times 3$ matrix $\mathbb{R}_r(\cdot)$ will be called *the right–reflection matrix.* So, if the vectors $X(\cdot)$ and $Y(\cdot)$ are connected through relation (5.4), the corresponding function (4.37) satisfies equation (5.1) and three boundary conditions at the right end.

It can be easily verified that to satisfy all six boundary conditions, the following equation must be satisfied for those of $\lambda$, where $\mathbb{R}_l^{-1}(\cdot)$ exists:

$$
\begin{pmatrix} \mathcal{A}(\lambda) \\ \mathcal{B}(\lambda) \\ \mathcal{C}(\lambda) \\ \mathcal{D}(\lambda) \\ \mathcal{E}(\lambda) \\ \mathcal{F}(\lambda) \end{pmatrix} = \left[ \begin{array}{c|c} 0 & \mathbb{R}_r(\lambda) \\ \hline \mathbb{R}_l^{-1}(\lambda) & 0 \end{array} \right] \begin{pmatrix} \mathcal{A}(\lambda) \\ \mathcal{B}(\lambda) \\ \mathcal{C}(\lambda) \\ \mathcal{D}(\lambda) \\ \mathcal{E}(\lambda) \\ \mathcal{F}(\lambda) \end{pmatrix}. \tag{5.5}
$$

Eq.(5.5) is certainly equivalent to the following one:

$$
\left( \mathbb{I} - \left[ \begin{array}{c|c} 0 & \mathbb{R}_r(\lambda) \\ \hline \mathbb{R}_l^{-1}(\lambda) & 0 \end{array} \right] \right) \begin{pmatrix} \mathcal{A}(\lambda) \\ \mathcal{B}(\lambda) \\ \mathcal{C}(\lambda) \\ \mathcal{D}(\lambda) \\ \mathcal{E}(\lambda) \\ \mathcal{F}(\lambda) \end{pmatrix} = 0, \tag{5.6}
$$

where $\mathbb{I}$ is the identity matrix. We notice that a solution of Eq.(5.6) is nontrivial if and only if

$$
\det \left( \mathbb{I} - \left[ \begin{array}{c|c} 0 & \mathbb{R}_r(\lambda) \\ \hline \mathbb{R}_l^{-1}(\lambda) & 0, \end{array} \right] \right) = 0, \tag{5.7}
$$

or equivalently

$$
\det(\mathbb{I} - \mathbb{R}_l^{-1}(\lambda)\mathbb{R}_r(\lambda)) = 0. \tag{5.8}
$$

We may factor out $\mathbb{R}_l^{-1}$ and obtain

$$
\det(\mathbb{R}_l^{-1}(\lambda)) \det(\mathbb{R}_l(\lambda) - \mathbb{R}_r(\lambda)) = 0, \tag{5.9}
$$

so that since $\mathbb{R}_l^{-1}$ exists (as will be shown later), we reduce Eq.(5.7) to

$$
\det(\mathbb{R}_l(\lambda) - \mathbb{R}_r(\lambda)) = 0. \tag{5.10}
$$

Thus we have reduced the problem involving a $6 \times 6$ matrix to a similar problem for a $3 \times 3$ matrix. From now on, we will carry out the following steps:

- calculate the approximations to the left– and right–reflection matrices,
- find approximations for the roots of Eq.(5.10).

Using (4.29)–(4.31), we can introduce the following fundamental system of solutions of Eq.(5.1).

$$\Psi_1(x,\lambda) = exp\left\{ i\left(\frac{\tilde{I}_\alpha}{GJ}\right)^{1/2} \lambda\left(1+O(\lambda^{-2})\right)(x+l)\right\}, \tag{5.11}$$

$$\Psi_2(x,\lambda) = exp\left\{ \left(\frac{\Delta}{EI\,\tilde{I}_\alpha}\right)^{1/4} \lambda^{1/2}\left(1+O(\lambda^{-2})\right)(x+l)\right\}, \tag{5.12}$$

$$\Psi_3(x,\lambda) = exp\left\{ i\left(\frac{\Delta}{EI\,\tilde{I}_\alpha}\right)^{1/4} \lambda^{1/2}\left(1+O(\lambda^{-2})\right)(x+l)\right\}, \tag{5.13}$$

$$\Psi_4(x,\lambda) = exp\left\{ -i\left(\frac{\tilde{I}_\alpha}{GJ}\right)^{1/2} ),\lambda\left(1+O(\lambda^{-2})\right)(x+l)\right\}, \tag{5.14}$$

$$\Psi_5(x,\lambda) = exp\left\{ -\left(\frac{\Delta}{EI\,\tilde{I}_\alpha}\right)^{1/4} \lambda^{1/2}\left[\left(1+O(\lambda^{-2})\right)(x+l)\right\}, \tag{5.15}$$

$$\Psi_6(x,\lambda) = exp\left\{ -i\left(\frac{\Delta}{EI\,\tilde{I}_\alpha}\right)^{1/4} \lambda^{1/2}\left(1+O(\lambda^{-2})\right)(x+l)\right\}. \tag{5.16}$$

Therefore, the general solution of Eq. (5.1) can be represented in the form

$$\begin{aligned}\Psi(x,\lambda) =& \mathcal{A}(\lambda)\Psi_1(x,\lambda) + \mathcal{B}(\lambda)\Psi_2(x,\lambda) + \mathcal{C}(\lambda)\Psi_3(x,\lambda) \\ &+ \mathcal{D}(\lambda)\Psi_4(x,\lambda) + \mathcal{E}(\lambda)\Psi_5(x,\lambda) + \mathcal{F}(\lambda)\Psi_6(x,\lambda).\end{aligned} \tag{5.17}$$

## 5.2 Left-Reflection Matrix

In this section, we will derive the asymptotic approximation for the left-reflection matrix $\mathbb{R}_l$. Let us substitute the general solution $\Psi(\cdot)$ given in (5.17) into each of the three left-hand boundary conditions of the operator pencil. Applying the first condition, $\Psi(-L,\lambda) = 0$, to the general solution gives us the first equation

$$\mathcal{A}(\lambda) + \mathcal{B}(\lambda) + \mathcal{C}(\lambda) + \mathcal{D}(\lambda) + \mathcal{E}(\lambda) + \mathcal{F}(\lambda) = 0. \tag{5.18}$$

The second boundary condition is $\Psi''(-L,\lambda) = 0$. Taking into account (5.11)–(5.16), we obtain the following asymptotical form for the second boundary condition:

$$\begin{aligned} &-\frac{\tilde{I}_\alpha}{GJ}\lambda^2\left(1+O(\lambda^{-2})\right)\mathcal{A}(\lambda) + \left(\frac{\Delta}{EI\,\tilde{I}_\alpha}\right)^{1/2}\lambda\left(1+O(\lambda^{-2})\right)\mathcal{B}(\lambda) \\ &-\left(\frac{\Delta}{EI\,\tilde{I}_\alpha}\right)^{1/2}\lambda\left(1+O(\lambda^{-2})\right)\mathcal{C}(\lambda) - \frac{\tilde{I}_\alpha}{GJ}\lambda^2\left(1+O(\lambda^{-2})\right)\mathcal{D}(\lambda) \\ &+\left(\frac{\Delta}{EI\,\tilde{I}_\alpha}\right)^{1/2}\lambda\left(1+O(\lambda^{-2})\right)\mathcal{E}(\lambda) - \left(\frac{\Delta}{EI\,\tilde{I}_\alpha}\right)^{1/2}\lambda\left(1+O(\lambda^{-2})\right)\mathcal{F}(\lambda) = 0. \end{aligned} \tag{5.19}$$

The third boundary condition is $GJ\Psi'''(-L,\lambda) + (\lambda^2\tilde{I}_\alpha + \pi\rho u^2)\Psi'(-L,\lambda) = 0$. The latter equation can be written in the asymptotical form as follows:

$$
\begin{aligned}
&\left[GJ\left(\frac{\tilde{I}_\alpha}{GJ}\right)^{3/2} i\lambda^3 - (\lambda^2\tilde{I}_\alpha + \pi\rho u^2)\left(\frac{\tilde{I}_\alpha}{GJ}\right)^{1/2} i\lambda\right]\left(1+O(\lambda^{-2})\right)\mathcal{A}(\lambda) \\
&+\left[-GJ\left(\frac{\Delta}{EI\,\tilde{I}_\alpha}\right)^{3/4} \lambda^{3/2} - (\lambda^2\tilde{I}_\alpha + \pi\rho u^2)\left(\frac{\Delta}{EI\,\tilde{I}_\alpha}\right)^{1/4} \lambda^{1/2}\right]\left(1+O(\lambda^{-1})\right)\mathcal{B}(\lambda) \\
&+\left[GJ\left(\frac{\Delta}{EI\,\tilde{I}_\alpha}\right)^{3/4} i\lambda^{3/2} - (\lambda^2\tilde{I}_\alpha + \pi\rho u^2)\left(\frac{\Delta}{EI\,\tilde{I}_\alpha}\right)^{1/4} i\lambda^{1/2}\right]\left(1+O(\lambda^{-1})\right)\mathcal{C}(\lambda) \\
&+\left[-GJ\left(\frac{\tilde{I}_\alpha}{GJ}\right)^{3/2} i\lambda^3 + (\lambda^2\tilde{I}_\alpha + \pi\rho u^2)\left(\frac{\tilde{I}_\alpha}{GJ}\right)^{1/2} i\lambda\right]\left(1+O(\lambda^{-2})\right)\mathcal{D}(\lambda) \\
&+\left[GJ\left(\frac{\Delta}{EI\,\tilde{I}_\alpha}\right)^{3/4} \lambda^{3/2} + (\lambda^2\tilde{I}_\alpha + \pi\rho u^2)\left(\frac{\Delta}{EI\,\tilde{I}_\alpha}\right)^{1/4} \lambda^{1/2}\right]\left(1+O(\lambda^{-1})\right)\mathcal{E}(\lambda) \\
&+\left[-GJ\left(\frac{\Delta}{EI\,\tilde{I}_\alpha}\right)^{3/4} i\lambda^{3/2} + (\lambda^2\tilde{I}_\alpha + \pi\rho u^2)\left(\frac{\Delta}{EI\,\tilde{I}_\alpha}\right)^{1/4} i\lambda^{1/2}\right]\left(1+O(\lambda^{-1})\right)\mathcal{F}(\lambda) \quad = 0.
\end{aligned}
\tag{5.20}
$$

