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Supersonic Survival Envelope Analysis for High Altitude Free Fall

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High altitude operations are always associated with a number of physical conditions challenging human survival scenarios, specifically during emergency egress for stratospheric and space flights. One of those conditions is presented through transition from supersonic to subsonic velocities in free fall from altitudes higher than 40 km. Research suggests mathematical model applications are applicable, outlining the survival envelope for human subjects involved in free fall and parachuting from different altitudes and subsequent transition from supersonic to subsonic velocities accompanied by aerodynamic shock waves impact on human body. Theoretical analysis for free fall equations shows that altitude of transonic transition on reentry decreases non-linearly and approaching a certain limit as initial altitude of free fall increases. This transonic transition limit is determined by planetary atmospheric characteristics and can be elevated by properly applied parachuting technology. Preliminary computation results show a distinct possibility for human subjects survival during emergency egresses from protective vehicle at altitudes around and higher than 100 km ~ 300,000 ft depending on their level of physical training. A discussion related to further research and extension of theoretical results into the area of direct experimentation with stratospheric sky-diving is also provided.

Nomenclature

V	=	velocity of free fall
Vt	=	terminal velocity of fall in uniform atmosphere
Vs	=	speed of sound in Earth atmosphere
m	=	mass of falling object
g	=	free fall acceleration, 9.81 $m*s^{-2}$
z	=	vertical coordinate of falling object
t	=	time
Cd	=	drag coefficient in Earth atmosphere at sea level
ρ	=	atmosphere density at sea level
A	=	cross-sectional area of falling object
K	=	$0.5*Cd*\rho*A$ = integral parameter introduced for simplicity of math transformations, it can be considered as an integrative drag/friction coefficient in the expression for drag forces in Laplace's atmosphere, or as an expression for drag forces at sea level if multiplied by V^2 (free fall velocity in square)
U	=	(V/Vt) = intermediate variable introduced to simplify math transformations for original differential

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equation describing forces balance in free fall

λ	=	characteristic height for standard atmosphere (Laplace's isothermal atmosphere approximation)
F(z)	=	intermediate function introduced for math equations compaction/shortening
Zo	=	free-fall initial altitude (initial velocity at this altitude is assumed to equal zero in this research which generally is not the case)
a	=	$2g\lambda/V(t)^2$ = non-dimensional constant specific for planetary atmosphere and gravity, for Earth environment numerical value is calculated as a = 46.686
Zs	=	altitude of transition from supersonic to subsonic velocities during a free fall
l	=	characteristic size of Laplacian's atmosphere exponentially changing density with altitude
FFA	=	free fall (initial altitudes)
MDA	=	maximum deceleration altitudes
MDAL	=	maximum deceleration altitudes limit

I. Introduction

The physical problem of free fall velocity profiles for objects moving from high altitudes was theoretically considered before.[1-3] But analysis and numerical calculations were conducted for relatively low altitudes (between 10 km and 30 km) where atmospheric drag forces are insignificant and do not significantly impact falling objects. Also within this range of initial free fall altitudes, the speed of sound cannot be reached which excludes from analysis such an important factor as transonic transition and following from this condition shock waves impact on relatively unprotected human body. The number of experimental implementations performed in this area is also very limited. The only stratospheric jumps known up to date were:

- Attempted in August 1960 by Col. J. Kittinger, USAF, from altitude approximately 31 km [4], and
- Several decades later in October 2012 by Felix Baumgartner from altitude approximately 39 km; during this attempt Baumgartner reached speed of sound barrier.[5]

At the same time the most interesting range for consideration is located above altitudes of 39 km, because of the number of challenges faced by skydivers and also human subjects during space rescue operations. Those challenges include, but are not limited to, transition from supersonic to subsonic velocities and impact of sound shock waves (not analyzed for human subjects).

Among those stress drag-forces, impacts to objects falling from different altitudes were analyzed theoretically [3] but transonic transitions remain as one of the least researched areas (even at the level of theoretical analysis). In this research a theoretical analysis is attempted to outline major approaches and obtain preliminary results regarding effects of transonic transitions on objects free falling from stratospheric altitudes around 40 km and higher. These results are supposed to help in development of further experimental approaches for emergency and rescue scenarios applied to human missions in space. Extreme skydiving, as a new emerging sport could also benefit from this analysis. Other challenges of stratospheric skydiving and their impacts on human subjects are going to be considered in more details in future upcoming research and publications.

