ON MINIMIZING EXPECTED WARRANTY COSTS IN 1-DIMENSION AND 2-DIMENSIONS WITH DIFFERENT REPAIR OPTIONS

by

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ABSTRACT

We study warranty models of one-dimension and two-dimensions with different options available to the manufacturer for repair-replacement upon product failure. Product failures during warranty period incur additional costs to the manufacturer and leads to customer dissatisfaction when suitable service action is not adopted. To forecast and minimize the expected cost of warranty servicing is of considerable interest to product manufacturers and decision makers. The results of this research will enable manufacturers, reliability engineers and warranty managers to make better decisions in developing suitable servicing strategies and towards significant cost savings in administering effective management of warranty programs.

In one-dimensional model we study a non-renewing combination warranty policy with initial base-warranty period (BWP) followed by pro-rata period (PRP). Product failures during BWP incur no cost to the customer, while the customer can purchase a new product at pro-rata price during PRP period. The manufacturer has three options available during BWP, namely minimal repair, general repair and replacement to address product failures and each option varies based on degree of repair, and the cost associated. We obtain optimal product price, pro-rata period, and sales volume for each type of repair-replacement option employed, derive stationary points and necessary second conditions to minimize the manufacturers cost of warranty servicing. We numerically illustrate the results obtained when the lifetime distribution of the product follows: (i) Gamma order-2, and (ii) extended weibull distribution.

In two-dimensional model we study a non- renewing, combination warranty policy defined by rectangular region, which is composed of three disjoint subregions. Product failures in each subregion incur variable costs based on the usage, age of the product, and repair strategy adopted. We assume pro-rated costs for servicing which is dependent on the age and the usage of the product at the time of failure. We consider the warranty duration limits set by the manufacturer for restricted and unrestricted product usage and compare the effects of two servicing strategies based on when the manufacturer can exercise replacement option. Using an estimated failure intensity and usage distribution we derive expressions for expected costs of

warranty servicing based on conditioning arguments. We numerically compare the effect of restricted and unrestricted cases for both the servicing strategies to obtain minimal expected costs to the manufacturer.

Finally, we conclude with a brief discussion about extensions to the models developed and future research directions.

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CHAPTER 1
INTRODUCTION

Warranty policies have been in existence for many years now with appearance across products and services. Product warranty is an agreement between manufacturer and customer by which the manufacturer agrees to repair-replace the product when the purchased product fails to work as intended. Customers value these policies for their time limited protection against early product failures, but for manufacturers the warranty has become a popular marketing tool. Other common interpretations of warranties emphasize quality, limitations of liability, reliability, product desirability, and superior craftsmanship. The present research focuses on minimization of the expected cost for servicing warranties in 1-dimension and 2-dimensions with an assortment of repair actions offered by the manufacturer, as well as the development of amenable strategies from the manufacturers viewpoint. Many product from small to large carries a warranty, the cost of which is observed in terms of quality and customer satisfaction of the product. The quality aspect for the cost of warranty is realized through product failures, repair, labor cost, and salvage value etc., and customer satisfaction is measured through acceptance of failure rate, manufacturers response time, and treatment meted to the customer by support crew. Simply put, manufacturer should be mindful that if their product keeps returning back to manufacturer, their customers will not.

The primary motivation of this research came from the large- scale impact of warranty policies on product sales, brand reputation, and competitiveness. Especially in the automobile industry, manufacturers are currently facing huge product recalls, increased warranty claim rates that lead to sinking profits, and loss of market capitalization due to tougher competition [1]. A closer look at a manufacturer's warranty management program reveals many telling details and suggests the scope of improvements. The estimated amount of warranty administration cost for many products runs to the tune of several billion dollars. A small reduction of costs associated with warranty servicing could translate to significant savings for the product manufacturers. Manufacturers should adopt ideal

[1] Auto Warranty vs. Quality. Industry Update
URL:http://www.warrantyweek.com/archive/ww20060620.html

strategies towards warranty servicing by optimizing the expected costs and warranty period duration so that the revenue generated from the sale of products with the warranty program will be higher than the revenue generated with the sale without a warranty. In short, warranties are an enterprizing asset, that when planned and administered properly, can provide huge benefits, but when managed poorly, may lead to severe loss of reputation and poor market capitalization.