Eq.(5.20) can be simplified to the following:

$$
\begin{aligned}
&\left[-\pi\rho u^2\left(\frac{\tilde{I}_\alpha}{GJ}\right)^{1/2} i\lambda\left(1+O(\lambda^{-2})\right)\right]\mathcal{A}(\lambda)\\
&+\left[-\tilde{I}_\alpha^{3/4}\left(\frac{\Delta}{EI}\right)^{1/4}\lambda^{5/2}-GJ\left(\frac{\Delta}{EI\,\tilde{I}_\alpha}\right)^{3/4}\lambda^{3/2}\right]\left(1+O(\lambda^{-1})\right)\mathcal{B}(\lambda)\\
&+\left[-\tilde{I}_\alpha^{3/4}\left(\frac{\Delta}{EI}\right)^{1/4} i\lambda^{5/2}+GJ\left(\frac{\Delta}{EI\,\tilde{I}_\alpha}\right)^{3/4} i\lambda^{3/2}\right]\left(1+O(\lambda^{-1})\right)\mathcal{C}(\lambda)\\
&+\left[\pi\rho u^2\left(\frac{\tilde{I}_\alpha}{GJ}\right)^{1/2} i\lambda\left(1+O(\lambda^{-2})\right)\right]\mathcal{D}(\lambda)\\
&+\left[\tilde{I}_\alpha^{3/4}\left(\frac{\Delta}{EI}\right)^{1/4}\lambda^{5/2}GJ\left(\frac{\Delta}{EI\,\tilde{I}_\alpha}\right)^{3/4}\lambda^{3/2}\right]\left(1+O(\lambda^{-1})\right)\mathcal{E}(\lambda)\\
&+\left[\tilde{I}^{3/4}\left(\frac{\Delta}{EI}\right)^{1/4} i\lambda^{5/2}-GJ\left(\frac{\Delta}{EI\,\tilde{I}_\alpha}\right)^{3/4} i\lambda^{3/2}\right]\left(1+O(\lambda^{-1})\right)\mathcal{F}(\lambda)=0.
\end{aligned}
\tag{5.21}
$$

It is convenient to rewrite Eqs.(5.18), (5.19), and (5.21) in the following manner:

$$
\begin{aligned}
\mathcal{A}(\lambda)+\mathcal{B}(\lambda)+\mathcal{C}(\lambda) &= -\mathcal{D}(\lambda)-\mathcal{E}(\lambda)-\mathcal{F}(\lambda),\\
Q_1(\lambda)\mathcal{A}(\lambda)+Q_2(\lambda)\mathcal{B}(\lambda)-Q_2(\lambda)\mathcal{C}(\lambda) &= -Q_1(\lambda)\mathcal{D}(\lambda)-Q_2(\lambda)\mathcal{E}(\lambda)+Q_2(\lambda)\mathcal{F}(\lambda),\\
Q_3(\lambda)\mathcal{A}(\lambda)+Q_4(\lambda)\mathcal{B}(\lambda)+Q_5(\lambda)\mathcal{C}(\lambda) &= Q_3(\lambda)\mathcal{D}(\lambda)+Q_4(\lambda)\mathcal{E}(\lambda)+Q_5(\lambda)\mathcal{F}(\lambda),
\end{aligned}
\tag{5.22}
$$

where $Q_i(\cdot)$, $i = 1, 2, \ldots, 5$, are new functions defined by the formulae

$$Q_1(\lambda) = -\left(\frac{\tilde{I}_\alpha}{GJ}\right)\lambda^2\left(1 + O(\lambda^{-2})\right),$$

$$Q_2(\lambda) = \left(\frac{\Delta}{EI\,\tilde{I}_\alpha}\right)^{1/2}\lambda\left(1 + O(\lambda^{-1})\right),$$

$$Q_3(\lambda) = -\,\pi\rho u^2\left(\frac{\tilde{I}_\alpha}{GJ}\right)^{1/2} i\lambda\left(1 + O(\lambda^{-2})\right), \tag{5.23}$$

$$Q_4(\lambda) = \left[-\tilde{I}_\alpha^{3/4}\left(\frac{\Delta}{EI}\right)^{1/4}\lambda^{5/2} - GJ\left(\frac{\Delta}{EI\,\tilde{I}_\alpha}\right)^{3/4}\lambda^{3/2}\right]\left(1 + O(\lambda^{-1})\right),$$

$$Q_5(\lambda) = \left[-\tilde{I}_\alpha^{3/4}\left(\frac{\Delta}{EI}\right)^{1/4} i\lambda^{5/2} + GJ\left(\frac{\Delta}{EI\,\tilde{I}_\alpha}\right)^{3/4} i\lambda^{3/2}\right]\left(1 + O(\lambda^{-1})\right).$$

Dividing through the second equation by $Q_2$, and the third equation by $Q_4$ we can simplify system (5.23) as follows:

$$\mathcal{A}(\lambda) + \mathcal{B}(\lambda) + \mathcal{C}(\lambda) = -\mathcal{D}(\lambda) - \mathcal{E}(\lambda) - \mathcal{F}(\lambda),$$

$$\begin{aligned} &Q_6\lambda\left(1 + O(\lambda^{-1})\right)\mathcal{A}(\lambda) + \mathcal{B}(\lambda) - \left(1 + O(\lambda^{-1})\right)\mathcal{C}(\lambda) \\ &= -Q_6\lambda\left(1 + O(\lambda^{-1})\right)\mathcal{D}(\lambda) - \mathcal{E}(\lambda) + \left(1 + O(\lambda^{-1})\right)\mathcal{F}(\lambda), \end{aligned} \tag{5.24}$$

$$\begin{aligned} &O(\lambda^{-3/2})\mathcal{A}(\lambda) + \mathcal{B}(\lambda) + i\left(1 + O(\lambda^{-1})\right)\mathcal{C}(\lambda) \\ &= O(\lambda^{-3/2})\mathcal{D}(\lambda) + \mathcal{E}(\lambda) + i\left(1 + O(\lambda^{-1})\right)\mathcal{F}(\lambda), \end{aligned}$$

where $Q_6 = -\dfrac{\tilde{I}_\alpha^{3/2}(EI)^{1/2}}{GJ\Delta^{1/2}}$.

Clearly, the three equations (5.24) can be written as one matrix equation

$$\begin{bmatrix} 1 & 1 & 1 \\ Q_6\lambda\,(1+O(\lambda^{-1})) & 1 & -(1+O(\lambda^{-1})) \\ O(\lambda^{-3/2}) & 1 & i\,(1+O(\lambda^{-1})) \end{bmatrix} \begin{bmatrix} \mathcal{A}(\lambda) \\ \mathcal{B}(\lambda) \\ \mathcal{C}(\lambda) \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 \\ -Q_6\lambda\,(1+O(\lambda^{-1})) & -1 & (1+O(\lambda^{-1})) \\ O(\lambda^{-3/2}) & 1 & i\,(1+O(\lambda^{-1})) \end{bmatrix} \begin{bmatrix} \mathcal{D}(\lambda) \\ \mathcal{E}(\lambda) \\ \mathcal{F}(\lambda) \end{bmatrix}. \tag{5.25}$$

Thus matrix equation (5.25) can be represented in the form

$$\mathbb{A}(\lambda)X(\lambda) = \mathbb{B}(\lambda)Y(\lambda), \tag{5.26}$$

where the vectors $X(\cdot)$ and $Y(\cdot)$ are defined in (5.2). If we solve Eq.(5.26) for $X(\cdot)$

$$X(\lambda) = \mathbb{A}^{-1}(\lambda)\mathbb{B}(\lambda)Y(\lambda), \tag{5.27}$$

then we observe that the left–reflection matrix as introduced in (5.3) can be given as

$$\mathbb{R}_l(\lambda) = \mathbb{A}^{-1}(\lambda)\mathbb{B}(\lambda). \tag{5.28}$$

Our next goal for now is to find an asymptotic approximation for $\mathbb{R}_l$. While the straightforward calculation of the above left–reflection matrix is possible, we exploit the similarity of the entries of $\mathbb{A}$ and $\mathbb{B}$ to make the calculation easier. We notice that

$$\mathbb{B}(\lambda) = -\mathbb{A}(\lambda) + \mathbb{V}(\lambda), \tag{5.29}$$

where

$$\mathbb{V}(\lambda) = 2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ O(\lambda^{-3/2}) & 1 & i\,(1 + O(\lambda^{-1})) \end{bmatrix}. \tag{5.30}$$

Thus the calculation of $\mathbb{R}_l(\cdot)$ can be simplified, for using this expression for $\mathbb{B}(\cdot)$ in terms of $\mathbb{A}(\cdot)$ we have that

$$\mathbb{R}_l(\lambda) = \mathbb{A}^{-1}(\lambda)\mathbb{B}(\lambda) = \mathbb{A}^{-1}(\lambda)(-\mathbb{A}(\lambda) + \mathbb{V}(\lambda)) = -\mathbb{I} + \mathbb{A}^{-1}(\lambda)\mathbb{V}(\lambda). \tag{5.31}$$

This alternate expression for $\mathbb{R}_l(\cdot)$ will make its calculation easier because only the middle column of $\mathbb{A}^{-1}(\cdot)$ is needed for the calculation of $\mathbb{A}^{-1}(\cdot)\mathbb{V}(\cdot)$ because of the fact that $\mathbb{V}(\cdot)$ has only one non–zero row. So recalling that

$$\mathbb{A}(\lambda) = \begin{bmatrix} 1 & 1 & 1 \\ Q_6\lambda\,(1 + O(\lambda^{-1})) & 1 & -\,(1 + O(\lambda^{-1})) \\ O(\lambda^{3/2}) & 1 & i\,(1 + O(\lambda^{-1})) \end{bmatrix}, \tag{5.32}$$

we calculate asymptotic approximations to the middle column of $\mathbb{A}^{-1}(\cdot)$ using the Cramer's Rule. First, we calculate $det\,\mathbb{A}$ by expansion with respect to the top row entries as

$$(\det \mathbb{A})(\lambda) = \begin{vmatrix} 1 & -(1+O(\lambda^{-1})) \\ 1 & i\,(1+O(\lambda^{-1})) \end{vmatrix} - \begin{vmatrix} Q_6\lambda\,(1+O(\lambda^{-1})) & -(1+O(\lambda^{-1})) \\ O(\lambda^{3/2}) & i\,(1+O(\lambda^{-1})) \end{vmatrix}$$
$$+ \begin{vmatrix} Q_6\lambda\,(1+O(\lambda^{-1})) & 1 \\ O(\lambda^{3/2}) & 1 \end{vmatrix}. \tag{5.33}$$

Simplifying, we get

$$(\det \mathbb{A})(\lambda) = Q_6\lambda(1-i)\left(1+O(\lambda^{-1})\right). \tag{5.34}$$

Now we proceed to find each specific entry of the last column of the matrix $\mathbb{A}^{-1}(\cdot)$ given in (5.28). Let $C_{3j}$, $j = 1, 2, 3$, be a cofactor corresponding to the third row, and the $j$-th entry of $\mathbb{A}(\cdot)$. Beginning with the first entry of $\mathbb{A}^{-1}(\cdot)$ we have

$$\mathbb{A}^{-1}_{13}(\lambda) = \frac{C_{31}}{(\det \mathbb{A})(\lambda)} = \frac{1}{(\det \mathbb{A})(\lambda)} \begin{vmatrix} 1 & 1 \\ 1 & -(1+O(\lambda^{-1})) \end{vmatrix} = \frac{-2\,(1+O(\lambda^{-1}))}{(\det \mathbb{A})(\lambda)}. \tag{5.35}$$

Recalling $(\det \mathbb{A})(\lambda)$ from (5.34), we can write

$$\mathbb{A}^{-1}_{13}(\lambda) = \frac{-2\,(1+O(\lambda^{-1}))}{(\det \mathbb{A})(\lambda)} = \frac{-2\,(1+O(\lambda^{-1}))}{Q_6\lambda(1-i)\,(1+O(\lambda^{-1}))}. \tag{5.36}$$

Now using the fact that

$$\frac{1}{1+O(\lambda^m)} = 1 + O(\lambda^m) + O(\lambda^{2m}) + \ldots = 1 + O(\lambda^m),\ m < 0, \tag{5.37}$$

we may finally write

$$\mathbb{A}^{-1}_{13}(\lambda) = \frac{-2}{Q_6\lambda(1-i)}\left(1+O(\lambda^{-1})\right). \tag{5.38}$$