II. Major Concepts from Previous Study

A. Free fall equation

The free fall differential equation for Laplace's isothermal atmosphere can be presented [1, 2] in the form:

$$\frac{mdV}{dt} = -m * g + K * V^2 * e^{\left(-\frac{z}{\lambda}\right)}$$
$$K = 0.5 * Cd * \rho * A$$

where mathematical symbols are clarified in 'Nomenclature' paragraph for this paper.

The parameter λ = characteristic height for standard atmosphere (Laplace's isothermal atmosphere), introduced for theoretical consideration. Within this project for the Earth's atmosphere an exponential decrease in pressure and density is assumed with characteristic Laplace's parameter $\lambda = 7.4621*10^3$ m = 7.46 km. [1, 6]

By substitution dt = dz/V, it is easy to obtain another form for the same equation:

$$\frac{mVdV}{dz} = -mg + KV^2 * e^{-\frac{z}{\lambda}},$$

or

$$\frac{VdV}{dz} = -g\left[1 - \left(\frac{K}{mg}\right) * V^2 * e^{-\frac{z}{\lambda}}\right],$$

$$\frac{VdV}{dz} = -g[1 - \left(\frac{V}{Vt}\right)^2 * e^{-\frac{z}{\lambda}}], \qquad (1)$$

where:: $Vt = \sqrt{\frac{mg}{K}} = \sqrt{\frac{mg}{0.5*Cd*\rho*A}}$,

and this parameter has a special name of 'terminal velocity' (for uniform, equal density at different altitudes, atmosphere), velocity which is reached in uniform sea level atmosphere when a balance between free fall acceleration and drag acceleration is achieved [1, 7], atmospheric density ρ is assumed at sea level here. The solution for (1) can be obtained by standard analytical methods of math analysis.

B. Solution

Solution for differential equation (1), omitting sign (–) as indicator that our z - axis direction is selected toward the Earth, can be written as [1, 2]:

$$U = \frac{V}{Vt} = a^{\frac{1}{2}} * \exp\left[-(\frac{a}{2})exp\left(-\frac{z}{\lambda}\right)\right] * F(z),$$

$$F(z) = [\frac{Zo-z}{\lambda} + \sum_{n=1}^{\infty} a^n * (e^{-\frac{nz}{\lambda}} - e^{-\frac{nZo}{\lambda}})/nn!]\hat{}(1/2)\ , \qquad (2)$$

where:
a = 2gλ/(Vt)^2 = constant specific for planetary atmosphere and gravity; for Earth environment numerical value is calculated as a = 46.686;
Zo = initial altitude of free fall (where initial velocity of free fall V = 0).

From numerical representation of (2) it is easy to see that free fall velocities increase to a certain level, then after entering dense atmospheric layers they asymptotically approach 'terminal velocity' of free fall in uniform atmosphere with density at sea level. Figure 1 supports this conclusion for different altitudes z and different initial free fall altitudes Zo. Three major altitudes Zo were selected for comparative calculations: 31 km [8-10], 39 km [5, 11], and 100 km (Karman Line, officially accepted boundary between space and Earth environment). To unify different initial altitudes to one scale (horizontal axis on Figure 1) a new variable is introduced here y = (1 – z/Zo) which is changing between 0 and 1 for different initial altitudes Zo.

