A METHOD OF SOLUTION OF THE WEIGHTED HEAT EQUATION

by

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CONTENTS

ABSTRACT

In this paper, a method of finding solutions of the heterogeneous heat equation is developed. Taking Fourier transforms, then Laplace transforms, of the heterogeneous heat equation yields an equation that involves a convolution with the solution. In order to make sense of what it would mean to "invert" the convolution (so that the solution is isolated), the "mollified" Fourier transform is introduced. This transform is used in the development of the method of inverting the convolution.

CHAPTER I

INTRODUCTION

Much attention has been paid to the homogeneous diffusion problem, i.e., the "heat equation,"

$$\frac{\partial^2}{\partial x^2}u(x,t) = \frac{\partial}{\partial t}u(x,t) \ . \tag{1.1}$$

If u is the temperature of a body at position x and time t, then this equation describes the behaviour of heat in homogeneous media (heat flows in the direction of decreasing temperature). This same problem can describe the behaviour of porous media flow in homogeneous media, since fluid flows in the direction of decreasing pressure. In order to consider fluid flow through heterogeneous media, however, a weight $w(x)$ must be added to equation (1.1) . This weight function is a measure of the conductivity of the medium at position x. The equation

$$\frac{\partial}{\partial x}\left[w(x)\frac{\partial}{\partial x}u(x,t)\right] = \frac{\partial}{\partial t}u(x,t) \tag{1.2}$$

describes the flow of fluid (or heat) through heterogeneous media. Compared with (1.1), very little work has been done on the heterogeneous diffusion problem, i.e., the "weighted heat equation."

CHAPTER II

TRANSFORMS

2.1 Fourier Transform

Definition 2.1 (Fourier Transform) *Let $\tilde{u}$ denote the Fourier transform of u, defined by*

$$\tilde{u}(\phi, t) = \mathcal{F}\left\{u(x, t)\right\}(\phi, t) = \int_{-\infty}^{\infty} e^{-2\pi i \phi x} u(x, t) dx \ .$$

Note that the Fourier transform is linear. In this paper. the Fourier transform will always take x to ϕ. The following are well-known[7] properties of the Fourier transform.

Theorem 2.2 (Inversion Theorem)

$$f(x) = \int_{-\infty}^{\infty} e^{2\pi i \phi x} \tilde{f}(\phi)\, d\phi$$

Theorem 2.3 (Derivative Theorem) *Let f be a function of x, then*

$$\mathcal{F}\left\{\frac{df}{dx}\right\} = 2\pi i \phi \mathcal{F}\left\{f\right\} \ .$$

Definition 2.4 (Convolution) *The* convolution *binary operator "$*$" is defined by*

$$\begin{aligned} f(\phi) * g(\phi) &= \int_{-\infty}^{\infty} f(\phi - \psi) g(\psi) d\psi \\ &= \int_{-\infty}^{\infty} f(\psi) g(\phi - \psi) d\psi \\ &= g(\phi) * f(\phi) \ . \end{aligned} \tag{2.1}$$

The convolution operator is commutative by (2.1).

Theorem 2.5 (Convolution Theorem)

$$\mathcal{F}\left\{f * g\right\} = \mathcal{F}\left\{f\right\} \mathcal{F}\left\{g\right\} = \tilde{f}\tilde{g}$$

and

$$\mathcal{F}^{-1}\left\{\tilde{f} * \tilde{g}\right\} = \mathcal{F}^{-1}\left\{\tilde{f}\right\} \mathcal{F}^{-1}\left\{\tilde{g}\right\} = fg \ .$$

2.2 Laplace Transform

Definition 2.6 (Laplace Transform) *Let U denote the Laplace transform of u, defined by*

$$U(x,s) = \mathcal{L}\left\{u(x,t)\right\}(x,s) = \int_0^\infty e^{-st}u(x,t)dt \ .$$

Note that the Laplace transform is linear. In this paper, the Laplace transform will always take t to s. Let $\mathcal{L}\left\{\mathcal{F}\left\{u\right\}\right\}$ be denoted by $\tilde{U}$. The following are well-known[1] properties of the Laplace transform.

Theorem 2.7 (Inversion Theorem)

$$f(t) = \lim_{b\to\infty} \frac{1}{2\pi i}\int_{c-ib}^{c+ib} e^{ts}F(s)\,ds \ .$$

Theorem 2.8 (Derivative Theorem) *Let f be a function of t, then*

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = s\mathcal{L}\left\{f\right\} - f(0) \ .$$

2.3 Delta Function

The Dirac Delta function "δ" has the following properties.

$$\begin{aligned}
\mathcal{F}\left\{x^k\right\} &= \frac{\delta^{(k)}(\omega)}{(-2\pi i)^k} \\
f * \delta^{(k)} &= f^{(k)} \\
\delta(-x) &= \delta(x) \\
\delta^{(k)}(cx) &= c^{-(k+1)}\delta^{(k)}(x) \qquad\qquad c \neq 0 \\
x^n\delta^{(k)}(x) &= \begin{cases} (-1)^n k!\delta^{(k-n)}(x) & n \leq k \\ 0 & otherwise \end{cases} \ .
\end{aligned} \tag{2.2}$$

2.4 Heterogeneous Heat Equation

In order to remove the outer derivative in the heterogeneous heat equation (1.2), the Fourier transform is applied. By the derivative theorem (theorem 2.3), the left hand side becomes

$$\mathcal{F}\left\{\frac{\partial}{\partial x}\left[w\frac{\partial u}{\partial x}\right]\right\} = \int_{-\infty}^{\infty} e^{-2\pi i\omega x}\frac{\partial}{\partial x}\left[w\frac{\partial u}{\partial x}\right]dx$$

$$= 2\pi i\phi \int_{-\infty}^{\infty} e^{-2\pi i\phi x} w \frac{\partial u}{\partial x} dx$$
$$= 2\pi i\phi \mathcal{F}\left\{ w \frac{\partial u}{\partial x} \right\} .$$

hence the heterogeneous heat equation (1.2) is transformed into

$$\frac{\partial}{\partial t}\tilde{u}(\phi,t) = 2\pi i\phi \mathcal{F}\left\{ w(x)\frac{\partial}{\partial x}u(x,t)\right\}(\phi,t) .$$

In order to integrate by parts, let $f = e^{-2\pi i\phi x}w(x)$ and $dg = \frac{\partial u}{\partial x}dx$, which gives $df = [w'(x) - 2\pi i\phi w(x)]e^{-2\pi i\phi x}dx$ and $g = u$. Assuming the boundary terms of fg go to zero,

$$\begin{aligned}
2\pi i\phi \mathcal{F}\left\{ w\frac{\partial u}{\partial x}\right\} &= 2\pi i\phi \int_{-\infty}^{\infty} e^{-2\pi i\phi x} w \frac{\partial u}{\partial x} dx \\
&= 2\pi i\phi \int_{-\infty}^{\infty} f\, dg \\
&= 2\pi i\phi\, fg|_{-\infty}^{\infty} - 2\pi i\phi \int_{-\infty}^{\infty} g\, df \\
&= -2\pi i\phi \int_{-\infty}^{\infty} u \left[e^{-2\pi i\phi x}w' - 2\pi i\phi e^{-2\pi i\phi x}w\right] dx \\
&= -2\pi i\phi \int_{-\infty}^{\infty} e^{-2\pi i\phi x} w' u\, dx \\
&\quad -4\pi^2\phi^2 \int_{-\infty}^{\infty} e^{-2\pi i\phi x} wu\, dx \\
&= -2\pi i\phi \mathcal{F}\{zu\} - 4\pi^2\phi^2\mathcal{F}\{wu\} \\
&= -2\pi i\phi(\tilde{z} * \tilde{u}) - 4\pi^2\phi^2(\tilde{w} * \tilde{u})
\end{aligned}$$

where $z = w'$. The heterogeneous heat equation (1.2) therefore has the form

$$\frac{\partial \tilde{u}}{\partial t} = -2\pi i\phi(\tilde{z} * \tilde{u}) - 4\pi^2\phi^2(\tilde{w} * \tilde{u}) . \tag{2.3}$$

The Laplace transform can be used to remove the derivative from the left hand side of (2.3). Taking the Laplace transform of each term in (2.3) gives

$$\left.\begin{aligned}
\mathcal{L}\left\{\tfrac{\partial}{\partial t}\tilde{u}(\phi,t)\right\}(\phi,s) &= s\mathcal{L}\{\tilde{u}(\phi,t)\}(\phi,s) - \tilde{h}(\phi) \\
&= s\tilde{U}(\phi,s) - \tilde{h}(\phi) \\
\mathcal{L}\{-2\pi i\phi(\tilde{z}(\phi) * \tilde{u}(\phi,t))\}(\phi,s) &= -2\pi i\phi(\tilde{z}(\phi) * \tilde{U}(\phi,s)) \\
\mathcal{L}\{-4\pi^2\phi^2(\tilde{w}(\phi) * \tilde{u}(\phi,t))\}(\phi,s) &= -4\pi^2\phi^2(\tilde{w}(\phi) * \tilde{U}(\phi,s))
\end{aligned}\right\} \tag{2.4}$$

where $\tilde{U}$ denotes $\mathcal{L}\{\mathcal{F}\{u\}\}$ and $h(x) = u(x,0)$ is the initial data. Combining (2.3) and (2.4) gives

$$s\tilde{U} - \tilde{h} = -2\pi i\phi(\tilde{z} * \tilde{U}) - 4\pi^2\phi^2(\tilde{w} * \tilde{U})$$

or

$$\tilde{h} = s\tilde{U} + 2\pi i\phi(\tilde{z} * \tilde{U}) + 4\pi^2\phi^2(\tilde{w} * \tilde{U}) \ . \tag{2.5}$$

In order to factor out $\tilde{U}$ in equation (2.5), let $\psi = \phi$, then

$$\tilde{h} = \left[s\delta * \tilde{U} + 2\pi i\psi(\tilde{z} * \tilde{U}) + 4\pi^2\psi^2(\tilde{w} * \tilde{U})\right]_{\psi=\phi} \tag{2.6}$$

$$= \left[\left(s\delta + 2\pi i\psi\tilde{z} + 4\pi^2\psi^2\tilde{w}\right) * \tilde{U}\right]_{\psi=\phi} \ . \tag{2.7}$$

Since the ϕ's in (2.5) are not involved in the convolution with $\tilde{U}$, they had to be re-named ψ in (2.6) in order to avoid confusion in (2.7) as to what should be convolved with $\tilde{U}$. When evaluating (2.7), ψ can be taken to be ϕ only after the convolution with $\tilde{U}$ has been performed. Now, since

$$\tilde{z} = \mathcal{F}\{w'\} = 2\pi i\phi\mathcal{F}\{w\} = 2\pi i\phi\tilde{w} \ ,$$

equation (1.2) becomes

$$\tilde{h} = \left[\left(s\delta - 4\pi^2\psi\phi\tilde{w} + 4\pi^2\psi^2\tilde{w}\right) * \tilde{U}\right]_{\psi=\phi}$$

$$= \left\{\left[s\delta + 4\pi^2\psi(\psi - \phi)\tilde{w}\right] * \tilde{U}\right\}_{\psi=\phi} \ . \tag{2.8}$$

Much would be accomplished if equation (2.8) could be solved for $\tilde{U}$. The presence of the Dirac delta function "δ," however, makes this difficult. A method will be developed (in Chapter III) that avoids this difficulty.