Thus we have an asymptotic expression of $\mathbb{A}_{13}^{-1}$. In what follows, it is convenient to use the notation

$$\hat{\omega}_{ij}(\lambda) = 1 + O(\lambda^{-1}), \tag{5.39}$$

which means that on the intersection of the $i-th$ row and the $j-th$ column, there is a factor $1 + O(\lambda^{-1})$. Thus, we may finally write that

$$\mathbb{A}_{13}^{-1} = \frac{-2}{Q_6\lambda(1-i)}\hat{\omega}_{13}. \tag{5.40}$$

Calculations of the other two needed entries of $\mathbb{A}^{-1}(\cdot)$ will proceed in a similar manner. Calculating $\mathbb{A}_{23}^{-1}(\cdot)$, we have

$$\begin{aligned}\mathbb{A}_{23}^{-1}(\lambda) = \quad & \frac{C_{32}}{(\det \mathbb{A})(\lambda)} = \frac{-1}{Q_6\lambda(1-i)\hat{\omega}_{23}(\lambda)}\begin{vmatrix} 1 & 1 \\ Q_6\lambda\,(1+O(\lambda^{-1})) & -(1+O(\lambda^{-1}))\end{vmatrix} \\ = \quad & \frac{(1+Q_6\lambda)}{Q_6\lambda(1-i)}\hat{\omega}_{23}(\lambda).\end{aligned} \tag{5.41}$$

Taking into account formula for $Q_6$ (see below (5.24)), we may rewrite (5.41) and have

$$\mathbb{A}_{23}^{-1}(\lambda) = \frac{i+1}{2}\hat{\omega}_{23}(\lambda). \tag{5.42}$$

Calculating the remaining entry, we have

$$\begin{aligned}\mathbb{A}_{33}^{-1}(\lambda) = & \frac{C_{23}}{(\det \mathbb{A})(\lambda)} = \frac{1}{Q_6\lambda(1-i)\hat{\omega}_{33}(\lambda)}\begin{vmatrix} 1 & 1 \\ Q_6\lambda\,(1+O(\lambda^{-1})) & 1\end{vmatrix} \\ = & \frac{1-Q_6\lambda}{Q_6\lambda(1-i)\hat{\omega}_{33}(\lambda)}.\end{aligned} \tag{5.43}$$

Using the expression for $Q_6$, we can see that

$$\mathbb{A}_{33}^{-1}(\lambda) = -\frac{1+i}{2}\hat{\omega}_{33}(\lambda). \tag{5.44}$$

Using formula (5.30) for $\mathbb{V}$ and (5.31) for $\mathbb{R}_l$, we obtain

$$\mathbb{R}_l(\lambda) = -\mathbb{I} + \mathbb{A}^{-1}(\lambda)\mathbb{V}(\lambda)$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + 2 \begin{bmatrix} * & * & \mathbb{A}_{13}^{-1}(\lambda) \\ * & * & \mathbb{A}_{23}^{-1}(\lambda) \\ * & * & \mathbb{A}_{33}^{-1}(\lambda) \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ O(\lambda^{3/2}) & 1 & i\,(1+O(\lambda^{-1})) \end{bmatrix}. \tag{5.45}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + 2 \begin{bmatrix} \mathbb{A}_{13}^{-1}(\lambda)O(\lambda^{3/2}) & \mathbb{A}_{13}^{-1}(\lambda) & i\mathbb{A}_{13}^{-1}(\lambda)\hat{\omega}_{13}(\lambda) \\ \mathbb{A}_{23}^{-1}(\lambda)O(\lambda^{3/2}) & \mathbb{A}_{23}^{-1}(\lambda) & i\mathbb{A}_{23}^{-1}(\lambda)\hat{\omega}_{23}(\lambda) \\ \mathbb{A}_{33}^{-1}(\lambda)O(\lambda^{3/2}) & \mathbb{A}_{33}^{-1}(\lambda) & i\mathbb{A}_{33}^{-1}(\lambda)\hat{\omega}_{33}(\lambda) \end{bmatrix}.$$

In (5.45), we have used the notation "*" for those entries, which are immaterial for us. Finally, we have the following representation for the left–reflection matrix:

$$\mathbb{R}_l(\lambda) = \begin{bmatrix} -1 + 2\mathbb{A}_{13}^{-1}(\lambda)O(\lambda^{3/2}) & 2\mathbb{A}_{13}^{-1}(\lambda) & 2i\mathbb{A}_{13}^{-1}(\lambda)\hat{\omega}_{13}(\lambda) \\ 2\mathbb{A}_{23}^{-1}(\lambda)O(\lambda^{3/2}) & -1 + 2\mathbb{A}_{23}^{-1}(\lambda) & 2i\mathbb{A}_{23}^{-1}(\lambda)\hat{\omega}_{23}(\lambda) \\ 2\mathbb{A}_{33}^{-1}(\lambda)O(\lambda^{3/2}) & 2\mathbb{A}_{33}^{-1}(\lambda) & -1 + 2i\mathbb{A}_{33}^{-1}(\lambda)\hat{\omega}_{33}(\lambda) \end{bmatrix}. \tag{5.46}$$

Using formulae (5.40 ) for $\mathbb{A}_{13}^{-1}(\cdot)$, we can write

$$-1+2\mathbb{A}_{13}^{-1}(\lambda)O(\lambda^{3/2}) = -1+2\frac{-2}{Q_6\lambda(1-i)}\hat{\omega}_{13} = -\hat{\omega}_{13}. \tag{5.47}$$

A similar calculation for $\mathbb{A}_{13}^{-1}(\lambda)$ yields

$$\mathbb{A}_{13}^{-1}(\lambda) = O(\lambda^{-1}). \tag{5.48}$$

Also $\mathbb{A}_{13}^{-1}(\lambda)i\hat{\omega}_{33}$ yields

$$\mathbb{A}_{13}^{-1}(\lambda)i\hat{\omega}_{33} = O(\lambda^{-1}). \tag{5.49}$$

Now we move on to the second row of the matrix $\mathbb{R}_l$. Using formulae for $Q_6$, we calculate that

$$\frac{1+Q_6\lambda}{Q_6\lambda} = 1+O(\lambda^{-1}). \tag{5.50}$$

Thus we substitute this expression into the entire expression for $(\mathbb{R}_l)_{22}$ in (5.40) and simplify to have

$$(\mathbb{R}_l)_{22}(\lambda) = -1+2\mathbb{A}_{23}^{-1} = -1+2\frac{(Q_6\lambda+1)}{Q_6\lambda(1-i)}1+O(\lambda^{-1}) = i\hat{\omega}_{22}. \tag{5.51}$$

Now considering the entry $(\mathbb{R}_l)_{21}$, we substitute (5.42) and have

$$(\mathbb{R}_l)_{21}(\lambda) = 2\mathbb{A}_{23}^{-1}O(\lambda^{-3/2}) = O(\lambda^{-3/2}). \tag{5.52}$$

Turning now to $(\mathbb{R}_l)_{23}$, we calculate

$$(\mathbb{R}_l)_{23}(\lambda) = 2i\mathbb{A}_{22}^{-1}(\lambda)\hat{\omega} = \frac{2i}{1-i}\hat{\omega}_{23}(\lambda) = (i-1)\hat{\omega}_{23}(\lambda). \tag{5.53}$$

Finally we evaluate the third row of $\mathbb{R}_l(\cdot)$. We calculate a portion of the first entry

$$(\mathbb{R}_l)_{31}(\lambda) = 2\mathbb{A}_{13}^{-1}(\lambda)O(\lambda^{-3/2} = -\frac{1+i}{2}O(\lambda^{-3/2} = O(\lambda^{-3/2}). \tag{5.54}$$

Similarly, we calculate the second entry as

$$(\mathbb{R}_l)_{32}(\lambda) = 2\mathbb{A}_{13}^{-1}(\lambda) = -2\frac{1+i}{2}1 + O(\lambda^{-1}) = -(1+i)\hat{\omega}_{32}. \tag{5.55}$$

And finally we calculate a portion of the last entry as

$$(\mathbb{R}_l)_{33}(\lambda) = -1 + 2i\mathbb{A}_{13}^{-1}(\lambda)1 + O(\lambda^{-1}) = -1 - 2\frac{1+i}{2}\hat{\omega} = -i\hat{\omega}. \tag{5.56}$$

Substituting the results of (5.47)–(5.56) into (5.39) for $\mathbb{R}_l(\cdot)$ and simplifying, we obtain the following asymptotic approximation to the left–reflection matrix:

$$\mathbb{R}_l(\lambda) = \begin{bmatrix} -i\hat{\omega}_{11}(\lambda) & O(\lambda^{-1}) & O(\lambda^{-1}) \\ O(\lambda^{3/2}) & i\hat{\omega}_{22}(\lambda) & (i-1)\hat{\omega}_{23}(\lambda) \\ O(\lambda^{3/2}) & -(1+i)\hat{\omega}_{32}(\lambda) & -i\hat{\omega}_{33}(\lambda) \end{bmatrix}. \tag{5.57}$$

## CHAPTER VI

## THE RIGHT–REFLECTION MATRIX

Now we will look for the right–reflection matrix $\mathbb{R}_r(\cdot)$ by substituting the general solution $\Psi(\cdot)$ from (4.37) into the right–end boundary conditions (1.7), (3.46), and (3.48). In what follows, it is convenient to introduce new notation

$$\begin{aligned} \exp\{i\gamma(\lambda)L\} &\equiv e(\lambda) \equiv e^{i\gamma(\lambda)L}, \\ \exp\{\hat{\gamma}(\lambda)L\} &\equiv \hat{e}(\lambda) \equiv e^{\hat{\gamma}(\lambda)L}, \\ \exp\{i\Gamma(\lambda)L\} &\equiv e_+(\lambda) \equiv e^{i\Gamma(\lambda)L}. \end{aligned} \tag{6.1}$$

We also recall that by (4.36)

$$P = \left[\frac{\Delta}{I_\alpha EI}\right]^{1/4}, \quad R^{1/2} = Q = \left[\frac{I_\alpha}{GJ}\right]^{1/2}. \tag{6.2}$$

Beginning with the right–end boundary condition (3.46)

$$\begin{aligned} &ig_h\tilde{I}_\alpha\lambda^3(1+O(\lambda^{-2}))\Psi'(0) + EI\,\tilde{I}_\alpha\lambda^2(1+O(\lambda^{-2}))\Psi''(0) \\ &\quad +i\lambda g_h GJ\,\Psi'''(0) + EI\,GJ\Psi^{IV}(0) = 0, \end{aligned} \tag{6.3}$$

we realize that we need the function $\Psi(\cdot)$, along with its first, second, third, and fourth derivatives. The first derivative of the general solution evaluated at $x = 0$ is

$$\begin{aligned} \Psi'(\lambda,0) =& i\gamma(\lambda)\mathcal{A}(\lambda)e(\lambda) + \hat{\gamma}(\lambda)\mathcal{B}(\lambda)\hat{e}(\lambda) + i\Gamma(\lambda)\mathcal{C}(\lambda)e_+(\lambda) - \\ & i\gamma(\lambda)\mathcal{D}(\lambda)e(\lambda)^{-1} - \hat{\gamma}(\lambda)\mathcal{E}(\lambda)\hat{e}(\lambda)^{-1} - i\Gamma(\lambda)\mathcal{F}(\lambda)e_+(\lambda)^{-1}. \end{aligned} \tag{6.4}$$