In the next section, we provide an introduction to the concept of warranty modeling and its relevance and implications. We discuss some important concepts regarding warranty policies with regard to warranty reserve management, addressing product recalls, outsourced service contracts and impact of globalization on warranty policies. In Section 1.2, different classifications of warranty policies are reviewed, followed by section 1.3, which highlights the broad objectives and the problem statement of this research. Finally, section 1.4 contains the chapter summary.

1.1 Background and Relevance

Warranty markets are estimated to be worth over $25 billion per year [2]. The significance of this, however, was either understated or underreported until a few years ago. Poor warranty management programs have resulted in many organizations continually losing money, despite their best efforts to increase the quality of their products through stringent quality compliance programs and practices. In addition, warranty programs are dictated by prevailing market conditions, and costs associated with them appear in post-hoc analysis and not a part of the initial business model of the manufacturer. All these factors show the need for better warranty management and analysis. In today's terms, warranty markets are given much attention, predominantly due to the increasing costs, signal of product quality, and other factors like variation of warranty coverage period, higher claim rates, warranty reserve size determination, increased product safety recall, and recent laws. We shall discuss each of the factors as we proceed further.

For example, according to US automotive trade report, warranty markets are driven by the costs of warranty servicing, which depends on product lifetimes and

[2] $http://www.accenture.com/Global/Services/By_Industry$
$/Electronics_and_High_Tech/R_and_I/MaximizingManagement.htm$

terms of warranty that have been offered. The product manufacturer is quite well advised of the above, but the cost for servicing the warranty becomes inherently complex with factors like unpredictable defects, outsourced services, and different product usage rates among different types of customers. The 2006-2007 original equipment manufacturers industry report highlights the above factors as key elements for survival in business and strongly emphasizes the need to continuous improvement. Similar trends are felt and observed in electronic products and several other product industries.

Product warranties are often perceived by customers as signals of product quality, and the strategies employed by the manufacturer to service the product greatly influences this perception [6]. It is a commonly agreed notion in industry that customer dissatisfaction leads to catastrophic failure in terms of loss of reputation and lost opportunity. According to recent research study, conducted by the American Society for Quality (ASQ), American Customer Satisfaction Index (ACSI), the leading measure of customer satisfaction, provides evidence of a dramatic drop in service quality of product warranty. In most industries, quality improvement has failed to keep pace with customer expectations, indicating there is significant room for opportunity to improve quality systems and processes [3]. Jack and Murthy [49] discuss ideal servicing strategies to increase customer satisfaction and obtain optimal parameters for those strategies. This current research focuses on the manufacturer's viewpoint concerning the development of analytical models and of ideal strategies towards warranty servicing.

The warranty coverage period offered varies greatly from one manufacturer to another, and the coverage period and type of coverage during warranty influences whether or not the customer buys the product. With the increase of options for the customer in product purchasing, manufacturers often aggressively market their product with better warranty terms than their competitors. Sometimes simple strategies adopted by manufacturers lead to surprising results. For example, Dell Inc. reduced the warranty coverage of its desktop and laptop product range from three years to one, yet it turned out that this strategy boosted Dell's revenue because customers chose to pay to lengthen and strengthen their coverage period [4].

[3] http://www.asq.org/quality-report
[4] http://www.warrantyweek.com/archive/ww20051025.html