2.5 Legendre Example

In the case where $w(x) = 1 - x^2$, the heterogeneous diffusion problem (1.2) is known to have as a solution $u(x,t) = xe^{-2t}$. This can be verified by direct substitution of w and u into equation (1.2):

$$\frac{\partial}{\partial x}\left[w\frac{\partial u}{\partial x}\right] = \frac{\partial}{\partial x}\left[(1 - x^2)\frac{\partial}{\partial x}(xe^{-2t})\right]$$

$$
\begin{aligned}
&= \frac{\partial}{\partial x}\left[(1-x^2)e^{-2t}\right] \\
&= \frac{\partial}{\partial x}(e^{-2t} - x^2 e^{-2t}) \\
&= -2xe^{-2t} = \frac{\partial u}{\partial t} \ .
\end{aligned}
$$

In order to show that

$$
\left.\begin{aligned}
w(x) &= 1 - x^2 \\
u(x,t) &= xe^{-2t}
\end{aligned}\right\} \tag{2.9}
$$

satisfies equation (2.8), compute each term, and substitute (2.9) for w and u:

$$
\begin{aligned}
\hat{h} &= \mathcal{F}\{h\} = \mathcal{F}\{x\} = \frac{i\delta'}{2\pi} && (2.10)\\
\hat{u} &= \mathcal{F}\{u\} = \mathcal{F}\left\{xe^{-2t}\right\} = \frac{ie^{-2t}\delta'}{2\pi} \\
\hat{U} &= \mathcal{L}\{\hat{u}\} = \mathcal{L}\left\{\frac{ie^{-2t}\delta'}{2\pi}\right\} = \frac{i\delta'}{2\pi(s+2)} \\
\hat{w} &= \mathcal{F}\{w\} = \mathcal{F}\left\{1 - x^2\right\} = \delta + \frac{\delta''}{4\pi^2} \ .
\end{aligned}
$$

Now, substituting the above terms into equation (2.8),

$$
\begin{aligned}
\hat{h} &= \left[s\delta + 4\pi^2 v(v-\phi)\hat{w}\right] * \hat{U}\Big|_{v=\phi} \\
&= s\hat{U} + 4\pi^2[v(v-\phi)\hat{w}] * \hat{U}\Big|_{v=\phi} \\
&= \frac{is\delta'}{2\pi(s+2)} - \frac{i}{2\pi(s+2)}[4\pi^2 v\delta + v\delta'' + 4\pi^2\phi v\delta' \\
&\quad + \phi v\delta''' - v^2\delta''' - 4\pi^2 v^2\delta']\Big|_{v=\phi} \\
&= \frac{is\delta'}{2\pi(s+2)} - \frac{i}{2\pi(s+2)}[4\pi^2\phi\delta + \phi\delta'' + 4\pi^2\phi^2\delta' \\
&\quad + \phi^2\delta''' - \phi^2\delta''' - 4\pi^2\phi^2\delta'] \\
&= \frac{i}{2\pi(s+2)}\left[s\delta' - 4\pi^2\phi\delta - \phi\delta''\right] \ .
\end{aligned}
$$

Now, by (2.2),

$$
\begin{aligned}
4\pi^2\phi\delta &= 0 \\
\phi\delta'' &= -2\delta' \ .
\end{aligned}
$$

hence

$$
\begin{aligned}
\tilde{h} &= \frac{i}{2\pi(s+2)}\left[s\delta' - 4\pi^2 0\delta - 0\delta''\right] \\
&= \frac{i}{2\pi(s+2)}\left[s\delta' + 2\delta'\right] \\
&= \frac{i\delta'}{2\pi} ,
\end{aligned}
$$

which agrees with equation (2.10).

CHAPTER III
MOLLIFIED TRANSFORMS

3.1 Fourier Transform

Definition 3.1 (Mollified Fourier Transform) *Let $\alpha \in \mathbb{R}$, then the "Mollified Fourier Transform" is defined to be*

$$\mathcal{F}_\alpha \{f(x)\}(\phi) = \tilde{f}_\alpha(\phi) = \mathcal{F}\left\{e^{-\alpha^2 x^2} f(x)\right\}(\phi) = \int_{-\infty}^{\infty} e^{-\alpha^2 x^2} e^{-2\pi i \phi x} f(x) dx \ .$$

Note that this is a generalization of the Fourier transform, since $\mathcal{F}_0 \{f\} = \mathcal{F}\{f\}$. The linearity of the mollified Fourier transform is immediate, since the Fourier transform itself is linear. Inversion of the mollified Fourier transform is an easy matter:

Theorem 3.2 (Mollified Inversion Theorem)

$$\mathcal{F}_\alpha^{-1} = e^{\alpha^2 x^2} \mathcal{F}^{-1} \ .$$

Proof:

$$\begin{aligned} e^{\alpha^2 x^2} \mathcal{F}^{-1} \{\mathcal{F}_\alpha \{f\}\} &= e^{\alpha^2 x^2} \mathcal{F}^{-1} \left\{\mathcal{F}\left\{e^{-\alpha^2 x^2} f\right\}\right\} \\ &= e^{\alpha^2 x^2} e^{-\alpha^2 x^2} f \\ &= f \ . \end{aligned}$$

□

The mollified Fourier transform is useful in evaluating transforms that would otherwise be problematic. As an example, $\mathcal{F}\{1\}$ is undefined, whereas [7, page 279]

$$\begin{aligned} \mathcal{F}_\alpha \{1\} &= \int_{-\infty}^{\infty} e^{-\alpha^2 x^2} e^{-2\pi i \phi x} dx \\ &= \mathcal{F}\left\{e^{-\alpha^2 x^2}\right\} \\ &= \frac{\sqrt{\pi}}{\alpha} \exp\left(\frac{-\pi^2 \phi^2}{\alpha^2}\right) \qquad (3.1) \end{aligned}$$

is defined whenever $\alpha \neq 0$. Since it will be used often, $\mathcal{F}_\alpha \{1\}$ will be denoted by δ_α.

Lemma 3.3 *Let p be a nonnegative integer. If*

$$G_p = \int_{-\infty}^{\infty} \exp\left(-\eta^2\right) \eta^p d\eta \tag{3.2}$$

then

$$G_p = \begin{cases} 0 & p \text{ odd} \\ \Gamma\left(\frac{p-1}{2}\right) & p \text{ even} \end{cases} .$$

Proof: It is known [3. page 180] that for $\Re\zeta > 0$.

$$\Gamma(\zeta) = \int_0^{\infty} \exp(-t)t^{\zeta-1}dt \ .$$

so that using the substitution $t = \eta^2$,

$$\begin{aligned} \frac{1}{2}\Gamma\left(\frac{p+1}{2}\right) &= \frac{1}{2}\int_0^{\infty} \exp(-t)t^{\frac{p-1}{2}}dt \\ &= \int_0^{\infty} \exp(-t)t^{\frac{p}{2}}\frac{dt}{2\sqrt{t}} \\ &= \int_0^{\infty} \exp(-\eta^2)\eta^p d\eta \ . \end{aligned} \tag{3.3}$$

Substituting $-\theta$ for η. (3.3) becomes

$$\begin{aligned} &\int_0^{\infty} \exp(-\eta^2)\eta^p d\eta \\ &\quad = \int_{-\infty}^{0} \exp(-\theta^2)(-\theta)^p d\theta \\ &\quad = (-1)^p \int_{-\infty}^{0} \exp(-\theta^2)\theta^p d\theta \ . \end{aligned} \tag{3.4}$$

Now. by (3.3) and (3.4). the improper integral (3.2) converges, and

$$\begin{aligned} G_p &= \int_{-\infty}^{\infty} \exp\left(-\eta^2\right) \eta^p d\eta \\ &= \int_0^{\infty} \exp\left(-\eta^2\right) \eta^p d\eta + \int_{-\infty}^{0} \exp\left(-\eta^2\right) \eta^p d\eta \\ &= \frac{\Gamma\left(\frac{p+1}{2}\right) + (-1)^p \Gamma\left(\frac{p+1}{2}\right)}{2} \ . \end{aligned}$$

which completes the proof. □

Lemma 3.4 *Let $\zeta \in \mathbb{C}$. Let k be a nonnegative integer. then*

$$\int_{-\infty}^{\infty} \exp(-\xi^2)\xi^k d\xi = \int_{-\infty+\zeta}^{\infty+\zeta} \exp(-\xi^2)\xi^k d\xi$$

Proof: Fix $a > 0$, and define the paths in $\mathbb{C}$

$$\begin{aligned}\sigma_0 &= [-a, a]\\ \sigma_1 &= [a, \zeta + a]\\ \sigma_2 &= [\zeta + a, \zeta - a]\\ \sigma_3 &= [\zeta - a, -a] \ .\end{aligned}$$

Define $\sigma = \sum \sigma_j$, then

$$\int_\sigma \exp(-\xi^2)\xi^k d\xi = 0 \tag{3.5}$$

by Cauchy's theorem. Now, if it can be shown that

$$\begin{aligned}\lim_{a\to\infty}\int_{\sigma_1} \exp(-\xi^2)\xi^k d\xi &= \lim_{a\to\infty}\int_a^{\zeta+a} \exp(-\xi^2)\xi^k d\xi\\ &= 0 \end{aligned}\tag{3.6}$$

(and similarly for $\int_{\sigma_3}$), then equation (3.5) would give

$$\begin{aligned}0 &= \lim_{a\to\infty}\left[\int_{\sigma_0} + \int_{\sigma_1} + \int_{\sigma_2} + \int_{\sigma_3}\right]\\ &= \lim_{a\to\infty}\left[\int_{\sigma_0} + \int_{\sigma_2}\right]\\ &= \int_{-\infty}^{\infty} \exp(-\xi^2)\xi^k d\xi + \int_{\infty+\zeta}^{-\infty+\zeta} \exp(-\xi^2)\xi^k d\xi \ .\end{aligned}$$

which would prove the lemma.