The second derivative of the function $\Psi(\cdot)$ evaluated at $x = 0$ is

$$\begin{aligned} \Psi''(\lambda,0) =& -\gamma^2(\lambda)\mathcal{A}(\lambda)e(\lambda) + \hat{\gamma}^2(\lambda)\mathcal{B}(\lambda)\hat{e}(\lambda) - \Gamma^2(\lambda)\mathcal{C}(\lambda)e_+(\lambda) - \\ & \gamma^2(\lambda)\mathcal{D}(\lambda)e(\lambda)^{-1} + \hat{\gamma}^2(\lambda)\mathcal{E}(\lambda)\hat{e}(\lambda)^{-1} - \Gamma^2(\lambda)\mathcal{F}(\lambda)e_+(\lambda)^{-1}. \end{aligned} \tag{6.5}$$

The third derivative of the function $\Psi(\cdot)$ evaluated at $x = 0$ is

$$\begin{aligned} \Psi'''(\lambda,0) =& -i\gamma^3(\lambda)\mathcal{A}(\lambda)e(\lambda) + \hat{\gamma}^3(\lambda)\mathcal{B}(\lambda)\hat{e}(\lambda) - i\Gamma^3(\lambda)\mathcal{C}(\lambda)e_+(\lambda) + \\ & i\gamma^3(\lambda)\mathcal{D}(\lambda)e(\lambda)^{-1} - \hat{\gamma}^3(\lambda)\mathcal{E}(\lambda)\hat{e}(\lambda)^{-1} + i\Gamma^3(\lambda)\mathcal{F}(\lambda)e_+(\lambda)^{-1}. \end{aligned} \tag{6.6}$$

The fourth derivative of the function $\Psi(\cdot)$ evaluated at $x=0$ is

$$\begin{aligned}\Psi^{IV}(\lambda,0) =& \gamma^4(\lambda)\mathcal{A}(\lambda)e(\lambda)+\hat{\gamma}^4(\lambda)\mathcal{B}(\lambda)\hat{e}(\lambda)+\Gamma^4(\lambda)\mathcal{C}(\lambda)e_+(\lambda)+ \\ & \gamma^4(\lambda)\mathcal{D}(\lambda)e(\lambda)^{-1}+\hat{\gamma}^4(\lambda)\mathcal{E}(\lambda)\hat{e}(\lambda)^{-1}+\Gamma^4(\lambda)\mathcal{F}(\lambda)e_+(\lambda)^{-1}.\end{aligned} \tag{6.7}$$

After substituting results (6.1)–(6.7) into the boundary condition, we get

$$\begin{aligned}
&[ig_h\tilde{I}_\alpha\lambda^3(1+o(\lambda^{-2}))(i\gamma)+EI\,\tilde{I}_\alpha\lambda^2(1+O(\lambda^{-2}))(i\gamma)^2 \\
&\qquad +i\lambda g_h\,GJ(i\gamma)^3+EI\;GJ(i\gamma)^4]e(\lambda)\mathcal{A}(\lambda) \\
&+[ig_h\tilde{I}_\alpha\lambda^3(1+o(\lambda^{-2}))(\hat{\gamma})+EI\,\tilde{I}_\alpha\lambda^2(1+O(\lambda^{-2}))(\hat{\gamma})^2+i\lambda g_h\;GJ(\hat{\gamma})^3 \\
&\qquad +EI\,GJ(\hat{\gamma})^4]\hat{e}(\lambda)\mathcal{B}(\lambda) \\
&+[ig_h\tilde{I}_\alpha\lambda^3(1+o(\lambda^{-2}))(i\Gamma)+EI\,\tilde{I}_\alpha\lambda^2(1+O(\lambda^{-2}))(i\Gamma)^2+i\lambda g_hGJ(i\Gamma)^3 \\
&\qquad +EI\,GJ(i\Gamma)^4]e_+(\lambda)\mathcal{C}(\lambda) \\
&-[ig_h\tilde{I}_\alpha\lambda^3(1+o(\lambda^{-2}))(-i\gamma)+EI\,\tilde{I}_\alpha\lambda^2(1+O(\lambda^{-2}))(-i\gamma)^2 \\
&\qquad +i\lambda g_h\;GJ(-i\gamma)^3+EI\;GJ(-i\gamma)^4]e^{-1}(\lambda)\mathcal{D}(\lambda) \\
&-[ig_h\tilde{I}_\alpha\lambda^3(1+o(\lambda^{-2}))(-\hat{\gamma})+EI\,\tilde{I}_\alpha\lambda^2(1+O(\lambda^{-2}))(-\hat{\gamma})^2 \\
&\qquad +i\lambda g_hGJ(-\hat{\gamma})^3+EI\,GJ(-\hat{\gamma})^4]\hat{e}^{-1}(\lambda)\mathcal{E}(\lambda) \\
&-[ig_hI_\alpha\lambda^3(1+o(\lambda^{-2}))(-i\Gamma)+EI\,I_\alpha\lambda^2(1+O(\lambda^{-2}))(-i\Gamma)^2+i\lambda g_hGJ(-i\Gamma)^3 \\
&\qquad +EI\,GJ(-i\Gamma)^4]e_+^{-1}(\lambda)\mathcal{F}(\lambda)=0.
\end{aligned} \tag{6.8}$$

Thus after simplifying, and dividing by $ig_hI_\alpha P(1+O(\lambda^{-1/2}))\lambda^{7/2}$, we obtain

$$\mathcal{A}(\lambda)O(\lambda^{-3/2})e(\lambda)++\mathcal{B}(\lambda)\hat{e}(\lambda)+\mathcal{C}(\lambda)i(1+O(\lambda^{-1/2}))e_{+}(\lambda)$$

$$-\mathcal{D}(\lambda)O(\lambda^{-3/2})e^{-1}(\lambda)-\mathcal{E}(\lambda)(1+O(\lambda^{-1/2})\hat{\gamma}^{-1}(\lambda)+\mathcal{F}(\lambda)i(1+O(\lambda^{-1/2}))e_{+}^{-1}=0. \tag{6.9}$$

Let us leave the terms containing $\mathcal{A}(\cdot)$, $\mathcal{B}(\cdot)$, and $\mathcal{C}(\cdot)$ on the left side of equation (6.9), while moving the terms with $\mathcal{D}(\cdot)$, $\mathcal{E}(\cdot)$, and $\mathcal{F}(\cdot)$ to the right side to obtain

$$\begin{aligned}&\mathcal{A}(\lambda)O(\lambda^{-3/2})e(\lambda)+\mathcal{B}(\lambda)\hat{e}(\lambda)+\mathcal{C}(\lambda)i(1+O(\lambda^{-1/2}))e_{+}(\lambda)=\\&\mathcal{D}(\lambda)O(\lambda^{-3/2})e(\lambda)^{-1}+\mathcal{E}(\lambda)(1+O(\lambda^{-1/2})\hat{\gamma}^{-1}(\lambda)+\mathcal{F}(\lambda)i(1+O(\lambda^{-1/2}))e_{+}^{-1}.\end{aligned} \tag{6.10}$$

Now we turn to the second right–hand side boundary condition of the operator pencil given in (1.7) as

$$GJ\Psi'(\lambda,0)+i\lambda g_{\alpha}\Psi(\lambda,0)=0. \tag{6.11}$$

Substituting in $\Psi(\cdot)$ from (4.36) and using notation (6.1), we obtain

$$[GJ(i\gamma)+i\lambda g_{\alpha}]e(\lambda)\mathcal{A}(\lambda)+[GJ(\hat{\gamma})+i\lambda g_{\alpha}]\hat{e}(\lambda)\mathcal{B}(\lambda)+[GJ(i\Gamma)+i\lambda g_{\alpha}]e_{+}(\lambda)\mathcal{C}(\lambda)$$

$$[GJ(-i\gamma)+i\lambda g_{\alpha}]e^{-1}(\lambda)\mathcal{D}(\lambda)+[GJ(-\hat{\gamma}^{-1})+i\lambda g_{\alpha}]\hat{e}^{-1}(\lambda)\mathcal{E}(\lambda)$$

$$+[GJ(-i\Gamma)+i\lambda g_{\alpha}]e_{+}^{-1}(\lambda)\mathcal{F}(\lambda)=0. \tag{6.12}$$

We divide (6.12) by $i\lambda g_{\alpha}(1-g_{\alpha}^{-1}GJ(\Delta^{-1}EI,I_{\alpha})^{-1/4}\lambda^{-1/2}$ and then collect together terms involving each of $\mathcal{A}(\cdot)$, $\mathcal{B}(\cdot)$, $\mathcal{C}(\cdot)$, $\mathcal{D}(\cdot)$, $\mathcal{E}(\cdot)$ and $\mathcal{F}(\cdot)$, respectively. Again we leave the terms involving $\mathcal{A}(\cdot)$, $\mathcal{B}(\cdot)$, and $\mathcal{C}(\cdot)$ on the left side and take the other terms to the right side to rewrite Eq.(6.12) in the form

$$
\begin{gathered}
[1+g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ}(1+O(g_\alpha^{-1}\lambda^{-1/2}))e(\lambda)\mathcal{A}(\lambda)+\hat{e}(\lambda)\mathcal{B}(\lambda) \\
+(1+O(g_\alpha^{-1}\lambda^{-1/2}))e_+(\lambda)\mathcal{C}(\lambda) = -[1-g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ}(1+O(g_\alpha^{-1}\lambda^{-1/2}))e^{-1}(\lambda)\mathcal{D}(\lambda) \\
-(1+O(g_\alpha^{-1}\lambda^{-1/2}))\hat{e}^{-1}(\lambda)\mathcal{E}(\lambda)-(1+O(g_\alpha^{-1}\lambda^{-1/2}))e_+^{-1}(\lambda)\mathcal{F}(\lambda).
\end{gathered} \tag{6.13}
$$

Examining the third right-hand boundary condition given in (3.48), we realize that we also need the fifth derivative of $\Psi$ evaluated at zero. The fifth derivative of the function $\Psi(\cdot)$ evaluated at $x=0$ is

$$
\begin{gathered}
\Psi^V(\lambda,0)=\gamma^5(\lambda)\mathcal{A}(\lambda)e(\lambda)+i\hat{\gamma}^5(\lambda)\mathcal{B}(\lambda)\hat{e}(\lambda)+i\Gamma^5(\lambda)\mathcal{C}(\lambda)e_+(\lambda)- \\
\gamma^5(\lambda)\mathcal{D}(\lambda)e(\lambda)^{-1}-i\hat{\gamma}^5(\lambda)\mathcal{E}(\lambda)\hat{e}(\lambda)^{-1}-i\Gamma^5(\lambda)\mathcal{F}(\lambda)e_+(\lambda)^{-1}.
\end{gathered} \tag{6.14}
$$

After substituting results (6.4)–(6.6) and (6.14) into the boundary condition (3.48), we divide the resulting equation by $\lambda^{7/2}I_\alpha P^3(1+O(\lambda^{-1}))$. The following approximation is valid for Eq.(6.13):

$$
\begin{gathered}
\mathcal{A}(\lambda)(1+O(\lambda^{-1}))O(\lambda^{-1/2})e(\lambda)+\mathcal{B}(\lambda)\hat{e}(\lambda)-\mathcal{C}(\lambda)i(1+O(\lambda^{-1}))e_+(\lambda)- \\
\mathcal{D}(\lambda)(1+O(\lambda^{-1}))O(\lambda^{-1/2})e^{-1}(\lambda)-\mathcal{E}(\lambda)(1+O(\lambda^{-1}))\hat{e}^{-1}(\lambda) \\
+\mathcal{F}(\lambda)i(1+O(\lambda^{-1}))e_+^{-1}(\lambda)=0.
\end{gathered} \tag{6.15}
$$

where P and Q are defined in (6.2).