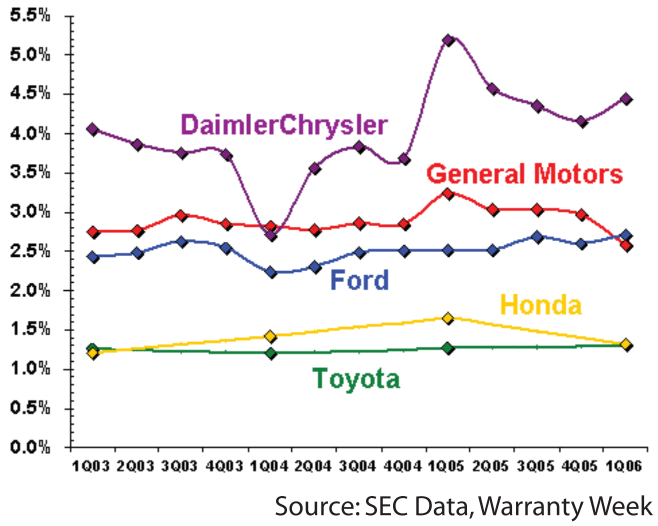


Figure 1.1. Warranty Claim Rates of Major Automobile Makers

In Figure 1, we present the warranty claim rates of leading automobile makers for six quarters in the years 2003-2006; the implications of a higher claim rate does not suggest that the manufacturer is doing very well. We can see that claim rate percentiles vary widely. For the case of Daimler Chrysler the claim rate never was below 3%, while Toyota had registered a very steady claim rate of 1.25%. We can now understand why some manufacturers perform better in automobile sales, retain higher market capitalization, and record higher profits than others. In 1st quarter of 2005, Chrysler recorded the highest claim rate 5.25%, while almost all other companies were still below 3.5%. If we compare the relative performance of the stock price of the manufacturers during 1st quarter of 2005, we are assured Honda and Toyota would have outperformed other manufacturers. Also, during 1st quarter of 2005, if Daimler Chrysler would have set aside their warranty reserves based on previous year's performance, then they would have reported lower profits compared to the other four companies. Based on the above facts it is clear that warranty costs are highly correlated to product quality and warranty claim rates severely affect the revenue, brand equity of the product, and in setting aside the manufacturer's cost towards warranty reserves.

Mc Guire [69] analyzed warranty costs as percentiles of total net sales across

Warranty Servicing Costs in Different Product Sectors

Warranty Servicing costs as a percentage of net sales	All companies	Type of Product Manufactured						
		Business equipment	Electrical equipment	Heating and refrigeration	Machinery products	Scientific equip and controls	Mobile equipment	All other Products
Less than 1%	24	23	20	21	33	23	21	30
1.0-1.9%	37	26	38	41	33	21	54	38
2.0-2.9%	20	26	20	24	22	16	17	16
3.0-4.9%	11	20	20	10	13	16	2	7
5.0-6.9%	4	3	0	3	0	12	2	4
7.0-9.9%	4	3	2	0	2	12	4	1
10% or more	1	0	0	0	0	0	0	3
Total*	100	100	100	100	100	100	100	100

* Does not add to exact value due to rounding

Source-Mc Guire E P. Industrial Product Warranties: Policies and Practices

Table 1.1. Warranty Servicing Costs in Different Product Sectors

Warranty Servicing costs as a percentage of net sales	All Warranties	Length of warranty period (In months)						
		0-3	4-9	10-12	13-18	19-24	25-36	49-60
Less than 1%	24	18	23	25	11	35	29	33
1.0-1.9%	37	45	40	34	67	29	43	33
2.0-2.9%	20	13	14	22	22	24	14	0
3.0-4.9%	11	18	17	10	0	6	14	33
5.0-6.9%	4	3	3	5	0	0	0	0
7.0-9.9%	5	5	3	4	0	6	0	0
10% or more	1	0	0	1	0	0	0	0
Total*	100	100	100	100	100	100	100	100

* Does not add to exact value due to rounding

Source-Mc Guire E P. Industrial Product Warranties: Policies and Practices

Table 1.2. Warranty Servicing Costs Versus Warranty Duration

different products based on a survey from 369 US manufacturers. Findings of the survey are presented in Table 1.1 and Table 1.2. Warranty servicing costs typically vary from 1-10% or more, with the vast majority of firms having a cost of less than 5%. Using the same survey, Mc Guire also analyzed the warranty duration offered for the products and found that the majority of the firms whose servicing costs were around 5% had policies of durations between 3 to 60 months. This tells how duration of warranty coverage impacts the cost of warranty programs. The survey illustrates the variability present in terms of protection period, price, and need to continually change the type of warranty models that are being employed by the manufacturer, where even very small percentile gains would lead to significant savings for the manufacturer. Next, we shall discuss the scope and currency of this research in regards to the context of managing reserve funds, unavoidable product recalls, impacts of outsourced warranty contracts, and the significance of globalization on product warranties.

1.1.1 Warranty Reserve Management

Warranty reserve funds are an estimated sum of money set aside by manufacturer towards the costs of servicing the warranty obligation in the event of product failure during the warranty duration. Reserve funds depend on the expected cost for warranty servicing due to product failure, apart from the number of products that are on the market, the spread of risk in warranty claims due to non-conformance of product quality and how many of these products are still under warranty coverage. Given the past usage history, it is very difficult to make an intelligent guess about what the reserve fund would look like without knowing the expected costs for servicing. Setting aside too many or too few warranty reserve funds would not be favorable to the manufacturer, as it would lead to lowering profits and disreputability in the event of a huge liability lawsuit. This research derives the expected servicing costs for one dimensional warranty policy based on time or usage, and two dimensional policies with respect to both time and usage of the product. Expected cost measure will help the manufacturer in setting aside realistic amounts of funds required for servicing product warranties, planning warranty policies effectively, and forecasting future requirements.