To prove (3.6), choose $a > 0$ such that

$$a^2 - a - 2a\,|\zeta| > |\zeta|^2 \ . \tag{3.7}$$

then for $0 \le t \le 1$ and θ real,

$$\left(a + |\zeta|\, te^{i\theta}\right)^2 \ \geq \ a^2 - 2a\,|\zeta| - |\zeta|^2 > a \ .$$

hence $\Re(\xi^2) > a$ for all $\xi \in [a, \xi + a]$. Now, by [3, page 65],

$$\left|\int_a^{\zeta+a} \exp(-\xi^2)\xi^k \, d\xi\right|$$

$$\begin{aligned}
&\leq |\zeta| \sup\left\{\left|\exp(-\xi^2)\xi^k\right| : \xi \in [a.\zeta+a]\right\} \\
&= |\zeta| \sup\left\{\exp[\Re(-\xi^2)]\,|\xi|^k : \xi \in [a.\zeta+a]\right\} \\
&\leq |\zeta| \sup\left\{|\xi|^k : \xi \in [a.\zeta+a]\right\} \exp\left(\sup\{\Re(-\xi^2) : \xi \in [a.\zeta+a]\}\right) \\
&= |\zeta| \max\{|a|.|\zeta+a|\}^k \exp\left(-\inf\{\Re(\xi^2) : \xi \in [a.\zeta+a]\}\right) \\
&< |\zeta| \max\{|a|.|\zeta+a|\}^k \exp(-a) \quad . \qquad (3.8)
\end{aligned}$$

As $a \to \infty$. (3.8) vanishes, hence the lemma is proven. □

Theorem 3.5 (Mollified Transform of Powers) *Let* $r > 0$. *then*

$$\mathcal{F}_\alpha\{x^r\} = \frac{\delta_\alpha}{\sqrt{\pi}\alpha^r}\sum_{k=0}^{\infty}\binom{r}{2k}\left(\frac{-\pi i\phi}{\alpha}\right)^{r-2k}\Gamma\left(k+\frac{1}{2}\right) .$$

Proof: Let $\zeta = \frac{\pi i\phi}{\alpha}$. then

$$\begin{aligned}
\mathcal{F}_\alpha\{x^r\} &= \int_{-\infty}^{\infty} e^{-\alpha^2x^2}e^{-2\pi i\phi x}x^r\,dx \\
&= e^{\zeta^2}\int_{-\infty}^{\infty} e^{-(\alpha x+\zeta)^2}x^r\,dx \\
&= \lim_{\substack{a\to-\infty \\ b\to\infty}} e^{\zeta^2}\int_a^b e^{-(\alpha x+\zeta)^2}x^r\,dx \ .
\end{aligned}$$

Now. using the substitution $\xi = \alpha x + \zeta$.

$$\begin{aligned}
\mathcal{F}_\alpha\{x^r\} &= \frac{e^{\zeta^2}}{\alpha}\lim_{\substack{a\to-\infty \\ b\to\infty}}\int_{\alpha a+\zeta}^{\alpha b+\zeta} e^{-\xi^2}\left(\frac{\xi-\zeta}{\alpha}\right)^r d\xi \\
&= \frac{e^{\zeta^2}}{\alpha^{r+1}}\lim_{\substack{a\to-\infty \\ b\to\infty}}\int_{\alpha a+\zeta}^{\alpha b+\zeta} e^{-\xi^2}(\xi-\zeta)^r d\xi \ .
\end{aligned}$$

and by the binomial theorem.

$$\mathcal{F}_\alpha\{x^r\} = \frac{e^{\zeta^2}}{\alpha^{r+1}}\lim_{\substack{a\to-\infty \\ b\to\infty}}\int_{\alpha a+\zeta}^{\alpha b+\zeta} e^{-\xi^2}\sum_{k=0}^{\infty}\binom{r}{k}\xi^k(-\zeta)^{r-k}d\xi \ . \qquad (3.9)$$

The uniform convergence in (3.9) justifies the switching of the limits. so that

$$\mathcal{F}_\alpha\{x^r\} = \frac{e^{\zeta^2}}{\alpha^{r+1}}\sum_{k=0}^{\infty}\binom{r}{k}(-\zeta)^{r-k}\lim_{\substack{a\to-\infty \\ b\to\infty}}\int_{\alpha a+\zeta}^{\alpha b+\zeta} e^{-\xi^2}\xi^k d\xi \ .$$

Now, applying lemma (3.4),

$$\begin{aligned}
\mathcal{F}_\alpha\{x^r\} &= \frac{e^{\zeta^2}}{\alpha^{r+1}}\sum_{k=0}^{\infty}\begin{pmatrix} r \\ k \end{pmatrix}(-\zeta)^{r-k}G_k \\
&= \frac{e^{\zeta^2}}{\alpha^{r+1}}\sum_{k=0}^{\infty}\left[\begin{pmatrix} r \\ 2k \end{pmatrix}(-\zeta)^{r-2k}G_{2k}\right. \\
&\quad \left. +\begin{pmatrix} r \\ 2k+1 \end{pmatrix}(-\zeta)^{r-2k-1}G_{2k+1}\right] . \qquad (3.10)
\end{aligned}$$

where the sum was broken into its even and odd terms. By (lemma 3.3), $G_{2k+1} = 0$ for all nonnegative integers k, hence all the odd terms in equation (3.10) are zero. This gives

$$\begin{aligned}
\mathcal{F}_\alpha\{x^r\} &= \frac{e^{\zeta^2}}{\alpha^{r+1}}\sum_{k=0}^{\infty}\begin{pmatrix} r \\ 2k \end{pmatrix}(-\zeta)^{r-2k}G_{2k} \\
&= \frac{e^{\zeta^2}}{\alpha^{r+1}}\sum_{k=0}^{\infty}\begin{pmatrix} r \\ 2k \end{pmatrix}(-\zeta)^{r-2k}\Gamma\left(\frac{2k+1}{2}\right) \\
&= \frac{e^{\left(\frac{-\pi^2 o^2}{\alpha^2}\right)}}{\alpha^{r+1}}\sum_{k=0}^{\infty}\begin{pmatrix} r \\ 2k \end{pmatrix}\left(\frac{-\pi i o}{\alpha}\right)^{r-2k}\Gamma\left(k+\frac{1}{2}\right) \\
&= \frac{\delta_\alpha}{\sqrt{\pi}\alpha^r}\sum_{k=0}^{\infty}\begin{pmatrix} r \\ 2k \end{pmatrix}\left(\frac{-\pi i o}{\alpha}\right)^{r-2k}\Gamma\left(k+\frac{1}{2}\right) . \qquad (3.11)
\end{aligned}$$

□

Corollary 3.6 *If n is a nonnegative integer, then*

$$\mathcal{F}_\alpha\{x^n\} = \frac{\delta_\alpha}{\sqrt{\pi}\alpha^n}\sum_{k=0}^{\left[\frac{n}{2}\right]}\begin{pmatrix} n \\ 2k \end{pmatrix}\left(\frac{-\pi i o}{\alpha}\right)^{n-2k}\Gamma\left(k+\frac{1}{2}\right) .$$

where $\left[\frac{n}{2}\right]$ is the greatest integer less than or equal to $\frac{n}{2}$.

Proof: For $k > \left[\frac{n}{2}\right]$, $\begin{pmatrix} n \\ 2k \end{pmatrix}$ is zero, hence the upper limit of the series in (3.11) need not be taken farther than $\left[\frac{n}{2}\right]$. □

Using (corollary 3.6), the mollified Fourier transform of the first five integer powers of x are as follows:

$$\mathcal{F}_\alpha\left\{x^0\right\} = \delta_\alpha \tag{3.12}$$

$$\begin{aligned}
\mathcal{F}_\alpha\left\{x^1\right\} &= \frac{\delta_\alpha}{\sqrt{\pi}\alpha^1}\sum_{k=0}^{0}\binom{1}{2k}\left(\frac{-\pi i\phi}{\alpha}\right)^{1-2k}\Gamma\left(k+\frac{1}{2}\right) \\
&= \frac{\delta_\alpha}{\sqrt{\pi}\alpha}\left(\frac{-\pi i\phi}{\alpha}\right)\Gamma\left(\frac{1}{2}\right) \\
&= \frac{-\pi i\phi\delta_\alpha}{\alpha^2}
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
\mathcal{F}_\alpha\left\{x^2\right\} &= \frac{\delta_\alpha}{\sqrt{\pi}\alpha^2}\sum_{k=0}^{1}\binom{2}{2k}\left(\frac{-\pi i\phi}{\alpha}\right)^{2-2k}\Gamma\left(k+\frac{1}{2}\right) \\
&= \frac{\delta_\alpha}{\sqrt{\pi}\alpha^2}\left(\frac{-\pi i\phi}{\alpha}\right)^2\Gamma\left(\frac{1}{2}\right)+\frac{\delta_\alpha}{\sqrt{\pi}\alpha^2}\Gamma\left(\frac{3}{2}\right) \\
&= \frac{-\pi^2\phi^2\delta_\alpha}{\alpha^4}+\frac{\delta_\alpha}{2\alpha^2}
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
\mathcal{F}_\alpha\left\{x^3\right\} &= \frac{\delta_\alpha}{\sqrt{\pi}\alpha^3}\sum_{k=0}^{1}\binom{3}{2k}\left(\frac{-\pi i\phi}{\alpha}\right)^{3-2k}\Gamma\left(k+\frac{1}{2}\right) \\
&= \frac{\delta_\alpha}{\sqrt{\pi}\alpha^3}\left(\frac{-\pi i\phi}{\alpha}\right)^3\Gamma\left(\frac{1}{2}\right)+\frac{3\delta_\alpha}{\sqrt{\pi}\alpha^3}\left(\frac{-\pi i\phi}{\alpha}\right)\Gamma\left(\frac{3}{2}\right) \\
&= \frac{\pi^3 i\phi^3\delta_\alpha}{\alpha^6}-\frac{3\pi i\phi\delta_\alpha}{2\alpha^4}
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
\mathcal{F}_\alpha\left\{x^4\right\} &= \frac{\delta_\alpha}{\sqrt{\pi}\alpha^4}\sum_{k=0}^{2}\binom{4}{2k}\left(\frac{-\pi i\phi}{\alpha}\right)^{4-2k}\Gamma\left(k+\frac{1}{2}\right) \\
&= \frac{\delta_\alpha}{\sqrt{\pi}\alpha^4}\left(\frac{-\pi i\phi}{\alpha}\right)^4\Gamma\left(\frac{1}{2}\right)+\frac{6\delta_\alpha}{\sqrt{\pi}\alpha^4}\left(\frac{-\pi i\phi}{\alpha}\right)^2\Gamma\left(\frac{3}{2}\right)+\frac{\delta_\alpha}{\sqrt{\pi}\alpha^4}\Gamma\left(\frac{5}{2}\right) \\
&= \frac{\pi^4\phi^4\delta_\alpha}{\alpha^8}-\frac{3\pi^2\phi^2\delta_\alpha}{\alpha^6}+\frac{3\delta_\alpha}{4\alpha^4}\ .
\end{aligned} \tag{3.16}$$

Theorem 3.7 (Recursion Formula)

$$(r+1)\mathcal{F}_\alpha\left\{x^r\right\} = 2\alpha^2\mathcal{F}_\alpha\left\{x^{r+2}\right\}+2\pi i\phi\mathcal{F}_\alpha\left\{x^{r+1}\right\}\ .$$

Before this can be proven, the following lemma is required.

Lemma 3.8

$$\begin{pmatrix} r+2 \\ 2k+2 \end{pmatrix} - \begin{pmatrix} r+1 \\ 2k+2 \end{pmatrix} = \frac{r+1}{2k+1} \begin{pmatrix} r \\ 2k \end{pmatrix}$$

Proof:

$$\begin{aligned}
&\begin{pmatrix} r+2 \\ 2k+2 \end{pmatrix} - \begin{pmatrix} r+1 \\ 2k+2 \end{pmatrix} \\
&= \frac{\Gamma(r+3)}{\Gamma(2k+3)\,\Gamma(r-2k+1)} - \frac{\Gamma(r+2)}{\Gamma(2k+3)\,\Gamma(r-2k)} \\
&= \frac{(r+2)(r+1)\Gamma(r+1)}{\Gamma(2k+3)\,\Gamma(r-2k+1)} - \frac{(r-2k)(r+1)\Gamma(r+1)}{\Gamma(2k+3)\,\Gamma(r-2k+1)} \\
&= (r+1)[(r+2)-(r-2k)]\frac{\Gamma(r+1)}{\Gamma(2k+3)\,\Gamma(r-2k+1)} \\
&= (r+1)(2k+2)\frac{\Gamma(r+1)}{(2k+2)(2k+1)\Gamma(2k+1)\,\Gamma(r-2k+1)} \\
&= \frac{r+1}{2k+1}\frac{\Gamma(r+1)}{\Gamma(2k+1)\,\Gamma(r-2k+1)} \\
&= \frac{r+1}{2k+1}\begin{pmatrix} r \\ 2k \end{pmatrix} .
\end{aligned}$$

□

Proof of (3.7):

$$\begin{aligned}
&\frac{2\alpha^2}{r+1}\mathcal{F}_\alpha\left\{x^{r+2}\right\} + \frac{2\pi i\phi}{r+1}\mathcal{F}_\alpha\left\{x^{r+1}\right\} \\
&= \frac{2\alpha^2}{r+1}\frac{\delta_\alpha}{\sqrt{\pi}\alpha^{r+2}}\sum_{k=0}^{\infty}\left(\frac{-\pi i\phi}{\alpha}\right)^{r+2-2k}\begin{pmatrix} r+2 \\ 2k \end{pmatrix}\Gamma\left(k+\frac{1}{2}\right) \\
&\quad + \frac{2\pi i\phi}{r+1}\frac{\delta_\alpha}{\sqrt{\pi}\alpha^{r+1}}\sum_{k=0}^{\infty}\left(\frac{-\pi i\phi}{\alpha}\right)^{r+1-2k}\begin{pmatrix} r+1 \\ 2k \end{pmatrix}\Gamma\left(k+\frac{1}{2}\right) \\
&= \frac{2\alpha^2}{r+1}\frac{\delta_\alpha}{\sqrt{\pi}\alpha^{r+2}}\sum_{k=-1}^{\infty}\left(\frac{-\pi i\phi}{\alpha}\right)^{r-2k}\begin{pmatrix} r+2 \\ 2k+2 \end{pmatrix}\Gamma\left(k+\frac{3}{2}\right) \\
&\quad + \frac{2\pi i\phi}{r+1}\frac{\delta_\alpha}{\sqrt{\pi}\alpha^{r+1}}\sum_{k=-1}^{\infty}\left(\frac{-\pi i\phi}{\alpha}\right)^{r-1-2k}\begin{pmatrix} r+1 \\ 2k+2 \end{pmatrix}\Gamma\left(k-\frac{3}{2}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{r+1}\frac{\delta_\alpha}{\sqrt{\pi}\alpha^r}\sum_{k=-1}^{\infty}\left(\frac{-\pi i o}{\alpha}\right)^{r-2k}\begin{pmatrix} r+2 \\ 2k+2 \end{pmatrix}\Gamma\left(k+\frac{3}{2}\right) \\
&\quad -\frac{2}{r+1}\frac{\delta_\alpha}{\sqrt{\pi}\alpha^r}\sum_{k=-1}^{\infty}\left(\frac{-\pi i o}{\alpha}\right)^{r-2k}\begin{pmatrix} r+1 \\ 2k+2 \end{pmatrix}\Gamma\left(k-\frac{3}{2}\right) \\
&= \frac{2}{r+1}\frac{\delta_\alpha}{\sqrt{\pi}\alpha^r}\sum_{k=-1}^{\infty}\left(\frac{-\pi i o}{\alpha}\right)^{r-2k}\left[\begin{pmatrix} r+2 \\ 2k+2 \end{pmatrix}-\begin{pmatrix} r-1 \\ 2k+2 \end{pmatrix}\right]\Gamma\left(k+\frac{3}{2}\right) .
\end{aligned}
$$

Applying (lemma 3.8),

$$
\begin{aligned}
&= \frac{2}{r+1}\frac{\delta_\alpha}{\sqrt{\pi}\alpha^r}\sum_{k=-1}^{\infty}\left(\frac{-\pi i o}{\alpha}\right)^{r-2k}\left[\frac{r+1}{2k+1}\begin{pmatrix} r \\ 2k \end{pmatrix}\right]\Gamma\left(k+\frac{3}{2}\right) \\
&= \frac{2\delta_\alpha}{\sqrt{\pi}\alpha^r}\sum_{k=-1}^{\infty}\left(\frac{-\pi i o}{\alpha}\right)^{r-2k}\left[\frac{1}{2\left(k+\frac{1}{2}\right)}\begin{pmatrix} r \\ 2k \end{pmatrix}\right]\left(k+\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right) \\
&= \frac{\delta_\alpha}{\sqrt{\pi}\alpha^r}\sum_{k=-1}^{\infty}\left(\frac{-\pi i o}{\alpha}\right)^{r-2k}\begin{pmatrix} r \\ 2k \end{pmatrix}\Gamma\left(k+\frac{1}{2}\right) .
\end{aligned}
$$

but since $\begin{pmatrix} r \\ -2 \end{pmatrix} = 0$,

$$
\begin{aligned}
&= \frac{\delta_\alpha}{\sqrt{\pi}\alpha^r}\sum_{k=0}^{\infty}\left(\frac{-\pi i o}{\alpha}\right)^{r-2k}\begin{pmatrix} r \\ 2k \end{pmatrix}\Gamma\left(k+\frac{1}{2}\right) \\
&= \mathcal{F}_\alpha\{x^r\} .
\end{aligned}
$$

□

Theorem 3.9 (Derivative Theorem)

$$
\mathcal{F}_\alpha\left\{\frac{df}{dx}\right\} = 2\alpha^2\mathcal{F}_\alpha\{xf\} + 2\pi i o\mathcal{F}_\alpha\{f\} .
$$

Proof: Let

$$
\begin{aligned}
u &= e^{-\alpha^2 x^2}e^{-2\pi i o x} \\
\frac{dv}{dx} &= \frac{df}{dx} .
\end{aligned}
$$

then

$$
\begin{aligned}
du &= (-2\alpha^2 x - 2\pi i o)e^{-\alpha^2 x^2} e^{-2\pi i o x} dx \\
v &= f \ .
\end{aligned}
$$

Using u and v to integrate by parts,

$$
\begin{aligned}
\mathcal{F}_\alpha \left\{ \frac{df}{dx} \right\} &= \int_{-\infty}^{\infty} e^{-\alpha^2 x^2} e^{-2\pi i o x} \frac{df}{dx} dx \\
&= \int_{-\infty}^{\infty} u \, dv \\
&= uv|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v \, du \\
&= e^{-\alpha^2 x^2} e^{-2\pi i o x} f \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (-2\alpha^2 x - 2\pi i o) e^{-\alpha^2 x^2} e^{-2\pi i o x} f \, dx \\
&= -\mathcal{F}_\alpha \left\{ (-2\alpha^2 x - 2\pi i o) f \right\} \\
&= 2\alpha^2 \mathcal{F}_\alpha \{xf\} + 2\pi i o \mathcal{F}_\alpha \{f\} \ .
\end{aligned}
$$

□

Note that the above derivative theorem applied to $f(x) = x^{r+1}$ gives the recursion theorem (theorem 3.7).