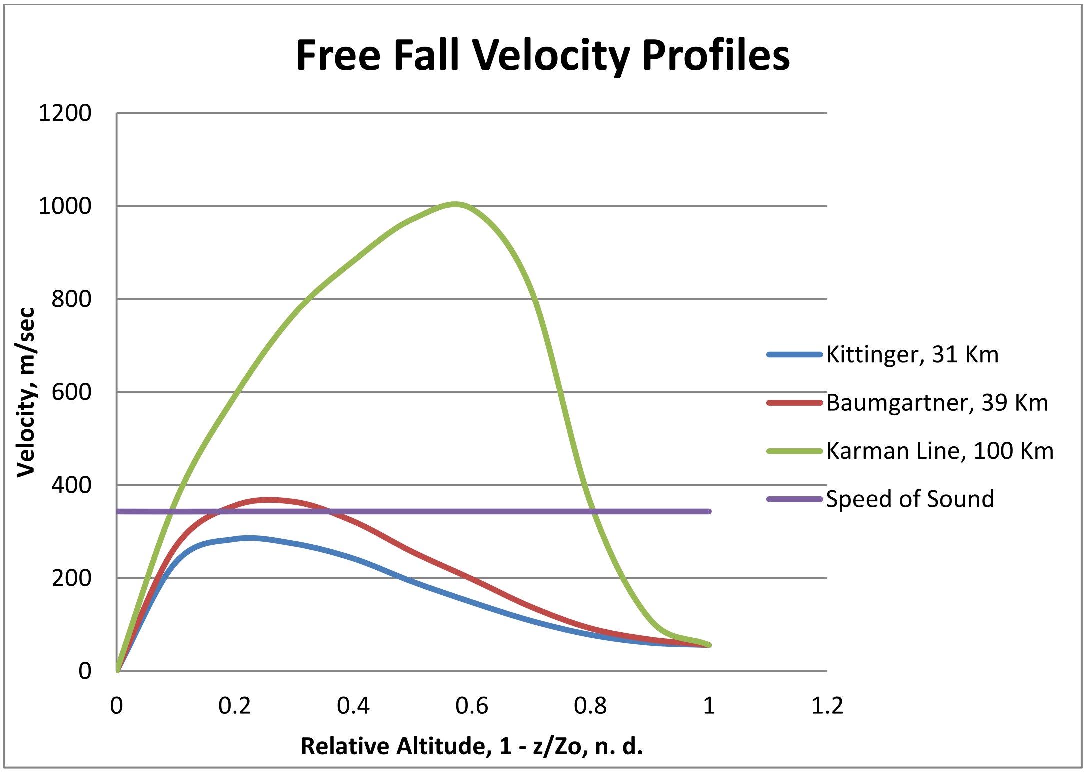

Figure 1. Comparison of free fall velocities profiles for objects moving in Laplace's (exponentially changing density) atmosphere. (Different initial altitudes are presented in different colors. When y increases from 0 to 1 coordinate z of free falling object changes from initial altitude Zo to 0 (zero) - level. Speed of sound 343.2 m/s is presented on the graph for reference purposes.)

Analyzing this graph allows us to formulate few important qualitative conclusions:

- Transonic transitions (crossing speed of sound barrier) are possible only in free fall above a certain critical altitude, such as for Baumgartner's sky diving, Zo = 38,969.3 m ≈ 39 km (which is determined by planetary atmospheric characteristics);
- Free fall velocities profile when sound barrier is being crossed (on reentry into the dense atmospheric layers) is becoming more and more steep as initial free fall altitude increases;
- Transonic transition altitude on reentry into dense atmospheric layers is changing nonlinearly and asymptotically approaching to a certain limit determined by characteristics of planetary (in this specific consideration the planet Earth) atmosphere and gravity (coordinates y for velocity profile graphs crossing with the speed of sound line are getting closer to the certain limit below value of 1.0 with initial altitude Zo increase).

These statements can be clearly supported by quantitative approximations following from equations (1, 2). The following below paragraphs provide more detail for some of the possible approximations.

III. Approximation for Altitude of Transonic Transition for Varying Free Fall Initial Altitudes Zo

In order to find transonic transition altitudes on reentry to dense atmosphere, the solution for the below equation has to be obtained:

$$\frac{Vs}{Vt} = a^{\frac{1}{2}} * \exp\left[-(\frac{a}{2})exp\left(-\frac{z}{\lambda}\right)\right] * F(Zm),$$

$$F(Zs) = \frac{Zo-Zs}{\lambda} + \sum_{n=1}^{\infty} a^n * [e^{-\frac{nZs}{\lambda}} - \ e^{-\frac{nZo}{\lambda}})/nn!]\hat{}(1/2)\ , \qquad (3),$$

where F(Zm) follows from original definition for F(z) (equation (2));
Vs = speed of sound, 343.2 m/s (under isothermal conditions, 20C);
Zs = altitude of transonic transition on reentry.