Once again we leave the terms involving $\mathcal{A}(\cdot)$, $\mathcal{B}(\cdot)$, and $\mathcal{C}(\cdot)$ on the left side, and move the terms involving $\mathcal{D}(\cdot)$, $\mathcal{E}(\cdot)$, and $\mathcal{F}(\cdot)$ to the right side, and have

$$\mathcal{A}(\lambda)(1+O(\lambda^{-1}))O(\lambda^{-1/2})e(\lambda)+\mathcal{B}(\lambda)\hat{e}(\lambda)-\mathcal{C}(\lambda)i(1+O(\lambda^{-1}))e_+(\lambda)=$$

$$\mathcal{D}(\lambda)(1+O(\lambda^{-1}))O(\lambda^{-1/2})e^{-1}(\lambda)+\mathcal{E}(\lambda)(1+O(\lambda^{-1}))\hat{e}^{-1}(\lambda) \tag{6.16}$$

$$-\mathcal{F}(\lambda)i(1+O(\lambda^{-1}))e_+^{-1}(\lambda).$$

Setting

$$\mathbb{E}(\lambda)=diag(e(\lambda),\hat{e}(\lambda),e_+(\lambda)), \tag{6.17}$$

we can write the three right–hand boundary conditions in the matrix form. This matrix form, which will appear below, contains two matrices denoted by $\mathbb{A}(\cdot)$ and $\mathbb{B}(\cdot)$. They are similar to the ones appearing in formula (5.26). Without any misunderstanding, we will use the same notation keeping in mind that the new matrices $\mathbb{A}(\cdot)$ and $\mathbb{B}(\cdot)$ are different from the ones which appeared in Section 5.2. Thus, we have

$$\mathbb{A}(\lambda)\,\mathbb{E}(\lambda)\,X(\lambda)=\mathbb{B}(\lambda)\,\mathbb{E}^{-1}(\lambda)\,Y(\lambda), \tag{6.18}$$

where the entries of the matrices of $\mathbb{A}(\cdot)$ and $\mathbb{B}(\cdot)$ are given in (6.10), (6.13), and (6.17).

To rewrite the matrix equation (6.18) explicitely, it is convenient to introduce the following notation: $\omega_{ij}(\lambda)$ and $\hat{\omega}_{ij}(\lambda)$; $\omega_{ij}(\lambda)$, with $i,j=1,2,3$, means that there is a factor $(1+O(\lambda^{-1/2}))$ on the intersection of the $i-th$ row and $j-th$ column in the matrix below, and $\hat{\omega}_{ij}(\lambda)$, with $i,j=1,2,3$, means that there is a factor $(1+O(\lambda^{-1}))$ on the intersection of the $i-th$ row and the $j-th$ column in the matrix below. With the above mentioned notation, the matrix equation (6.18) then becomes

$$
\begin{bmatrix} O(\lambda^{3/2}) & 1 & i\omega_{13} \\ \\ \left[1+g_\alpha^{-1}\sqrt{I_\alpha GJ}\right]\omega_{21} & 1 & \omega_{23} \\ \\ O(\lambda^{-1/2}\hat{\omega}_{31}) & 1 & -i\hat{\omega}_{33} \end{bmatrix} \mathbb{E} \begin{bmatrix} \mathcal{A}(\lambda) \\ \\ \mathcal{B}(\lambda) \\ \\ \mathcal{C}(\lambda) \end{bmatrix} =
$$

$$
\begin{bmatrix} O(\lambda^{3/2}) & \omega_{12} & i\omega_{13} \\ \\ -\left[1-g_\alpha^{-1}\sqrt{I_\alpha GJ}\right]\omega_{21} & -\omega_{22} & -\omega_{23} \\ \\ O(\lambda^{-1/2}\hat{\omega}_{31} & \hat{\omega}_{32} & -i\hat{\omega}_{33}) \end{bmatrix} \mathbb{E}^{-1} \begin{bmatrix} \mathcal{D}(\lambda) \\ \\ \mathcal{E}(\lambda) \\ \\ \mathcal{F}(\lambda) \end{bmatrix} \tag{6.19}
$$

Assuming that the matrix $\mathbb{A}^{-1}(\cdot)$ exists, we solve (6.18) for $X(\cdot)$ and obtain

$$
X(\lambda) = \mathbb{E}^{-1}(\lambda)\mathbb{A}^{-1}(\lambda)\mathbb{B}(\lambda)\,\mathbb{E}^{-1}(\lambda)Y(\lambda). \tag{6.20}
$$

Comparing (5.4) with (6.20), we conclude that the right–reflection matrix can be represented as

$$
\mathbb{R}_r(\lambda) = \mathbb{E}^{-1}(\lambda)\mathbb{A}^{-1}(\lambda)\mathbb{B}(\lambda)\,\mathbb{E}^{-1}(\lambda). \tag{6.21}
$$

It is this right–reflection matrix that will be calculated in the remainder of this section. While we could certainly compute $\mathbb{A}(\cdot)^{-1}\mathbb{B}(\cdot)$ in a straightforward manner, we instead notice that $\mathbb{B}(\cdot)$ can be thought of as a "perturbation" of $\mathbb{A}(\cdot)$, i.e., let

$$
\mathbb{B}(\lambda) = \mathbb{A}(\lambda) - \mathbb{V}(\lambda). \tag{6.22}
$$

For $\mathbb{V}(\cdot)$, we obtain the expression

$$\mathbb{V} = \begin{bmatrix} 0 & 1-\omega_{12}(\lambda) & 0 \\ \\ 2\omega_{21}(\lambda) & \omega_{22}(\lambda)+1 & 2\omega_{23}(\lambda) \\ \\ 0 & 1-\hat{\omega}_{32}(\lambda) & 0 \end{bmatrix} \tag{6.23}$$

Thus the matrix $\mathbb{A}(\cdot)^{-1}\mathbb{B}(\cdot)$ can be written as

$$\mathbb{A}(\lambda)^{-1}\mathbb{B}(\lambda) = \mathbb{A}(\lambda)^{-1}(\mathbb{A}(\lambda) - \mathbb{V}(\lambda)) = \mathbb{I} - \mathbb{A}(\lambda)^{-1}\mathbb{V}(\lambda). \tag{6.24}$$

Since $\mathbb{V}(\cdot)$ has a row of zeros, it will be more efficient to calculate $\mathbb{A}(\lambda)^{-1}\mathbb{V}(\lambda)$ instead of $\mathbb{A}(\lambda)^{-1}\mathbb{B}(\lambda)$.

Now we begin to calculate an asymptotic representation for the entries of the matrix $\mathbb{A}(\lambda)^{-1}$.

Setting

$$\mathbb{A}(\lambda) = \begin{bmatrix} O(\lambda^{3/2}) & \omega_{12}(\lambda) & i\omega_{13}(\lambda) \\ \\ -\left[1 - g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ}\right]\omega_{21}(\lambda) & -\omega_{22}(\lambda) & -\omega_{23}(\lambda) \\ \\ O(\lambda^{-1/2})\hat{\omega}_{31}(\lambda) & \hat{\omega}_{32}(\lambda) & -i\hat{\omega}_{33}(\lambda) \end{bmatrix}, \tag{6.25}$$

we calculate an asymptotic approximation to *det* $\mathbb{A}(\lambda)$ by expanding along the entries of the first column. We have

$$\begin{aligned} \det(\mathbb{A}(\lambda)) &= O(\lambda^{3/2})[i\hat{\omega}_{33}(\lambda) + \omega_{23}(\lambda)\hat{\omega}_{32}(\lambda)] \\ &\quad [1 - g_\alpha\sqrt{\tilde{I}_\alpha GJ}\omega_{21}(\lambda)[-O(\lambda^{3/2})i\hat{\omega}_{33}(\lambda) - O(\lambda^{-1/2})i\hat{\omega}_{31}(\lambda)\omega_{13}(\lambda) \\ &\quad + O(\lambda^{-1/2})\hat{\omega}_{31}(\lambda)(-i\omega_{12}(\lambda)\hat{\omega}_{33}(\lambda) - i\omega_{13}(\lambda)\hat{\omega}_{32}(\lambda) \\ &= (1 + O(\lambda^{-1/2})[2i(1 + g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ}]. \end{aligned} \tag{6.26}$$

Next we calculate each entry of $\mathbb{A}(\cdot)^{-1}$ by dividing the appropriate cofactor by $det\,\mathbb{A}(\lambda)$. Beginning with the top row, we calculate $(\mathbb{A}(\lambda)^{-1})_{11}$ and have

$$\begin{aligned}(\mathbb{A}(\lambda)^{-1})_{11} =& \frac{|\mathbb{A}(\lambda)|_{11}}{\det \mathbb{A}(\lambda)} = \frac{-(1+i)(1+O(\lambda^{-1/2}))}{\omega[2i(1+g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ})]} \\ =& \frac{-(1+i)}{2i(1+g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ})}(1+O(\lambda^{-1/2})),\end{aligned} \tag{6.27}$$

where $|\mathbb{A}(\cdot)|_{11}$ is a cofactor of $\mathbb{A}(\cdot)$ corresponding to $a_{11}$. Now we proceed to find the remaining entries in the first row. The second entry is

$$(\mathbb{A}(\lambda)^{-1})_{12} = -\frac{|\mathbb{A}(\lambda)|_{21}}{\det \mathbb{A}(\lambda)} = \frac{(1+O(\lambda^{-1/2})}{2(1+g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ})}, \tag{6.28}$$

and the third entry of the first row is calculated as

$$\begin{aligned}(\mathbb{A}(\lambda)^{-1})_{13} =& \frac{|\mathbb{A}(\lambda)|_{31}}{\det \mathbb{A}(\lambda)} = \frac{(1-i)(1+O(\lambda^{-1/2}))}{(1+O(\lambda^{-1/2})[2i(1+g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ})]} \\ =& \frac{1-i}{2i}\frac{(1+O(\lambda^{-1/2})}{1+g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ}}.\end{aligned} \tag{6.29}$$

Now we move to the second row of $\mathbb{A}(\lambda)^{-1}$, whose first entry is

$$(\mathbb{A}(\lambda)^{-1})_{21} = -\frac{|\mathbb{A}(\lambda)|_{12}}{\det \mathbb{A}(\lambda)} = \frac{i[1+g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ}](1+O(\lambda^{-1/2}))}{(1+O(\lambda^{-1/2})[2i(1+g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ})]} = \frac{1}{2}(1+O(\lambda^{-1/2}). \tag{6.30}$$

The second entry is

$$(\mathbb{A}(\lambda)^{-1})_{22} = \frac{|\mathbb{A}(\lambda)|_{22}}{\det \mathbb{A}(\lambda)} = \frac{-iO(\lambda^{-1/2})}{(1+O(\lambda^{-1/2})[2i(1+g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ})]} = O(\lambda^{-1/2}). \tag{6.31}$$