Theorem 3.10 (Convolution Theorem)

$$\mathcal{F}_\alpha \{fg\} = \mathcal{F}_\beta \{f\} * \mathcal{F}_\beta \{g\} = \mathcal{F}_\beta \{g\} * \mathcal{F}_\beta \{f\} \ . \tag{3.17}$$

where $\alpha = \sqrt{2}\beta$.

Proof:

$$
\begin{aligned}
\mathcal{F}_\alpha \{fg\} &= \int_{-\infty}^{\infty} e^{-\alpha^2 x^2} e^{-2\pi i o x} fg \, dx \\
&= \int_{-\infty}^{\infty} \left[exp\left(\frac{-\alpha^2 x^2}{2} \right) f \right] \left[exp\left(\frac{-\alpha^2 x^2}{2} \right) g \right] e^{-2\pi i o x} dx \\
&= \int_{-\infty}^{\infty} \left(e^{-\beta^2 x^2} f \right) \left(e^{-\beta^2 x^2} 2g \right) e^{-2\pi i o x} dx \\
&= \mathcal{F} \left\{ \left(e^{-\beta^2 x^2} f \right) \left(e^{-\beta^2 x^2} 2g \right) \right\} \\
&= \mathcal{F} \left\{ \left(e^{-\beta^2 x^2} f \right) \right\} * \mathcal{F} \left\{ \left(e^{-\beta^2 x^2} 2g \right) \right\} \\
&= \mathcal{F}_\beta \{f\} * \mathcal{F}_\beta \{g\} \ .
\end{aligned}
$$

hence the first equality in (3.17) holds. The commutativity of the convolution is obvious, hence the second equality also holds. □

3.2 Delta Function

In this section, the function "δ_α" is defined, and some properties are given.

Definition 3.11

$$\delta_\alpha(\omega) = \mathcal{F}_\alpha\{1\} = \frac{\sqrt{\pi}}{\alpha}\exp\left(-\frac{\pi^2\omega^2}{\alpha^2}\right) .$$

The "δ_α" function is useful when dealing with mollified Fourier transforms because it allows the parameter of a mollified transform to be reduced:

Theorem 3.12 (Parameter Reduction) *Let $\alpha = \sqrt{2}\beta$, then*

$$\mathcal{F}_\alpha\{f\} = \mathcal{F}_\beta\{f\} * \delta_\beta .$$

Proof: Substitute 1 for g in (theorem 3.10). □

It may be useful to be able to reduce δ_α itself. Three ways of doing this are:

Claim 3.13

$$\delta_\alpha = \delta_\beta * \delta_\beta .$$

Proof: Apply (theorem 3.12), substituting 1 for f. □

Claim 3.14

$$\delta_\alpha = \frac{\sqrt[4]{\pi}}{\sqrt{2\beta}}\sqrt{\delta_\beta} .$$

Proof:

$$\begin{aligned}
\delta_\alpha &= \frac{\sqrt{\pi}}{\alpha}\exp\left(-\frac{\pi^2\omega^2}{\alpha^2}\right) \\
&= \frac{\sqrt{\pi}}{\sqrt{2}\beta}\exp\left(-\frac{\pi^2\omega^2}{2\beta^2}\right) \\
&= \frac{\sqrt{\pi}}{\sqrt{2}\beta}\sqrt{\exp\left(-\frac{\pi^2\omega^2}{\beta^2}\right)} \\
&= \frac{\sqrt[4]{\pi}}{\sqrt{2\beta}}\frac{\sqrt{\sqrt{\pi}\exp\left(-\frac{\pi^2\omega^2}{\beta^2}\right)}}{\sqrt{\beta}} \\
&= \frac{\sqrt[4]{\pi}}{\sqrt{2\beta}}\sqrt{\delta_\beta} .
\end{aligned}$$

□

Claim 3.15

$$\delta_\alpha = \mathcal{F}_\beta\left\{e^{-\beta^2 x^2}\right\} .$$

Proof:

$$\begin{aligned}
\mathcal{F}_\beta\left\{e^{-\beta^2 x^2}\right\} &= \int_{-\infty}^{\infty} e^{-\beta^2 x^2} e^{-\beta^2 x^2} e^{-2\pi i \phi x} dx \\
&= \int_{-\infty}^{\infty} e^{-2\beta^2 x^2} e^{-2\pi i \phi x} dx \\
&= \int_{-\infty}^{\infty} e^{-\alpha^2 x^2} e^{-2\pi i \phi x} dx \\
&= \delta_\alpha .
\end{aligned}$$

□

3.3 Heterogeneous Heat Equation

The equation for which solutions are being sought is

$$\frac{\partial}{\partial x}\left[w(x)\frac{\partial u}{\partial x}\right] = \frac{\partial u}{\partial t} . \tag{3.18}$$

Let $z = xw$, and take the mollified Fourier transform of equation (3.18) to get

$$\begin{aligned}
\frac{\partial \tilde{u}_\alpha}{\partial t} &= 2\alpha^2 \mathcal{F}_\alpha\left\{z\frac{\partial u}{\partial x}\right\} + 2\pi i \phi \mathcal{F}_\alpha\left\{w\frac{\partial u}{\partial x}\right\} \\
&= 2\alpha^2 \mathcal{F}_\beta\{z\} * \mathcal{F}_\beta\left\{\frac{\partial u}{\partial x}\right\} + 2\pi i \phi\left(\mathcal{F}_\beta\{w\} * \mathcal{F}_\beta\left\{\frac{\partial u}{\partial x}\right\}\right) .
\end{aligned} \tag{3.19}$$

Now, in order to factor $\mathcal{F}_\beta\left\{\frac{\partial u}{\partial x}\right\}$ out of each term of (3.19), the ϕ in the second term must be replaced with a dummy variable, say ψ_1, so that it is clear that ϕ is not involved in the convolution with $\mathcal{F}_\beta\left\{\frac{\partial u}{\partial x}\right\}$. Hence,

$$\begin{aligned}
\frac{\partial \tilde{u}_\alpha}{\partial t} &= (2\alpha^2 \tilde{z}_\beta + 2\pi i \psi_1 \tilde{w}_\beta) * \mathcal{F}_\beta\left\{\frac{\partial u}{\partial x}\right\} \\
&= (2\alpha^2 \tilde{z}_\beta + 2\pi i \psi_1 \tilde{w}_\beta) * (2\beta^2 \mathcal{F}_\beta\{xu\} + 2\pi i \phi \mathcal{F}_\beta\{u\}) \\
&= (2\alpha^2 \tilde{z}_\beta + 2\pi i \psi_1 \tilde{w}_\beta) * (2\beta^2 \mathcal{F}_\beta\{xu\} + 2\pi i \phi \tilde{u}_\beta) ,
\end{aligned}$$

where it is understood that ψ_1 should be taken to be ϕ once the convolution is performed. Let $\beta = \sqrt{2}\gamma$, then

$$\frac{\partial \tilde{u}_\alpha}{\partial t} = (2\alpha^2 \tilde{z}_\beta + 2\pi i \psi_1 \tilde{w}_\beta) * [2\beta^2(\mathcal{F}_\gamma\{x\} * \tilde{u}_\gamma) + 2\pi i \phi \tilde{u}_\beta] . \tag{3.20}$$

Taking Laplace transforms of (3.20) gives

$$s\tilde{U}_\alpha - \tilde{h}_\alpha = (2\alpha^2 \tilde{z}_3 + 2\pi i v_1 \tilde{w}_3) * [2\beta^2(\mathcal{F}_\gamma\{x\} * \tilde{U}_\gamma) + 2\pi i \phi \tilde{U}_\beta] \ ,$$

and now using the properties of "δ_α," and the definition $\alpha = \sqrt{2}\beta = 2\gamma$,

$$\begin{aligned} s\tilde{U}_\alpha - \tilde{h}_\alpha &= (4\beta^2 \tilde{z}_3 + 2\pi i v_1 \tilde{w}_3) \\ &\quad *[4\gamma^2(\mathcal{F}_\gamma\{x\} * \tilde{U}_\gamma) + 2\pi i \phi(\delta_\gamma * \tilde{U}_\gamma)] \ , \end{aligned}$$

so now, (by 3.13),

$$\begin{aligned} s\tilde{U}_\alpha - \tilde{h}_\alpha &= (4\beta^2 \tilde{z}_3 + 2\pi i v_1 \tilde{w}_3) \\ &\quad *[4(\{-\pi i \phi \delta_\gamma\} * \tilde{U}_\gamma) + 2\pi i \phi(\delta_\gamma * \tilde{U}_\gamma)] \qquad (3.21) \\ &= S * [4(\{-\pi i \phi \delta_\gamma\} * \tilde{U}_\gamma) + 2\pi i \phi(\delta_\gamma * \tilde{U}_\gamma)] \ . \end{aligned}$$

Now, in order to factor $\tilde{U}_\gamma$ from each term, another dummy variable, v_2, must be introduced so that it is clear that the ϕ in the second term is not involved in the convolution with $\tilde{U}_\gamma$:

$$\begin{aligned} s\tilde{U}_\alpha - \tilde{h}_\alpha &= S * [(-4\pi i \phi \delta_\gamma + 2\pi i v_2 \delta_\gamma) * \tilde{U}_\gamma] \\ &= S * (T * \tilde{U}_\gamma) \ , \qquad (3.22) \end{aligned}$$

where

$$\begin{aligned} S &= 4\beta^2 \tilde{z}_3 + 2\pi i v_1 \tilde{w}_3 \\ T &= -4\pi i \phi \delta_\gamma + 2\pi i v_2 \delta_\gamma \ . \end{aligned}$$

It must be kept in mind that the v_2 in T is not involved in the convolution with $\tilde{U}_\gamma$, but is involved in the convolution with S. (Also, v_1 is not involved in any convolution.) Now, solving equation (3.22) for $\tilde{h}_\alpha$,

$$\tilde{h}_\alpha = s\tilde{U}_\alpha - S * (T * \tilde{U}_\gamma) \ . \qquad (3.23)$$