This equation is transcendent and the analytical formula connecting Zs and Zo, which can be obtained only through reasonable approximations. Those preliminary approximations can be made through a number of assumptions:

- a*exp(-Zs/λ) > 1, taking into account that a = 46.68;
- exp(-Zs/λ) → 0, which means exponent is small enough, which is reasonable for considered free fall altitudes Zo, between 40 km and higher (for example ~ 400km), and λ = 7.46 km;
- exp(-Zo/λ) → 0, which means it is even smaller than exp(-Zs/λ), because Zo » Zs for majority of considered altitudes;
- higher degree terms under sign of sum Σ approaching to 0 and can be neglected

Then original equation (3) could be approximated as:

$$\left(\frac{Vs}{Vt}\right)^2 = a * e^{-aexp\left(-\frac{Zs}{\lambda}\right)} * [\frac{Zo-Zm}{\lambda} + aexp\left(-\frac{Zm}{\lambda}\right)].$$

Substituting into last equation expression for a =2gλ/Vt^2 it is easy to get:

$$\frac{Vs^2}{2g\lambda} = \exp\left(-\mathrm{a} * \exp\left(-\frac{\mathrm{Zs}}{\lambda}\right)\right) * [\frac{\mathrm{Zo}}{\lambda} * \left(1 - \frac{\mathrm{Zs}}{\mathrm{Zo}}\right) + \mathrm{a} * \exp\left(-\frac{\mathrm{Zs}}{\lambda}\right)]$$

Logarithm math operation applied to both sides of this last equation gives:

$$Ln\left(\frac{Vs^2}{2g\lambda}\right) = -ae^{-\frac{Zs}{\lambda}} + Ln\{\frac{Zo}{\lambda} * \left(1 - \frac{Zs}{Zo}\right) + a * \exp\left(-\frac{Zs}{\lambda}\right)\}.$$

Because of Zo>Zs, and Zs>λ next step in approximations can be applied what gives:

$$Ln\left(\frac{Vs^2}{2g\lambda}\right) - \ Ln[\frac{Zo}{\lambda} * \left(1 - \frac{Zs}{Zo}\right)] = -ae^{-\frac{Zs}{\lambda}}$$

$$Ln\left(\frac{Vs^2}{2g\mathrm{Zo}}\right) = -ae^{-\frac{Zs}{\lambda}}$$

Approximate expression for Zs can be found then as:

$$Zs = \lambda * Ln[\frac{a}{Ln\left(\frac{2gZo}{Vs^2}\right)}] \qquad (4),$$

Results of the approximation from (4) and results from direct calculations for transonic transition altitudes from equation (1) are presented on Figure 2. Original calculations for transonic altitudes as a function of Zo are given within range of free fall altitudes between 40 km and 450 km.

Comparison for different approximations gives satisfactory agreement.