The third entry is

$$(\mathbb{A}(\lambda)^{-1})_{23} = -\frac{|\mathbb{A}(\lambda)|_{32}}{\det \mathbb{A}(\lambda)} = \frac{i[1+g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ}](1+O(\lambda^{-1/2}))}{(1+O(\lambda^{-1/2})[2i(1+g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ})]} = \frac{(1+O(\lambda^{-1/2})}{2}. \tag{6.32}$$

Finally we turn to the third row, whose first entry is

$$(\mathbb{A}(\lambda)^{-1})_{31} = \frac{|\mathbb{A}(\lambda)|_{13}}{\det \mathbb{A}(\lambda)} = \frac{[1 + g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ}](1 + O(\lambda^{-1/2}))}{(1 + O(\lambda^{-1/2})[2i(1 + g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ})]} = \frac{(1 + O(\lambda^{-1/2})}{2i}. \tag{6.33}$$

The second entry is

$$(\mathbb{A}(\lambda)^{-1})_{32} = -\frac{|\mathbb{A}(\lambda)|_{23}}{\det \mathbb{A}(\lambda)} = \frac{O(\lambda^{-1/2})}{(1 + O(\lambda^{-1/2})[2i(1 + g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ})]} = O(\lambda^{-1/2}). \tag{6.34}$$

The third entry is

$$(\mathbb{A}(\lambda)^{-1})_{33} = \frac{|\mathbb{A}(\lambda)|_{33}}{\det \mathbb{A}(\lambda)} = \frac{-[1 + g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ}](1 + O(\lambda^{-1/2}))}{(1 + O(\lambda^{-1/2})[2i(1 + g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ})]} = \frac{-(1 + O(\lambda^{-1/2})}{2i}. \tag{6.35}$$

Taking into account the definition of $\omega_{jk}(\lambda)$ and formulae (6.27)–(6.35), we obtain the following representation:

$$\mathbb{A}(\lambda)^{-1} = \begin{bmatrix} \frac{i-1}{2}\frac{\omega_{11}(\lambda)}{(1 + g_\alpha^{-1}\sqrt{I_\alpha GJ})} & \frac{\omega_{12}(\lambda)}{(1 + g_\alpha^{-1}\sqrt{I_\alpha GJ})} & \frac{-(1+i)}{2}\frac{\omega_{13}(\lambda)}{(1 + g_\alpha^{-1}\sqrt{I_\alpha GJ})} \\ \frac{\omega_{21}(\lambda)}{2} & O(\lambda^{-1/2}) & \frac{\omega_{23}(\lambda)}{2} \\ -i\frac{\omega_{31}(\lambda)}{2} & O(\lambda^{-1/2}) & i\frac{\omega_{33}(\lambda)}{2} \end{bmatrix}. \tag{6.36}$$

Now we are in a position to compute the right–reflection matrix. Using (6.21) and (6.24), we recall that

$$\mathbb{R}_r(\lambda) = \mathbb{E}^{-1}(\lambda)\left[\mathbb{I} - \mathbb{A}(\lambda)^{-1}\mathbb{V}(\lambda)\right]\mathbb{E}^{-1}(\lambda), \tag{6.37}$$

where $\mathbf{A}(\lambda)^{-1}$ is given by (6.48) and

$$\mathbb{V}(\lambda) = \begin{bmatrix} 0 & 1-\omega_{12}(\lambda) & 0 \\ 2\omega_{21}(\lambda) & \omega_{22}(\lambda)+1 & 2\omega_{23}(\lambda) \\ 0 & 1-\hat{\omega}_{32}(\lambda) & 0 \end{bmatrix}. \tag{6.38}$$

We compute $\mathbb{A}(\lambda)^{-1}\mathbb{V}(\lambda)$ as

$$\mathbb{A}(\lambda)^{-1}\mathbb{V}(\lambda) = \begin{bmatrix} \dfrac{2\omega_{11}(\lambda)}{1+g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ}} & \dfrac{2\omega_{12}(\lambda)}{1+g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ}} & \dfrac{2\omega_{13}(\lambda)}{1+g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ}} \\ O(\lambda^{-1/2}) & O(\lambda^{-1/2}) & O(\lambda^{-1/2}) \\ O(\lambda^{-1/2}) & O(\lambda^{-1/2}) & O(\lambda^{-1/2}) \end{bmatrix}. \tag{6.39}$$

Using the latter result, we calculate that

$$\mathbb{I} - \mathbb{A}(\lambda)^{-1}\mathbb{V}(\lambda) = \begin{bmatrix} 1-\dfrac{2\omega_{11}(\lambda)}{1+g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ}} & -\dfrac{2\omega_{12}(\lambda)}{1+g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ}} & -\dfrac{2\omega_{13}(\lambda)}{1+g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ}} \\ O(\lambda^{-1/2}) & (1+O(\lambda^{-1/2})) & O(\lambda^{-1/2}) \\ O(\lambda^{-1/2}) & O(\lambda^{-1/2}) & (1+O(\lambda^{-1/2})) \end{bmatrix}. \tag{6.40}$$

Now we are in a position to calculate the right–reflection matrix. We have

$$\mathbb{R}_r = \mathbb{E}^{-1}(\lambda)(\mathbb{I} - \mathbb{A}(\lambda)^{-1}\mathbb{V}(\lambda))\mathbb{E}^{-1}(\lambda) = \begin{bmatrix} e(\lambda)^{-1} & 0 & 0 \\ 0 & \hat{e}(\lambda)^{-1} & 0 \\ 0 & 0 & e_+(\lambda)^{-1} \end{bmatrix} \times$$

$$\begin{bmatrix} 1 - \dfrac{2\omega_{11}(\lambda)}{1 + g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ}} & -\dfrac{2\omega_{12}(\lambda)}{1 + g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ}} & -\dfrac{2\omega_{13}(\lambda)}{1 + g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ}} \\ O(\lambda^{-1/2}) & (1 + O(\lambda^{-1/2})) & O(\lambda^{-1/2}) \\ O(\lambda^{-1/2}) & O(\lambda^{-1/2}) & (1 + O(\lambda^{-1/2})) \end{bmatrix} \times \tag{6.41}$$

$$\begin{bmatrix} e(\lambda)^{-1} & 0 & 0 \\ 0 & \hat{e}(\lambda)^{-1} & 0 \\ 0 & 0 & e_+(\lambda)^{-1} \end{bmatrix}.$$

Thus, computation of the right–reflection matrix $\mathbb{R}_r$ is complete.

# CHAPTER VII

# SPECTRAL ASYMPTOTICS

## 7.1 Spectral Equation

In this section, we are in a position to give an asymptotic form for the equation, whose solutions will give us asymptotic representations for the spectrum. We reproduce the main equation from Section 5.1

$$\det(\mathbb{R}_l(\lambda) - \mathbb{R}_r(\lambda)) = 0. \tag{7.1}$$

Using asymptotic approximations for the reflection matrices from Chapters V and VI (see formulae (5.57) and (6.42)), we have

$$\mathbb{R}_l(\lambda) - \mathbb{R}_r(\lambda) = \begin{bmatrix} -i\hat{\omega}_{11}(\lambda) & O(\lambda^{-1}) & O(\lambda^{-1}) \\ O(\lambda^{3/2}) & i\hat{\omega}_{22}(\lambda) & (i-1)\hat{\omega}_{23}(\lambda) \\ O(\lambda^{3/2}) & -(1+i)\hat{\omega}_{32}(\lambda) & -i\hat{\omega}_{33}(\lambda) \end{bmatrix} -$$

$$\begin{bmatrix} \left(1 - \dfrac{2\omega_{11}(\lambda)}{1+g_\alpha^{-1}\sqrt{I_\alpha GJ}}\right)\dfrac{1}{e^2(\lambda)} & -\dfrac{2\omega_{12}(\lambda)}{1+g_\alpha^{-1}\sqrt{I_\alpha GJ}}\dfrac{1}{e(\lambda)\hat{e}(\lambda)} & -\dfrac{2\omega_{13}(\lambda)}{1+g_\alpha^{-1}\sqrt{I_\alpha GJ}}\dfrac{1}{e(\lambda)e_+(\lambda)} \\ O(\lambda^{-1/2})\dfrac{1}{e(\lambda)\hat{e}(\lambda)} & \dfrac{1+O(\lambda^{-1/2})}{\hat{e}^2(\lambda)} & O(\lambda^{-1/2})\dfrac{1}{\hat{e}(\lambda)e_+(\lambda)} \\ O(\lambda^{-1/2})\dfrac{1}{e(\lambda)e_+(\lambda)} & O(\lambda^{-1/2})\dfrac{1}{\hat{e}(\lambda)e_+(\lambda)} & \dfrac{1+O(\lambda^{-1/2})}{e_+^2(\lambda)} \end{bmatrix} \tag{7.2}$$

which is

$$\mathbb{R}_l - \mathbb{R}_r = \begin{bmatrix} i\hat{\omega} - e^{-2}\omega & (i-1)\hat{\omega} - e^{-1}\hat{e}^{-1}O(\lambda^{-1/2}) & (\frac{Q}{P})(i-1)\lambda^{1/2}\hat{\omega} + e^{-1}e_+^{-1}r_{13}\lambda^{3/2}\hat{\omega} \\ -(i+1)\hat{\omega} - e^{-1}\hat{e}^{-1}O(\lambda^{-1/2}) & -i\hat{\omega} - \hat{e}^{-2}\omega & (\frac{Q}{P})(1-i)\lambda^{1/2}\hat{\omega} + \hat{e}^{-1}e_+^{-1}r_{23}\lambda^{3/2}\hat{\omega} \\ O(\lambda^{-3}) - e^{-1}e_+^{-1}O(\lambda^{-2}) & O(\lambda^{-3}) - \hat{e}^{-1}e_+^{-1}O(\lambda^{-2}) & -1 + O(\lambda^{-2.5}) - e_+^{-2}(1-r_{33})\hat{\omega} \end{bmatrix}. \quad (7.3)$$

**Derivations of the spectral asymptotics.** We now recall that $\mathcal{L}_{g_h g_\alpha}$ is a dissipative operator, which means that its eigenvalues must be in the closed upper half–plane. So we may write

$$\lambda = \bar{x} + i\bar{y}, \quad \lambda^{1/2} = x + iy, \tag{7.4}$$

where $\bar{x} \in \mathbb{R}$, and $\bar{y}$, $x$, $y > 0$. Let us write the expressions for $e(\lambda)$, $\hat{e}(\lambda)$, and $e_+(\lambda)$ in terms of $x$ and $y$.

We recall definitions (6.1) and (4.35) to calculate $e(\lambda)$ as

$$e(\lambda) = e^{i\gamma(\lambda)L} = exp\left\{ i\lambda\sqrt{\frac{I_\alpha}{GJ}}L \right\}. \tag{7.5}$$

Notice that $e(\lambda)$ is bounded in the upper half–plane, but $e(\lambda)^{-1}$ is unbounded. In a similar manner, we calculate $\hat{e}(\lambda)$ as

$$\hat{e}(\lambda) = e^{\hat{\gamma}(\lambda)L} = exp\left\{ \lambda^{1/2}\left(\frac{\Delta}{EI\,I_\alpha}\right)^{1/4} L \right\} \tag{7.6}$$

Notice that $\hat{e}(\lambda)$ is unbounded in the upper half–plane, but $\hat{e}(\lambda)^{-1}$ is bounded. We calculate $e_+(\lambda)$ as

$$e_+(\lambda) = e^{i\Gamma(\lambda)L} = exp\left\{ i\lambda^{1/2}\left(\frac{\Delta}{EI\,I_\alpha}\right)^{1/4} L \right\}. \tag{7.7}$$

Notice that $e_+(\lambda)$ is bounded in the upper half–plane, but $e_+(\lambda)^{-1}$ is not.