Much will be accomplished if equation (3.23) can be solved for $\tilde{U}_\gamma$. The following convolutions will prove to be useful:

$$\begin{aligned} (\phi\delta_\gamma) * (\phi\delta_\gamma) &= \left[\left(\frac{-\gamma^2}{\pi i}\right)\left(\frac{-\pi i \phi \delta_\gamma}{\gamma^2}\right)\right] * \left[\left(\frac{-\gamma^2}{\pi i}\right)\left(\frac{-\pi i \phi \delta_\gamma}{\gamma^2}\right)\right] \\ &= \frac{-\gamma^4}{\pi^2}\mathcal{F}_\gamma\{x\} * \mathcal{F}_\gamma\{x\} \\ &= \frac{-\beta^4}{4\pi^2}\mathcal{F}_\beta\left\{x^2\right\} \ . \end{aligned}$$

and applying equation (3.14).

$$
\begin{aligned}
&= \frac{-\beta^4}{4\pi^2}\left(\frac{-\pi^2\phi^2\delta_\beta}{\beta^4} + \frac{\delta_\beta}{2\beta^2}\right) \\
&= \frac{\phi^2\delta_\beta}{4} - \frac{\beta^2\delta_\beta}{8\pi^2} \ . \qquad (3.24)
\end{aligned}
$$

$$
\begin{aligned}
\delta_\gamma * (\phi\delta_\gamma) &= \delta_\gamma * \left[\left(\frac{-\gamma^2}{\pi i}\right)\left(\frac{-\pi i\phi\delta_\gamma}{\gamma^2}\right)\right] \\
&= \frac{-\gamma^2}{\pi i}\delta_\gamma * \mathcal{F}_\gamma\{x\} \\
&= \frac{-\gamma^2}{\pi i}\mathcal{F}_\beta\{x\} \\
&= \frac{-\beta^2}{2\pi i}\mathcal{F}_\beta\{x\} \ .
\end{aligned}
$$

and applying equation (3.13).

$$
\begin{aligned}
&= \frac{-\beta^2}{2\pi i}\left(\frac{-\pi i\phi\delta_\beta}{\beta^2}\right) \\
&= \frac{\phi\delta_\beta}{2} \ . \qquad (3.25)
\end{aligned}
$$

$$
\begin{aligned}
[\phi\delta_\beta(\phi)] * \delta_\beta(\phi-\theta) &= \left[\left(\frac{-\beta^2}{\pi i}\right)\left(\frac{-\pi i\phi\delta_\beta}{\beta^2}\right)\right] * \mathcal{F}_\beta\left\{e^{2\theta\pi i x}\right\} \\
&= \frac{-\beta^2}{\pi i}\mathcal{F}_\beta\{x\} * \mathcal{F}_\beta\left\{e^{2\theta\pi i x}\right\} \\
&= \frac{-\beta^2}{\pi i}\mathcal{F}_\alpha\left\{xe^{2\theta\pi i x}\right\} \\
&= \frac{-\alpha^2}{2\pi i}\mathcal{F}_\alpha\left\{xe^{2\theta\pi i x}\right\} \ . \qquad (3.26)
\end{aligned}
$$

3.4 Legendre Example

As an example, let $w(x) = 1 - x^2$, then the heterogeneous heat equation (1.2) is known to have as one of its solutions $u(x,t) = xe^{-2t}$. In this case (using corollary 3.5, which gives a formula for $\mathcal{F}_\alpha\{x^n\}$),

$$
\tilde{w}_\alpha = \mathcal{F}_\alpha\left\{1 - x^2\right\} = \mathcal{F}_\alpha\{1\} - \mathcal{F}_\alpha\left\{x^2\right\}
$$

$$
\begin{aligned}
&= \delta_\alpha - \frac{\delta_\alpha}{\sqrt{\pi}\alpha^2}\left[\left(\frac{-\pi i \phi}{\alpha}\right)^2 \Gamma\left(\frac{1}{2}\right) + \Gamma\left(\frac{3}{2}\right)\right] \\
&= \left(\frac{\pi^2\phi^2}{\alpha^4} - \frac{1}{2\alpha^2} + 1\right)\delta_\alpha \ .
\end{aligned}
$$

Recalling that $z = xw$,

$$
\begin{aligned}
\tilde{z}_\alpha &= \mathcal{F}_\alpha\left\{x - x^3\right\} = \mathcal{F}_\alpha\{x\} - \mathcal{F}_\alpha\left\{x^3\right\} \\
&= \left(\frac{-\pi i \phi}{\alpha^2}\right)\delta_\alpha - \frac{\delta_\alpha}{\sqrt{\pi}\alpha^3}\left[\left(\frac{-\pi i \phi}{\alpha}\right)^3 \Gamma\left(\frac{1}{2}\right) + 3\left(\frac{-\pi i \phi}{\alpha}\right)\Gamma\left(\frac{3}{2}\right)\right] \\
&= \left(-\frac{\pi i \phi}{\alpha^2} - \frac{\pi^3 i \phi^3}{\alpha^6} + \frac{3\pi i \phi}{2\alpha^4}\right)\delta_\alpha \\
\tilde{h}_\alpha &= \mathcal{F}_\alpha\{x\} = \frac{\delta_\alpha}{\sqrt{\pi}\alpha}\sum_{k=0}^{1}\begin{pmatrix} 1 \\ 2k \end{pmatrix}\left(\frac{-\pi i \phi}{\alpha}\right)^{1-2k}\Gamma\left(k + \frac{1}{2}\right) \\
&= \frac{\delta_\alpha}{\sqrt{\pi}\alpha}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\left(\frac{-\pi i \phi}{\alpha}\right)\Gamma\left(\frac{1}{2}\right) \\
&= \frac{-\pi i \phi \delta_\alpha}{\alpha^2} \qquad (3.27) \\
\tilde{u}_\alpha &= \mathcal{F}_\alpha\left\{x e^{-2t}\right\} = e^{-2t}\mathcal{F}_\alpha\{x\} = \frac{-\pi i \phi \delta_\alpha e^{-2t}}{\alpha^2} \\
\tilde{U}_\alpha &= \mathcal{L}\{\tilde{u}_\alpha\} = \mathcal{L}\left\{\frac{-\pi i \phi \delta_\alpha e^{-2t}}{\alpha^2}\right\} \\
&= \frac{-\pi i \phi \delta_\alpha}{\alpha^2}\mathcal{L}\left\{e^{-2t}\right\} \\
&= \frac{-\pi i \phi \delta_\alpha}{(s+2)\alpha^2} \ . \qquad (3.28)
\end{aligned}
$$

Use these formulae to compute

$$
\begin{aligned}
T * \tilde{U}_\gamma &= (-4\pi i \phi \delta_\gamma + 2\pi i v_2 \delta_\gamma) * \frac{-\pi i \phi \delta_\gamma}{(s+2)\gamma^2} \\
&= \frac{-4\pi^2}{(s+2)\gamma^2}(\phi\delta_\gamma) * (\phi\delta_\gamma) + \frac{2\pi^2 v_2}{(s+2)\gamma^2}\delta_\gamma * (\phi\delta_\gamma) \ .
\end{aligned}
$$

and equations (3.24) and (3.25) give

$$
T * \tilde{U}_\gamma = \frac{-4\pi^2}{(s+2)\gamma^2}\left(\frac{\phi^2\delta_\beta}{4} - \frac{\beta^2\delta_\beta}{8\pi^2}\right) + \frac{2\pi^2 v_2}{(s+2)\gamma^2}\left(\frac{\phi\delta_\beta}{2}\right)
$$

$$\begin{aligned} &= \frac{-\pi^2\phi^2\delta_\beta}{(s+2)\gamma^2} + \frac{\beta^2\delta_\beta}{2(s+2)\gamma^2} + \frac{\pi^2\phi^2\delta_\beta}{(s+2)\gamma^2} \\ &= \frac{\beta^2\delta_\beta}{2(s+2)\gamma^2} \\ &= \frac{\delta_\beta}{s+2} \ , \end{aligned} \tag{3.29}$$

hence

$$\begin{aligned} S*(T*\tilde{U}_\gamma) &= (4\beta^2\tilde{z}_\beta + 2\pi i\psi_1\tilde{w}_\beta) * \frac{\delta_\beta}{s+2} \\ &= \frac{1}{s+2}(4\beta^2\tilde{z}_\alpha + 2\pi i\psi_1\tilde{w}_\alpha) \\ &= \frac{1}{s+2}(2\alpha^2\tilde{z}_\alpha + 2\pi i\phi\tilde{w}_\alpha) \\ &= \frac{1}{s+2}\left[2\alpha^2\left(\frac{-\pi i\phi}{\alpha^2} - \frac{\pi^3 i\phi^3}{\alpha^6} + \frac{3\pi i\phi}{2\alpha^4}\right)\delta_\alpha \right. \\ &\quad \left. + 2\pi i\phi\left(\frac{\pi^2\phi^2}{\alpha^4} - \frac{1}{2\alpha^2} + 1\right)\delta_\alpha\right] \\ &= \frac{2\pi i\phi\delta_\alpha}{s+2}\left(-1 - \frac{\pi^2\phi^2}{\alpha^4} + \frac{3}{2\alpha^2} + \frac{\pi^2\phi^2}{\alpha^4} - \frac{1}{2\alpha^2} + 1\right) \\ &= \frac{2\pi i\phi\delta_\alpha}{(s+2)\alpha^2} \ . \end{aligned} \tag{3.30}$$

Now, combining equation (3.28) with equation equation (3.30) gives

$$\begin{aligned} s\tilde{U}_\alpha - S*(T*\tilde{U}_\gamma) &= -\frac{s\pi i\phi\delta_\alpha}{(s+2)\alpha^2} - \frac{2\pi i\phi\delta_\alpha}{(s+2)\alpha^2} \\ &= -\frac{\pi i\phi\delta_\alpha}{\alpha^2} \ , \end{aligned}$$

hence by equation (3.27),

$$\begin{aligned} \tilde{h}_\alpha &= -\frac{\pi i\phi\delta_\alpha}{\alpha^2} \\ &= s\tilde{U}_\alpha - S*(T*\tilde{U}_\gamma) \ . \end{aligned}$$

which agrees with equation (3.23).