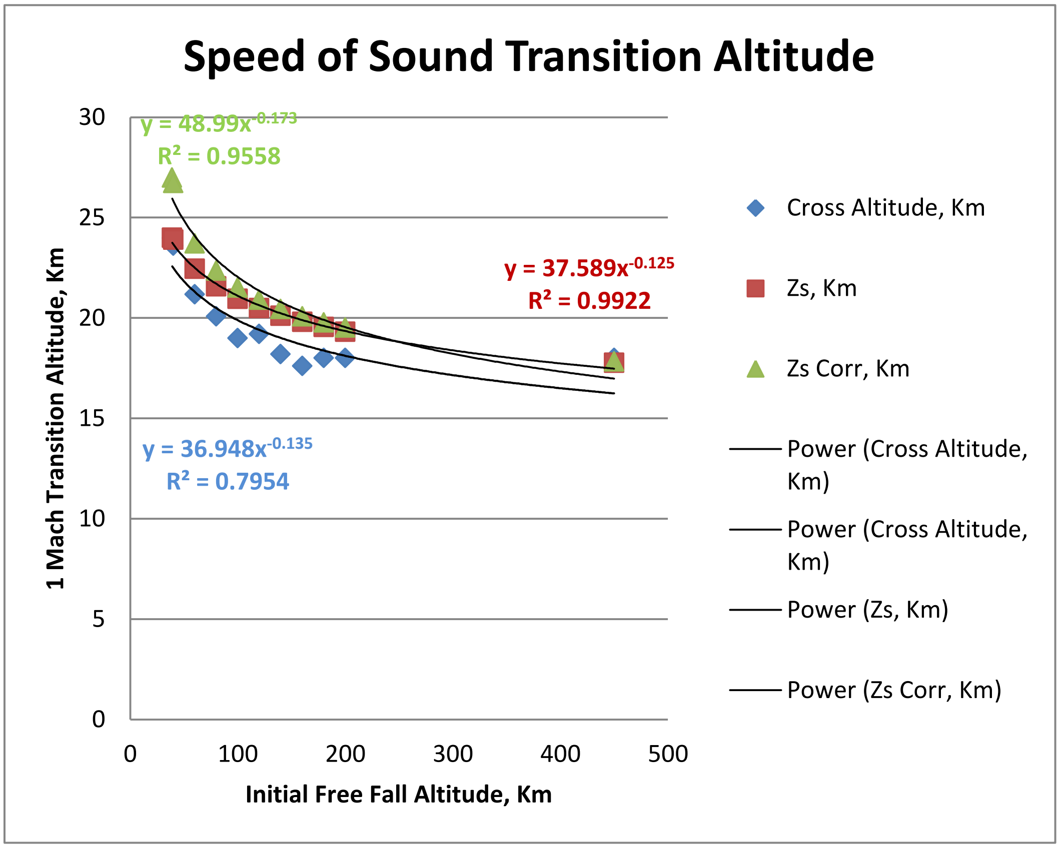

Figure 2. Transonic transition altitudes Zs as a function of initial free fall altitudes Zo. (Zs= values (red) calculated from approximate equation (4). Zs values (blue) calculated from original equation for velocity profiles (1). Zs corrected values (green) are presented for comparative demonstrations. Standard Excel Power approximation is applied for all results to demonstrate close proximity for different approximations.)

Exponential approximation from Excel do show good agreement (very much similar exponents) between calculations from original equation (2) and from approximate equation (4) within the range of free fall altitudes 40 km and 450 km.

IV. Effect of Parachuting in Free Fall from High Altitudes

The last formula (4) allows preliminary analysis for effects of parachuting on altitude of transonic transition. It is represented through constant a variations: $a = 2g\lambda/Vt^2$, and $Vt = (mg/0.5*\rho*Cd)^{1/2}*A^{-1/2}$ following from original free fall equation. Falling body mass m and parachute cross-sectional area A are the most widely changing variables here. So, a ~ A/m, which means that constant a increases when the cross-sectional area of falling object A increases, and constant a decreases when body mass increases and vice versa. Variation for a was applied within range of a = 5 to 100 and calculation of Zs from equation (4) have been done just for demonstrative purposes here. Furthermore specific research is required for wider variations around a = 46.686 (constant for Earth atmosphere and average human body mass), specifically above value a = 100 which results from an increase of falling object cross-sectional area A. Increase in a value by 2 means that parachute area A increases 2 times also.

Results are presented on Figure 3. They do show the increase in a value twice more leads to increase transonic transition altitude from approximately 18 km to 25 km where the atmosphere is more rarified:

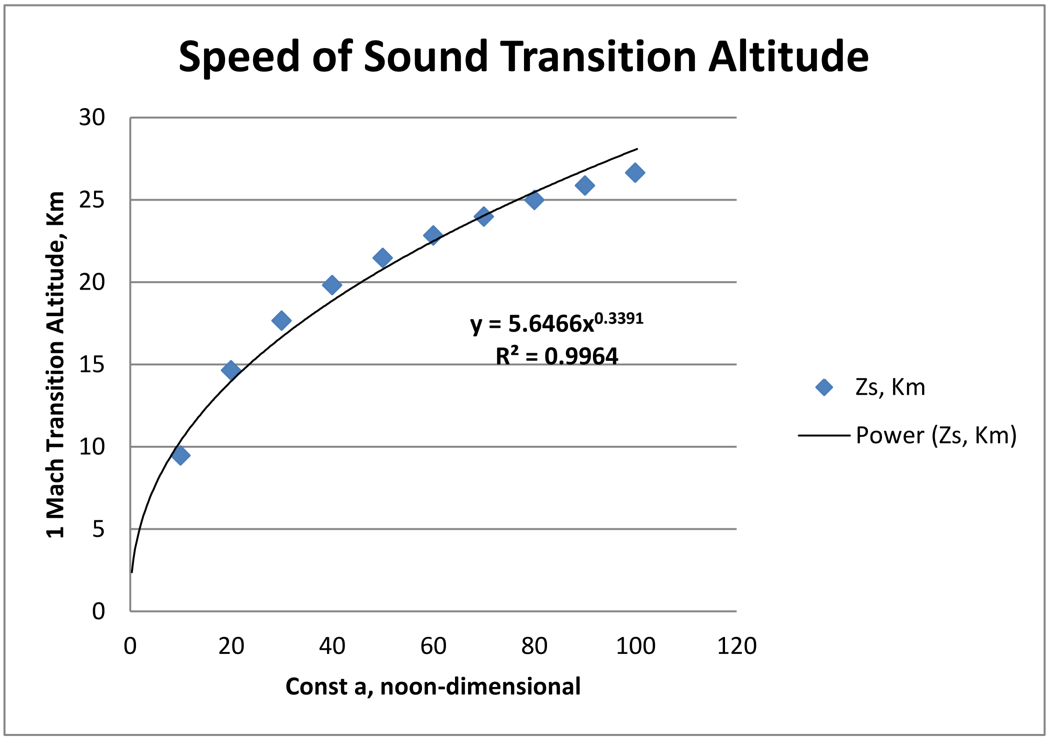

Figure 3. Parachuting effect (high cross-sectional area A and constant a values increase) on transonic transition altitude on reentry into dense atmospheric layers (calculations are done for initial free fall altitude 100 Km).

Figure 3 does show positive effects of parachuting on transonic transition altitudes bringing them to more rarified atmospheric layers. So, parachuting has to be considered as a countermeasure to mitigate effect of transonic shock waves on reentry into dense atmospheric layers. Here, parachute deployment is assumed since the start of the free fall, which is not a completely correct assumption and requires further analysis in terms of stability of deployment under low atmospheric densities. Parachute deployment scenario during free fall is a subject for independent study however; because it requires more specific considerations with regard to the parachute deployment profile in time.

V. Conclusions

A. Theoretical Developments

Theoretical approach and algorithms based on atmospheric drag equation are developed for analysis of velocity profiles for objects free falling from high (stratospheric) altitudes;
Analysis was conducted for obtaining formula approximating free fall initial altitudes and altitudes where transonic transitions occurs on reentry into dense atmospheres;
Reasonable mathematical approximation is founded for numerical estimates within range of initial free fall altitudes 10 km to 450km.

B. Quantitative Estimates of Impact

Object falling from higher altitudes has higher maximum velocity and reaches it at a higher altitude;
Transonic transition altitude decreases with initial free fall altitude increase nonlinearly and eventually approach a certain limit determined by planetary atmosphere characteristics. For Earth atmosphere this limit is approximately18 km. Because of very low density atmosphere at this altitude transonic shockwaves impact on the human body would not be too high;
Parachuting (properly applied) increases this transonic transition altitude significantly leading to further decrease in shockwaves impact.
Further research (theoretical and experimental) are required to validate these theoretical estimates and develop reasonable scenario of parachuting for emergency escape or extreme skydiving from stratospheric altitudes.

C Future Research Directions

Next phase of theoretical approximations for high altitudes free fall and parachuting could be provided through:

- Theoretical analysis of free fall math model in non-isothermal atmosphere;
- Theoretical analysis of heat load on the object for free fall from high altitudes.

Basic schematics for experimental testing could be suggested from above presented analysis and conclusions:

- Skydiving for human subjects could be provided with gradual increase of initial altitude starting from 30 km - 40 km (already tested initial altitudes for experimentation);
- Gradual increase with increments of 5 km could be provided up to altitudes of approximately100 km and probably higher (strategy applied in stratospheric skydiving by Felix Baumgartner currently in collaboration with Red Bull Stratos);
- Based on first few steps of super-high altitudes skydiving (range between 30 km and 40 km, which can be considered as transonic altitudes) next increase in test altitudes for free fall and following parachuting could be suggested at 50 km.
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