If we now multiply both sides of the reflection matrices by the non–singular matrix

$$\tilde{\mathbb{E}}(\lambda) = \begin{bmatrix} e(\lambda) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e_+(\lambda) \end{bmatrix}, \tag{7.8}$$

whose entries are bounded, then we will arrive at a matrix, whose entries are all bounded in the upper half–plane. Thus we have changed problem (7.1) into the one involving a matrix, whose entries are bounded in the upper half–plane. Thus, we are looking for the solutions of the following equation:

$$
\begin{aligned}
0 &= \det(\mathbb{R}_l(\lambda) - \mathbb{R}_r(\lambda)) = \det \tilde{\mathbb{E}}(\lambda) \det(\mathbb{R}_l(\lambda) - \mathbb{R}_r(\lambda)) \det \tilde{\mathbb{E}}(\lambda) \\
&= \det \left[\tilde{\mathbb{E}}(\lambda)(\mathbb{R}_l(\lambda) - \mathbb{R}_r(\lambda))\tilde{\mathbb{E}}(\lambda)\right]
\end{aligned}
$$

$$
= \det \left[\begin{array}{ccc}
-e^2(\lambda) - 1 + \dfrac{2\omega_{11}(\lambda)}{1 + g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ}} & \dfrac{2\omega_{12}(\lambda)}{1 + g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ}}\dfrac{1}{\hat{e}(\lambda)} + O(\lambda^{-1}) & \dfrac{2\omega_{13}(\lambda)}{1 + g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ}} + O(\lambda^{-1}) \\
O(\lambda^{-3/2}) & i - \dfrac{1 + O(\lambda^{-1/2})}{\hat{e}(\lambda)^2} & e_+(\lambda)(i-1)\hat{\omega}_{23}(\lambda) - \dfrac{1}{\hat{e}(\lambda)}O(\lambda^{-1/2}) \\
O(\lambda^{-3/2}) & -e_+(\lambda)(1+i)\hat{\omega}_{32}(\lambda) - \dfrac{1}{\hat{\gamma}(\lambda)}O(\lambda^{-1/2}) & -e_+^2 i - \omega_{33}(\lambda)
\end{array}\right].
\tag{7.9}
$$

Let us expand this determinant with respect to the entries of the first column.

$$
\left[-e^2(\lambda) - 1 + \frac{2\omega_{11}(\lambda)}{1 + g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ}}\right] \times
$$

$$
\det \begin{bmatrix}
i - \frac{\omega(\lambda)}{\hat{e}^2(\lambda)} & e_+(\lambda)(i-1)\hat{\omega}(\lambda) - \frac{1}{\hat{e}(\lambda)}O(\lambda^{-1/2}) \\
-e_+(\lambda)(i+1)\hat{\omega}(\lambda) - \frac{1}{\hat{e}(\lambda)}O(\lambda^{-1/2}) & -e_+^2(\lambda)i - \omega(\lambda)
\end{bmatrix} = 0.
\tag{7.10}
$$

## 7.2 The $\alpha$–branch of the Spectrum

In this section, first we will derive the leading term of the asymptotic approximation for the $\alpha$–branch of the spectrum. Then we will justify the estimate for the remainder term. We now return to Eq.(7.10).

We start with the equation

$$0 = \left[ -e^2(\lambda) - 1 + \frac{2\omega 11(\lambda)}{1 + g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ}} \right] \tag{7.11}$$

which can be reduced to the form

$$e^2(\lambda) = \frac{1 - g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ}}{1 + g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ}} + O(\lambda^{-1/2}). \tag{7.12}$$

Eq. (7.12) can be represented as

$$e^{2i\sqrt{\tilde{I}_\alpha/GJ}L\lambda} = \frac{1 - g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ}}{1 + g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ}} + g(\lambda), \tag{7.13}$$

where $g(\lambda)$ is an analytic function, which can be estimated in the following way: $|g(\lambda)| \le C|\lambda|^{-1/2}$, where $C$ is an absolute constant. If we replace $g(\lambda)$ with zero, we will have what we call the *model equation*

$$e^{2i\sqrt{\tilde{I}_\alpha/GJ}L\lambda} = \frac{1 - g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ}}{1 + g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ}}. \tag{7.14}$$

Obviously the solutions of Eq.(7.15) are given by the formula

$$2i\sqrt{\tilde{I}_\alpha/GJ}L\lambda_n = \ln\left[\frac{1 - g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ}}{1 + g_\alpha^{-1}\sqrt{\tilde{I}_\alpha GJ}} + O(\lambda^{-1/2})\right] + i2\pi n. \tag{7.15}$$

Thus for each $n \in \mathbb{Z}$, we will have a corresponding $\mathring{\lambda}_n^\alpha$. Solving for these $\mathring{\lambda}_n^\alpha$, we have

$$\mathring{\lambda}_n^\alpha = \frac{i}{2\sqrt{\tilde{I}_\alpha/GJL}} \ln\left[\frac{g_\alpha + \sqrt{\tilde{I}_\alpha GJ}}{g_\alpha - \sqrt{\tilde{I}_\alpha GJ}}\right] + \frac{\pi n}{\sqrt{\tilde{I}_\alpha/GJL}}. \tag{7.16}$$

Thus, formula (7.16) gives us the leading term in the asymptotical representation for the $\alpha$-branch of the eigenvalues. Note that under our assumption $g_\alpha \neq \sqrt{\tilde{I}_\alpha GJ}$, the logarithmic term is well-defined.

Now we will justify the remainder term appearing in the formula for the $\alpha$-branch. The main tool is Rouche's Theorem: *Assume that $F$ and $g$ are two analytic functions in the closed disk centered at the point $a$ of radius $r$ (which we denote as $B_r(a) \cup \partial B_r(a)$) If the following estimate is valid*

$$|F(\lambda)| > |g(\lambda)| > 0, \quad \lambda \in \partial B_r(a), \tag{7.17}$$

*then the number of zeros counting their multiplicities of the functions $(F+g)$ and $F$ coincide in $B_r(a)$.* By using this theorem, we will find circles of radii $\epsilon_n$ around the approximated eigenvalues $\overset{\circ}{\lambda}{}^\alpha_n$, so that the actual eigenvalue $\lambda^\alpha_n$ is in the circle of radius $\epsilon_n$ at $\overset{\circ}{\lambda}{}^\alpha_n$.

If we let $F(\lambda) = g(\lambda)$, for $\lambda \in B_{\epsilon_n}(\lambda^o_n)$, which means

$$\lambda = \lambda^o_n + \epsilon_\varepsilon e^{i\varphi} \tag{7.18}$$

where $\epsilon_\varepsilon < \epsilon_n$ and $0 \le \varphi < 2\pi$. We want to get an estimate for $|F(\lambda)|$, so for $\lambda \in B_{\epsilon_n}(\lambda^o_n)$ we have

$$\begin{aligned}
|F(\lambda)| &= \left| e^{2i\sqrt{I_\alpha/GJ}L\lambda^o_n} e^{2i\sqrt{I_\alpha/GJ}L\epsilon_\epsilon e^{i\varphi}} - \frac{1-g_\alpha^{-1}\sqrt{I_\alpha GJ}}{1+g_\alpha^{-1}\sqrt{I_\alpha GJ}} \right| \\
&= \left| e^{2i\sqrt{I_\alpha/GJ}L\left[\frac{i}{2\sqrt{I_\alpha/GJ}L}\ln\left[\frac{g_\alpha+\sqrt{I_\alpha GJ}}{g_\alpha-\sqrt{I_\alpha GJ}}\right]+\frac{\pi n}{\sqrt{I_\alpha/GJ}L}\right]} e^{2i\sqrt{I_\alpha/GJ}L\epsilon_\epsilon e^{i\varphi}} - \frac{1-g_\alpha^{-1}\sqrt{I_\alpha GJ}}{1+g_\alpha^{-1}\sqrt{I_\alpha GJ}} \right| \\
&= \left| \frac{1-g_\alpha^{-1}\sqrt{I_\alpha GJ}}{1+g_\alpha^{-1}\sqrt{I_\alpha GJ}} \left[ e^{2i\sqrt{I_\alpha/GJ}L\epsilon_\epsilon e^{i\varphi}} - 1 \right] \right|
\end{aligned} \tag{7.19}$$

Let us now consider $\epsilon_\varepsilon = \epsilon_n$

$$e^{2i\sqrt{I_\alpha/GJ}L\epsilon_\varepsilon e^{i\varphi}} - 1 = e^{\chi\epsilon_n} - 1 \tag{7.20}$$

where

$$\chi = 2i\sqrt{I_\alpha/GJ}Le^{i\varphi}. \tag{7.21}$$

If $\chi\epsilon_n$ is small, then the Taylor series for

$$e^{\chi\epsilon_n} = \sum_{m=0}^{\infty} \frac{(\chi\epsilon_n)^m}{m!} = 1 + \frac{\chi\epsilon_n}{1} + O(\epsilon_n^2). \tag{7.22}$$

If we use the result (7.22), we have

$$e^{\chi\epsilon_n} - 1 = \chi\epsilon_n + O(\epsilon_n^2). \tag{7.23}$$

So returning to the estimate for $|F(\lambda)|$ when $\lambda \in \partial B_{\epsilon_n}(\lambda_n^o)$, we have

$$|F(\lambda)|_{\lambda\in\partial B_{\epsilon_n}(\lambda_n^o)} = \left|\frac{1 - g_\alpha^{-1}\sqrt{I_\alpha GJ}}{1 + g_\alpha^{-1}\sqrt{I_\alpha GJ}}\right| |\chi|\epsilon_n \left|(1 + O(\epsilon_n))\right| = C_1|\chi|\epsilon_n \left|(1 + O(\epsilon_n))\right|. \tag{7.24}$$

Let us examine $g(\lambda)$ for $\lambda \in \partial B_{\epsilon_n}(\lambda_n^o)$ . We have

$$|g(\lambda)| \leq C_2|\lambda|^{-1/2} \leq 2C_2|\lambda_n^o|^{-1/2}. \tag{7.25}$$

If we compare (7.24) and (7.25), we can consider

$$\epsilon_n = \frac{3C_2|\lambda_n^o|^{-1/2}}{C_1(2L\sqrt{I_\alpha/GJ})}. \tag{7.26}$$

Using $\epsilon_n$ from (7.26) in the estimate for $|F(\lambda)|_{\lambda\in\partial B_{\epsilon_n}(\lambda_n^o)}$ in (7.24) we have :

$$|F(\lambda)|_{\lambda\in\partial B_{\epsilon_n}(\lambda_n^o)} = 3C_2|\lambda_n^o|^{-1/2} \left|(1 + O(\epsilon_n))\right| \geq 2.5C_2|\lambda_n^o|^{-1/2}, \tag{7.27}$$

where n is large enough. This gives

$$|F(\lambda)|_{\lambda\in\partial B_{\epsilon_n}(\lambda_n^o)} \geq 2.5C_2|\lambda_n^o|^{-1/2} > 2C_2|\lambda_n^o|^{-1/2} \geq |g(\lambda)|. \tag{7.28}$$

Using Rouche's Theorem, since $|F(\lambda)| > |g(\lambda)$ on the boundary, the analytic functions $F(\lambda)$ and $F(\lambda) + g(\lambda)$ have the same number of roots counting multiplicities. So the roots we found of $F(\lambda)$ correspond 1-1 of the roots of $f(\lambda) + g(\lambda)$, hence we arrive at the $\alpha$–branch of our roots:

$$\lambda_n^\alpha = \frac{i}{2\sqrt{I_\alpha/GJL}} \ln\left[\frac{g_\alpha + \sqrt{I_\alpha GJ}}{g_\alpha - \sqrt{I_\alpha GJ}}\right] + \frac{\pi n}{\sqrt{I_\alpha/GJL}} + O(\lambda^{-1/2}). \tag{7.29}$$

## 7.3 The $\beta$–branches of the Spectrum

In this section, we investigate the roots of the second part of the model equation, after simplifing we have

$$0 = \det\begin{bmatrix} i - \frac{1}{\hat{e}^2(\lambda)} & e_+(\lambda)(i-1) \\ -e_+(\lambda)(1+i) & -e_+^2 i - 1 \end{bmatrix} + O(\lambda^{-1/2}). \tag{7.30}$$

This holds when $\tilde{x}, \tilde{y} > 0$ and $C = (\frac{\Delta}{EI I_\alpha})^{1/4} L$

$$0 = \det\begin{bmatrix} i - e^{2C(-\tilde{x}-i\tilde{y})} & e^{C(i\tilde{x}-\tilde{y})}(i-1) \\ e^{C(i\tilde{x}-\tilde{y})}(1+i) & -ie^{2C(i\tilde{x}-\tilde{y})} - 1 \end{bmatrix}. \tag{7.31}$$

We need to consider three cases.