CHAPTER IV

INVERSION

4.1 Heterogeneous Heat Equation

Recall that $\alpha = \sqrt{2}\beta$ and $\beta = \sqrt{2}\gamma$. Consider Λ, defined by

$$\Lambda \tilde{U}_\gamma = \frac{1}{s} S * (T * \tilde{U}_\gamma) \ . \tag{4.1}$$

to be a linear operator on $\tilde{U}_\gamma$. Note that Λ transforms a γ-mollified expression into an α-mollified expression. By equation (3.23), the heterogeneous heat equation (1.2) becomes

$$\begin{aligned} \frac{1}{s}\tilde{h}_{\alpha_0} &= \tilde{U}_{\alpha_0} - \frac{1}{s} S * (T * \tilde{U}_{\alpha_2}) \\ \frac{1}{s}\tilde{h}_{\alpha_0} &= \tilde{U}_{\alpha_0} - \Lambda \tilde{U}_{\alpha_2} \ . \end{aligned} \tag{4.2}$$

where the relationship $\alpha_k = \sqrt{2}\alpha_{k+1}$ is used to generalize the definition of α, β, and γ. In order to maintain consistency in the mollification parameter, any expression falling within n occurrences of Λ should have a mollification parameter of α_{2n}, since n applications of Λ will transform an α_{2n}-mollified expression into an α_0-mollified expression.

Each iteration in the proposed method consists of two steps: (1) apply Λ to both sides of the previous iteration, and (2) add equation (4.2) to the result from step 1.

Now, for the first iteration, apply Λ to both sides of equation (4.2), then add the result to (4.2). This gives

$$\begin{aligned} \frac{1}{s}\tilde{h}_{\alpha_0} + \frac{1}{s}\Lambda\tilde{h}_{\alpha_2} &= \tilde{U}_{\alpha_0} - \Lambda\tilde{U}_{\alpha_2} + \Lambda\tilde{U}_{\alpha_2} - \Lambda^2\tilde{U}_{\alpha_4} \\ &= \tilde{U}_{\alpha_0} - \Lambda^2\tilde{U}_{\alpha_4} \ . \end{aligned} \tag{4.3}$$

Furthermore, applying Λ to both sides of equation (4.3), then adding the result to equation (4.2) gives

$$\begin{aligned} \frac{1}{s}\tilde{h}_{\alpha_0} + \frac{1}{s}\Lambda\tilde{h}_{\alpha_2} + \frac{1}{s}\Lambda^2\tilde{h}_{\alpha_4} &= \tilde{U}_{\alpha_0} - \Lambda\tilde{U}_{\alpha_2} + \Lambda\tilde{U}_{\alpha_2} - \Lambda^3\tilde{U}_{\alpha_6} \\ &= \tilde{U}_{\alpha_0} - \Lambda^3\tilde{U}_{\alpha_6} \ . \end{aligned} \tag{4.4}$$

Continuing this process n times gives

$$\frac{1}{s}\tilde{h}_{\alpha_0} + \frac{1}{s}\Lambda\tilde{h}_{\alpha_2} + \cdots + \frac{1}{s}\Lambda^n\tilde{h}_{\alpha_{2n}} = \tilde{U}_{\alpha_0} - \Lambda^{n+1}\tilde{U}_{\alpha_{2n-2}} \ . \tag{4.5}$$

If it is assumed that $\Lambda^n\tilde{U}_{\alpha_{2n}} \to 0$ as $n \to \infty$ then according to this process.

$$\tilde{U}_{\alpha_0} = \frac{1}{s}\tilde{h}_{\alpha_0} + \Lambda\left(\frac{1}{s}\tilde{h}_{\alpha_2} + \Lambda\left(\frac{1}{s}\tilde{h}_{\alpha_4} + \cdots + \Lambda\left(\frac{1}{s}\tilde{h}_{\alpha_{2n}} + \cdots\right)\cdots\right)\right) \ . \tag{4.6}$$

In order to illustrate equation (4.6), the first few iterations will be computed. But first.

Lemma 4.1

$$\Lambda\tilde{f}_{\alpha_2} = \frac{1}{s}\mathcal{F}_{\alpha_0}\left\{(f'w)'\right\} \ .$$

Proof:

$$\begin{aligned}
\Lambda\tilde{f}_{\alpha_2} &= \frac{1}{s}S * \left(T * \tilde{f}_{\alpha_2}\right) \\
&= \frac{1}{s}S * \left[(-4\pi i\phi\delta_{\alpha_2} + 2\pi i\psi_2\delta_{\alpha_2}) * \tilde{f}_{\alpha_2}\right] \\
&= \frac{1}{s}S * \left[\left(4\alpha_2^2\frac{-\pi i\phi\delta_{\alpha_2}}{\alpha_2^2} + 2\pi i\psi_2\delta_{\alpha_2}\right) * \tilde{f}_{\alpha_2}\right] \ .
\end{aligned}$$

Now, recall equations (3.12) and (3.13), then

$$\begin{aligned}
\Lambda\tilde{f}_{\alpha_2} &= \frac{1}{s}S * \left[\left(4\alpha_2^2\mathcal{F}_{\alpha_2}\{x\} + 2\pi i\psi_2\mathcal{F}_{\alpha_2}\{1\}\right) * \mathcal{F}_{\alpha_2}\{f\}\right] \\
&= \frac{1}{s}S * \left(4\alpha_2^2\mathcal{F}_{\alpha_1}\{xf\} + 2\pi i\psi_2\mathcal{F}_{\alpha_1}\{f\}\right) \\
&= \frac{1}{s}\left(4\alpha_1^2\bar{z}_{\alpha_1} + 2\pi i\psi_1\bar{w}_{\alpha_1}\right) * \left(2\alpha_1^2\mathcal{F}_{\alpha_1}\{xf\} + 2\pi i\phi\mathcal{F}_{\alpha_1}\{f\}\right) \ .
\end{aligned}$$

but by the derivative theorem (3.9).

$$\begin{aligned}
\Lambda\tilde{f}_{\alpha_2} &= \frac{1}{s}\left(4\alpha_1^2\mathcal{F}_{\alpha_1}\{xw\} + 2\pi i\psi_1\mathcal{F}_{\alpha_1}\{w\}\right) * \mathcal{F}_{\alpha_1}\{f'\} \\
&= \frac{1}{s}\left(4\alpha_1^2\mathcal{F}_{\alpha_0}\{xf'w\} + 2\pi i\psi_1\mathcal{F}_{\alpha_0}\{f'w\}\right) \\
&= \frac{1}{s}\left(2\alpha_0^2\mathcal{F}_{\alpha_0}\{xf'w\} + 2\pi i\phi\mathcal{F}_{\alpha_0}\{f'w\}\right) \ .
\end{aligned}$$

and applying the derivative theorem (3.9) once again,

$$\Lambda \bar{f}_{\alpha_2} = \frac{1}{s}\mathcal{F}_{\alpha_0}\{(f'w)'\} .$$

□

Now, by equation (4.1), the first iteration of (4.6) is

$$\begin{aligned}\tilde{U}_{1,\alpha_0} &= \frac{1}{s}\tilde{h}_{\alpha_0} + \Lambda\frac{1}{s}\tilde{h}_{\alpha_2} \\ &= \frac{1}{s}\mathcal{F}_{\alpha_0}\{h\} + \frac{1}{s^2}\mathcal{F}_{\alpha_0}\{(h'w)'\} \\ &= \mathcal{F}_{\alpha_0}\left\{\frac{h}{s} + \frac{(h'w)'}{s^2}\right\} . \end{aligned} \tag{4.7}$$

Let

$$\begin{aligned} Q_0 &= h \\ Q_1 &= h + \frac{(Q_0'w)'}{s} \\ Q_2 &= h + \frac{(Q_1'w)'}{s} \\ &\vdots \\ Q_n &= h + \frac{(Q_{n-1}'w)'}{s} , \end{aligned} \tag{4.8}$$

then the first iteration is

$$\tilde{U}_{1,\alpha_0} = \mathcal{F}_{\alpha_0}\left\{\frac{Q_1}{s}\right\} .$$

The second iteration of (4.6) is

$$\begin{aligned}\tilde{U}_{2,\alpha_0} &= \frac{1}{s}\tilde{h}_{\alpha_0} + \Lambda\left(\frac{1}{s}\tilde{h}_{\alpha_2} + \Lambda\frac{1}{s}\tilde{h}_{\alpha_4}\right) \\ &= \frac{1}{s}\tilde{h}_{\alpha_0} + \Lambda\left(\frac{1}{s}\tilde{Q}_{0,\alpha_2} + \Lambda\frac{1}{s}\tilde{Q}_{0,\alpha_4}\right) \\ &= \frac{1}{s}\tilde{h}_{\alpha_0} + \Lambda\left(\frac{1}{s}\mathcal{F}_{\alpha_2}\left\{Q_0 + \frac{(Q_0'w)'}{s}\right\}\right) \\ &= \frac{1}{s}\tilde{h}_{\alpha_0} + \Lambda\left(\frac{1}{s}\tilde{Q}_{1,\alpha_2}\right) \\ &= \frac{1}{s}\tilde{h}_{\alpha_0} + \frac{1}{s^2}\mathcal{F}_{\alpha_0}\{(Q_1'w)'\} \end{aligned}$$

$$= \mathcal{F}_{\alpha_0}\left\{\frac{h}{s} + \frac{(Q_1' w)'}{s^2}\right\} \tag{4.9}$$

$$= \mathcal{F}_{\alpha_0}\left\{\frac{Q_2}{s}\right\} . \tag{4.10}$$

Lemma 4.2 *In general, the nth iteration of (4.6) is*

$$\tilde{U}_{n,\alpha} = \mathcal{F}_\alpha\left\{\frac{Q_n}{s}\right\} \tag{4.11}$$

where Q_n is defined by (4.8).