1.1.2 Warranties and Product Recalls

Product recalls are truly a nightmare for a manufacturer, and send mixed messages of criticism and surprise about the manufacturer and place of the product in the market. Product safety recall has several factors associated with it, which include, product price, products made and assembled in developing nations, risk severity etc. Warranty costs associated with the products should account for factors mentioned above, planning and administering suitable policies to mitigate risk factors. Another important aspect of the study in warranty models stems from the increased safety recalls and laws governing the safety of products. Recently, for example the Transportation Recall, Enhancement Accountability, and Documentation (TREAD) Act [5] of 2000 requires manufacturers to record, aggregate, and report a broad collection of data and statistics regarding warranty claims and their component safety systems. This represents an enormous challenge for the manufacturer since each recall is very expensive, leads to tangible loss of reputation, and increase in the perceived risk factor for the manufacturer. The other relevant recent legislation, the Federal Accounting Standards Board Interpretation Number 45 [6] (FASB No. 45) makes warranty costs available to the public domain for publicly traded organizations. Both pieces of legislation give the investment community and stock analysts key information about the millions of dollars spent on warranty servicing. Furthermore, this information is being used as an important economic indicator due to the strong correlation between warranty expenses and earnings per share. In recent times product recalls has beens steadily increasing and Consumer Product Safety Commission (CPSC) acts as government regulator of product safety and issues safety recall notices for tainted and products nonconforming to safety standards. The extensions to this research to incorporate safety recall cost along with product pricing are discussed in conclusion and future reserach section of chapter 6.

1.1.3 Warranties and Outsourcing

The increase in the size of outsourcing business operations by the product manufacturers are quite prevalent in warranty services. The key objective in

[5] *http://www.citizen.org/documents/TREAD%20Act.pdf*
[6] *http://www.fasb.org/news/nr112502.shtml*

outsourcing services is to reduce or control the cost of warranty repair costs since the customer pays no money for products which fail during warranty duration and the manufacturer bears the entire cost for servicing. Also, outsourcing repairs and services gives product manufacturer improved outlook into developing more innovative products, getting better value for the money spent on servicing, forces business to define core competencies, and simultaneously utilizing the services of the vendor. A more detailed explanation about the benefits for outsourcing warranty services are provided in [72]. Manufacturers would naturally expect to use the geographically distributed vendor support, but doing so they are exposed to risk associated with customer satisfaction and delivery times. Ideally the manufacturer would like to achieve a balance between cost savings and acceptable levels of customer service through outsourced service contracts. Expected costs for servicing serve as bench mark standard for manufacturers and management in making crucial decisions regarding whether outsourced services will bring desired profits. The current research serves the purpose of knowing the estimated servicing costs assuming the manufacturer performs in-house warranty servicing.

1.1.4 Globalization and Product Warranties

The impact of globalization has changed the way bussiness are conducted across products and services, and it is totally not uncommon for products sold today to carry global warranties. Most common products which carry such global warranties include, laptop computers, peripherals, electronics, machinery, and consumer goods. The notion of global warranties enhances the brand value of the product, safeguards the broader interests of the purchaser and extends the business to atypical markets. Such practices bring host of associated risks and liabilities for the product manufacturer to address, which include service quality deterioration and the delay experienced by the consumer. The manufacturer often faces considerable challenges in strategizing the warranty terms when the product carries global warranty due to differential costs of spare parts and services among countries. In such cases, the manufacturer must rely on the expectation of warranty costs for the products sold, enhancing the scope for this research. We discuss several improvements to this study in chapter 6 to accomdate several cost variables associated with product warranties of one and two dimensions.