CASE I: $\tilde{y} > \tilde{x}$, and $\tilde{y} \to \infty$. The model for this is

$$\det\begin{bmatrix} i - e^{2C(-\tilde{x}-i\tilde{y})} & 0 \\ 0 & -1 \end{bmatrix} = 0. \tag{7.32}$$

The model equation reduces to

$$e^{2C(-\tilde{x}-i\tilde{y})} = i. \tag{7.33}$$

This means

$$e^{-2C(\tilde{x}_k+i\tilde{y}_k)} = e^{-i\frac{\pi}{2}(4k-1)}. \tag{7.34}$$

This gives

$$\tilde{x}_k = 0 \tag{7.35}$$

and

$$\tilde{y}_k = \frac{\pi}{4C}(4k-1). \tag{7.36}$$

But we know $\tilde{y}_k > 0$ so for $k = 1, 2, 3, \ldots$

$$u_k^o = i\frac{\pi}{4C}(4k-1). \tag{7.37}$$

CASE II: $\tilde{x} > \tilde{y}$, and $\tilde{x} \to \infty$. The model for this is

$$0 = \det \begin{bmatrix} i & e^{C(i\tilde{x}-\tilde{y})}(i-1) \\ -e^{C(i\tilde{x}-\tilde{y})}(1+i) & -ie^{2C(i\tilde{x}-\tilde{y})} - 1 \end{bmatrix}. \tag{7.38}$$

This gives the model equation as

$$e^{2C(i\tilde{x}-\tilde{y})} - i + e^{2C(i\tilde{x}-\tilde{y})}(1+i)(i-1) = 0, \tag{7.39}$$

or

$$-e^{2C(i\tilde{x}-\tilde{y})} = i. \tag{7.40}$$

This means

$$e^{2C(i\tilde{x}_k-\tilde{y}_k)} = e^{\frac{\pi}{2}i(4k-1)}. \tag{7.41}$$

This gives us that $\tilde{y}_k = 0$ and $\tilde{x}_k = \ frac\pi 4C(4k-1)$ but $\tilde{x}_k > 0$, so for $k = 1, 2, 3, \ldots$ we have

$$u_k^o = \frac{\pi}{4C}(4k-1). \tag{7.42}$$

CASE III: $\tilde{x} = \tilde{y} \to \infty$ The model for this case is

$$\det \begin{bmatrix} i & 0 \\ 0 & -1 \end{bmatrix} = 0 \tag{7.43}$$

but this cannot happen.

ROUCHE'S THEOREM: From the original model for the $\beta$–branch we have

$$\det \begin{bmatrix} i - \frac{1}{\hat{e}^2(\lambda)} & e_+(\lambda)(i-1) \\ -e_+(\lambda)(1+i) & -e_+^2 i - 1 \end{bmatrix} = \hat{g}(\lambda) \tag{7.44}$$

where $\hat{g}(\lambda)$ is an analytic function, which can be estimated in the following way $|\hat{g}(\lambda)| \le \hat{C}|\lambda|^{-1/2}$, where $\hat{C}$ is an absolute constant. Let

$$\hat{F}(\lambda) = \det \begin{bmatrix} i - \frac{1}{\hat{e}^2(\lambda)} & e_+(\lambda)(i-1) \\ -e_+(\lambda)(1+i) & -e_+^2 i - 1 \end{bmatrix}. \tag{7.45}$$

Notice $\hat{F}(\lambda) = \hat{g}(\lambda)$. Simplifying the determinate;

$$\hat{F}(\lambda) = -e^{2iC_1\lambda^{1/2}} + e^{-2C_1\lambda^{1/2}} + i(1 + e^{2C_1(i-1)\lambda^{1/2}}), \tag{7.46}$$

where $C_1 = \left(\frac{\Delta}{EI I_\alpha}\right)^{1/4} L$. Let $u = \lambda^{1/2}$. Consider $\hat{F}(u), \hat{G}(u)$ where $\hat{F}(u) = \hat{F}(\lambda)\mid_{u=\lambda^{-1/2}}$ $\hat{G}(u) = \hat{g}(\lambda)\mid_{u=\lambda^{-1/2}}$. This gives

$$\hat{F}(u) = -e^{2iC_1 u} + e^{-2C_1 u} + i(1 + e^{2C_1(i-1)u}) \tag{7.47}$$

and $|\hat{G}(u)| \le \hat{C}|u|^{-1}$. Let $u_n^o = \frac{\pi}{4c}(4n-1)$ Consider $u \in B_{\epsilon_n}(u_n^o)$, this means

$$u = u_n^o + \epsilon_\varepsilon e^{i\varphi}, \tag{7.48}$$

where $\epsilon_\varepsilon < \epsilon_n$ and $0 \le \varphi < 2\pi$. We want an estimate for $|\hat{F}(u)|$, so for $u \in B_{\epsilon_n}(u_n^o)$ and simplifying (7.47) we get

$$|\hat{F}(u)| = \left| e^{-2C_1\epsilon_\varepsilon e^{i\varphi}} \left( e^{-\frac{\pi}{2}(4n-1)} + e^{-\frac{\pi}{2}(4n-1)} e^{2C_1 i\epsilon_\varepsilon e^{i\varphi}} \right) + i + ie^{-2C_1 i\epsilon_\varepsilon e^{i\varphi}} \right|. \tag{7.49}$$

Let $\epsilon_\varepsilon = \epsilon_n$ and let $\chi = 2C_1 e^{i\varphi}$, this gives

$$|\hat{F}(u)| = \left|(e^{-\frac{\pi}{2}(4n-1)} + i)\right| \left|1 + e^{-\chi\epsilon_n}\right|. \tag{7.50}$$

If $-\chi\epsilon_n$ is small, then the Taylors series for

$$e^{-\chi\epsilon_n} = \sum_{m=0}^{\infty} \frac{(-\chi\epsilon_n)^m}{m!} = 1 - \frac{\chi\epsilon_n}{1} + O(\epsilon_n^2). \tag{7.51}$$

If we use this result, we have:

$$1 + e^{-chi\epsilon_n} = 2 - \chi\epsilon_n + O(\epsilon_n^2) \tag{7.52}$$

So returning to our estimate for $|\hat{F}(u)|$ we have

$$\begin{aligned} \left|\hat{F}(u)\right|_{u\in\partial B_{\epsilon_n}(u_n^o)} &= C_1\left|2 - \chi\epsilon_n + O(\epsilon_n^2)\right| \\ &= 2C_1\left|1 - \frac{\chi\epsilon_n}{2} + O(\epsilon_n^2)\right| \\ &= 2C_1\left|1 + O(\epsilon_n)\right| \\ &\geq 1.5C_1. \end{aligned} \tag{7.53}$$

This gives us for $n$ large enough

$$\left|\hat{F}(u)\right|_{u\in\partial B_{\epsilon_n}(u_n^o)} \geq 1.5C_1 \geq \frac{\hat{C}}{n} \geq \left|\hat{G}(u)\right|. \tag{7.54}$$

So by Rouche's Theorem since $\left|\hat{F}(u)\right| > \left|\hat{G}(u)\right|$ on the boundary, the analytic functions $\left|\hat{F}(u)\right|$ and $\left|\hat{F}(u)\right| + \left|\hat{G}(u)\right|$ have the same number of roots counting multiplicity, so the roots of $\left|\hat{F}(u)\right|$ correspond 1-1 to the roots of $\left|\hat{F}(u)\right| + \left|\hat{G}(u)\right|$, hence we have

$$u_n = u_n^o + O(u_n^{-1}) = u_n^o + O(|n|^{-1}) \tag{7.55}$$

but

$$\begin{aligned} \lambda_n^\beta &= (u_n)^2 \\ &= (u_n^o + O(|n|^{-1}))^2 \\ &= (u_n^o)^2 + O(u_n^o|n|^1) \\ &= \frac{\pi^2}{16L^2(\frac{\Delta}{EII_\alpha})^{1/2}}(4n-1)^2 + O(1). \end{aligned} \tag{7.56}$$

# CHAPTER VIII

# CONCLUSION AND DIRECTIONS OF FUTURE RESEARCH

As a way of summarizing we point out that in this dissertation a model of a finite beam that can bend and twist has been studied. The differential parts formed a coupled linear hyperbolic system. Our main interest is the dynamics generator of this system. We showed that its spectrum consists of two branches and derived the precise spectral asymptotics of those branches.

• Further work includes finding a numerical scheme to calculate the spectrum. The beginning work is is being done by looking at the decoupled system ($S = \mu = f_1 = f_2 = 0$.) The decoupled problem describes the individual independent vibrations of a beam and a string with nonstandard sets of the boundary conditions. Each is a challenging problem in its own right. In the numerical study we are planning to start with the string boundary–value problem. Looking for a solution that is not time dependent leads to the basic string problem with boundary conditions inherited from the coupled system. The dynamics generator for this string model is also a non-selfadjoint operator. The latter fact means that this nonselfadjoint operator might have multiple eigenvalues with the corresponding chains of eigenvectors and associate vectors. Our goal will be to find an appropriate numerical approach, which is capable "to check" the multiplicity of a multiple eigenvalue and "to feel" the difference between an eigenvector and an associate vector.

• The second numerical task will be to carry out a similar study of the boundary problem for the beam boundary–value problem. In this part, we expect to have a two branch spectrum. In its numerical part, we would like to locate the number of an eigenvalue, from which numerical results for the eigenvalues are close within given accuracy to the leading asymptotical terms obtained in analytical part of the problem.

• Finally, we would like to point out that numerically our problem is challenging as well because of the typical numerical data, i.e., the stiffness coefficients are of orders $O(10^6 - -10^7)$, while all other structural parameters are of orders $O(1)$. The latter fact could lead to instability of a specific numerical scheme.

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