Proof: For purpose of induction, assume that the $n-1$ iteration of (4.6) is

$$\tilde{U}_{n-1,\alpha_0} = \mathcal{F}_{\alpha_0}\left\{\frac{Q_{n-1}}{s}\right\} .$$

then the nth iteration is

$$\begin{aligned}
\tilde{U}_{n,\alpha_0} &= \frac{1}{s}\tilde{h}_{\alpha_0} + \Lambda\left(\frac{1}{s}\tilde{h}_{\alpha_2} + \Lambda\left(\frac{1}{s}\tilde{h}_{\alpha_4} + \cdots + \Lambda\left(\frac{1}{s}\tilde{h}_{\alpha_{2n-2}} + \Lambda\tilde{h}_{\alpha_{2n}}\right)\cdots\right)\right) \\
&= \frac{1}{s}\tilde{h}_{\alpha_0} + \Lambda\tilde{U}_{n-1,\alpha_2} \\
&= \frac{1}{s}\tilde{h}_{\alpha_0} + \Lambda\left(\frac{1}{s}\tilde{Q}_{n-1,\alpha_2}\right) \\
&= \frac{1}{s}\tilde{h}_{\alpha_0} + \frac{1}{s^2}\mathcal{F}_{\alpha_0}\left\{\left(Q_{n-1}' w\right)'\right\} \\
&= \mathcal{F}_{\alpha_0}\left\{\frac{h}{s} + \frac{\left(Q_{n-1}' w\right)'}{s^2}\right\} \\
&= \mathcal{F}_{\alpha_0}\left\{\frac{Q_n}{s}\right\} .
\end{aligned}$$

□

Theorem 4.3 (Method of Inversion) *If $\Lambda^n\tilde{U}_{\alpha_{2n}} \to 0$ as $n \to \infty$, then*

$$u = \lim_{n\to\infty} \mathcal{L}^{-1}\left\{\frac{Q_n}{s}\right\} . \tag{4.12}$$

Proof: Recalling the definition of $\tilde{U}_\alpha$, the nth iteration gives

$$\begin{aligned}
u_n &= \mathcal{L}^{-1}\left\{\mathcal{F}_\alpha^{-1}\left\{\mathcal{F}_\alpha\left\{\frac{Q_n}{s}\right\}\right\}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{Q_n}{s}\right\} .
\end{aligned} \tag{4.13}$$

By (4.6), $u_n \to u$ if $\Lambda^n\tilde{U}_{\alpha_{2n}} \to 0$. □

4.2 Legendre Example

Recall that a solution to the heterogeneous heat equation (1.2) when $w = 1-x^2$ is $u = xe^{-2t}$. This example can be used to illustrate the application of (4.6). So, in the case where $w = 1 - x^2$ and $u = xe^{-2t}$,

$$\bar{U}_{\alpha_0} = \mathcal{F}_{\alpha_0}\{\mathcal{L}\{u\}\} = \mathcal{F}_{\alpha_0}\left\{\mathcal{L}\left\{xe^{-2t}\right\}\right\} = \mathcal{F}_{\alpha_0}\left\{\frac{x}{s+2}\right\} \tag{4.14}$$

is the solution against which successive iterations of (4.6) should be compared. By (4.7), the first iteration for this example is

$$\begin{aligned}
u_1 &= \mathcal{L}^{-1}\left\{\frac{h}{s} + \frac{(h'w)'}{s^2}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{x}{s} + \frac{(x'w)'}{s^2}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{x}{s} + \frac{(1-x^2)'}{s^2}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{x}{s} - \frac{2x}{s^2}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{x}{s^2}(s-2)\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{x}{s+2}\frac{s^2-4}{s^2}\right\} \\
&= x(1-2t) \; .
\end{aligned}$$

The second iteration is (by 4.10)

$$\begin{aligned}
u_2 &= \mathcal{L}^{-1}\left\{\frac{h}{s} + \frac{(Q_1'w)'}{s^2}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{h}{s} + \frac{1}{s^2}\left[\left(h + \frac{(h'w)'}{s}\right)' w\right]'\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{h}{s} + \frac{1}{s^2}\left[\left(x + \frac{(x'w)'}{s}\right)' w\right]'\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{h}{s} + \frac{1}{s^2}\left[\left(x + \frac{(1-x^2)'}{s}\right)' w\right]'\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{h}{s} + \frac{1}{s^2}\left[\left(x - \frac{2x}{s}\right)'(1-x^2)\right]'\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{L}^{-1}\left\{\frac{h}{s}+\frac{1}{s^2}\left[\left(1-\frac{2}{s}\right)(1-x^2)\right]'\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{h}{s}+\frac{1}{s^2}\left(1-x^2-\frac{2}{s}+\frac{2x^2}{s}\right)'\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{h}{s}+\frac{1}{s^2}\left(-2x+\frac{4x}{s}\right)\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{x}{s^3}\left(s^2-2s+4\right)\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{x}{s+2}\frac{s^3+8}{s^3}\right\} \\
&= x\left(1-2t+2t^2\right) \quad .
\end{aligned}
$$

Finally, third iteration is

$$
\begin{aligned}
u_3 &= \mathcal{L}^{-1}\left\{\frac{h}{s}+\frac{(Q_2'w)'}{s^2}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{h}{s}+\frac{1}{s^2}\left[\left(h+\frac{(Q_1'w)'}{s}\right)'w\right]'\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{h}{s}+\frac{1}{s^2}\left[\left(h+\frac{1}{s}\left\{\left[1-\frac{2}{s}\right]w\right\}'\right)'w\right]'\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{h}{s}+\frac{1}{s^2}\left[\left(x+\frac{1}{s}\left\{\left[1-\frac{2}{s}\right][1-x^2]\right\}'\right)'w\right]'\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{h}{s}+\frac{1}{s^2}\left[\left(x+\frac{1}{s}\left\{1-x^2-\frac{2}{s}+\frac{2x^2}{s}\right\}'\right)'w\right]'\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{h}{s}+\frac{1}{s^2}\left[\left(\frac{x}{s^2}\{s^2-2s+4\}\right)'w\right]'\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{h}{s}+\frac{1}{s^2}\left[\left(\frac{1}{s^2}\{s^2-2s+4\}\right)(1-x^2)\right]'\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{x}{s}+\frac{1}{s^2}\left(\frac{1}{s^2}\{s^2-2s+4\}\right)(-2x)\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{x}{s}-\frac{2x}{s^4}\left(s^2-2s+4\right)\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{x}{s^4}\left(s^3-2s^2+4s-8\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \mathcal{L}^{-1}\left\{\frac{x}{s+2}\frac{s^4-16}{s^4}\right\} \\
&= x\left(1-2t+2t^2-\frac{4}{3}t^3\right) \quad . \qquad (4.15)
\end{aligned}
$$

If it is taken under consideration that, for this example, a known solution is

$$
\begin{aligned}
u &= xe^{-2t} \\
&= x\sum_{n=0}^{\infty}\frac{(-2t)^n}{n!} \\
&= x\left(1-2t+2t^2-\frac{4}{3}t^3+\cdots+\frac{(-2t)^n}{n!}+\cdots\right) \quad .
\end{aligned}
$$

it is rather convincing that (4.6) is correct, in light of (4.15).

CHAPTER V

CONCLUSION

The method developed here *appears* to be producing the correct results for the few known solutions to (1.2). It is suspected that if the iteration is carried out indefinitely, the method will converge to the actual solution; but this is by no means proven. If, however, convergence can be proven, then this method would be useful in finding solutions to (1.2). The fact that the method gives significant results for the known examples is encouraging.

The method is useful even without proof of convergence, since application of the iteration is easily implemented by computer software (see Appendix A). In order to find a solution u for the heterogeneous heat equation, a suitable number of iterations may be computed for a given weight equation w and initial data h. The output of the iteration can then be examined, so that the mathematician may determine the solution by recognizing the series, given the first few terms. The suspected solution then can be easily checked by substituting it into the actual heterogeneous heat equation.

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APPENDIX A
PROGRAMS

```
#!maple
# hheinv.ms  Maple Vr3 script
#
# REVISION: 1.0 14Jul95 eralston@math.ttu.edu (Eddie Ralston)
#
# This script is a tool that may aid in finding solutions to the
# heterogeneous heat equation:
#
#                            (wu )  = u
#                               x x    t
#
# where u is a function of (x,t) and w is a function of (x).
#
# This script implements the method developed in the thesis
# ``A Method of Solution of the Weighted Heat Equation'' by
# the author of this script.
#
# INSTRUCTIONS:
#   Once Maple is started, this script can be loaded by entering:
#
#                        read `hheinv.ms`;
#
#   The script will then be loaded into the user's current Maple session.
#   The three functions that the user should be concerned with are
#   w(), h(), and iterate().  w() and h() default to the Legendre
#   example discussed in the thesis.  In order to perform, for example,
#   ten iterations with a cosine weight function and sine initial
#   data, the user should enter the following commands, once this
#   script is loaded:
#
#     w := proc(x) cos(x); end;
#     h := proc(x) sin(x); end;
#     iterate(10);
```

```
#
#   The tenth iteration will be printed to the current output device.
#
#   The function compare() prints the left and right hand sides of
#   the heterogeneous heat equation, inserting w, h, and the u
#   generated by the most recent iterate().

#-----------------------------
# load the required libraries.
#-----------------------------
readlib(laplace):


#-----------------------------------
# define the default user parameters.
#-----------------------------------

# the weight equation w(x).
w := proc(x) 1-x^2; end;

# the initial data h(x).
h := proc(x) x; end;


#-------------------------------------------
# define the functions used in the iteration.
#-------------------------------------------
iterproc := proc(f,x,s)
  h(x) + diff( diff( f(x,s), x ) * w(x), x ) / s:
  end:

iterate := proc(iterations)
  global u, f, ftemp;
```

```
  local i;

  f := proc(x,s) h(x); end:
  for i from 1 to iterations do
    ftemp := iterproc( f, x, s ):
    f := proc(x,s) ftemp; end:
  od;

  u := invlaplace( f(x,s)/s, s, t ):
  u;
  end:

compare := proc()
  global u;

  # print the left hand side.
  print( diff( w(x) * diff( u, x ), x ) );

  # print the right hand side.
  print( diff( u, t ) );
  end:

#--------------------------------------------
# do the iterations with the current example.
#--------------------------------------------
iterate(3);

#--------------------------------------
# print the left and right hand sides
# of the hhe, using the iteration as u.
#--------------------------------------
compare();
```

```
# end of hheinv.ms
```

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Student's Signature Date

Disagree (Permission is not granted.)

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