This research will help warranty managers with the expected costs associated with warranty servicing, which will aid in making crucial decisions when product recall is imminent. Thus, the managers get a better idea of their reserve funds when a particular strategy is adopted. Also, this research is tailored to address decisions regarding repair-replacement during warranty period and their associated costs, which combinations of repair-replacement are ideal to minimize the expected costs, and the optimal length of the warranty period with the manufacturer. The conclusions of this research will balance the theoretical rigor and practical application, and are applicable to different industries with suitable modifications. They will aid the manufacturer in increasing his revenue, marketability of products, improved customer satisfaction, and enhanced brand equity.

In the next section we discuss the different classifications of warranty policies and how they are categorized based on types of repair actions initiated upon failure, number of variables in the model, costs associated with warranty servicing, and length of warranty coverage.

1.2 Warranty Policies and Its Classification

We broadly classify warranty policies based on four important decision variables which affect the terms and cost of warranty servicing, namely (i) type of repair action performed upon failure, (ii) number of classifying variables in the model, (iii) cost of servicing, and (iv) length of warranty policy coverage. Figure 1.2 gives the pictorial representation of product warranties based on the four classifications given above. We will discuss the general characteristics of the above factors in this section, and specific details regarding how they are mathematically modeled and analyzed will be seen in chapter 3.

1.2.1 Type of repair action

When a purchased product fails to perform as intended, the type of repair-replacement action performed is one of the important factors in warranty management; a complete replacement means substituting the failed product with a

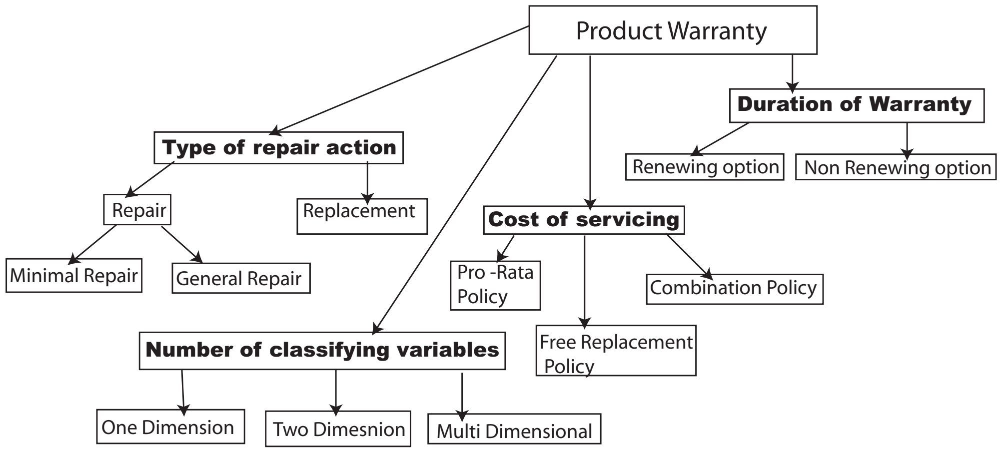


Figure 1.2. Product Warranty Classification

new or refurbished product. The replacement option might be mathematically convenient to model but might not be economical for the manufacturer, especially when only certain components need replacement instead of the whole product. For when the failed product can be repaired, we consider two types of possible repair action: minimal repair and general repair. Minimal repair restores the failed system to its functioning state just prior to failure. Minimal repair will be the most cost effective for the manufacturer but might not entirely favor customer satisfaction. General repair action is an intermediate between the extremes of minimal repair and replacement. We will describe general repair in more detail in section 3.1.3 and its application in section 4.2.3, of the one dimensional model.

1.2.2 Number of classifying variables in the model

Product warranty is modeled using the number of classifying variables associated with the model. Typical classifier variables include age of the product, usage of the product, number of parts produced from the purchased machinery, etc. Based on the number of variables considered in the model, it is classified as one-dimensional (1-D), two-dimensional (2-D), or multi-dimensional. In one-dimensional policy, the warranty coverage usually starts at the time of the purchase of product and ends after a pre-specified duration of time after the initial purchase. One dimensional policies commonly occur with consumer electronics and durables. In 2-D policies, the duration of the warranty policy is determined by two variables. The most commonly used variables are time and usage of the product. This is very reasonable since product failures are commonly attributed to usage of the product. Warranty expiration in 2-D is considered only when either or both of the variables used in the model are exceeded. Typically, 2-D policies can be seen in automobile warranties, for example, the 3 year and 50000 mile warranty. Warranty policy expires when either the vehicle is more than 3 years old, the mileage is more than 50000, or both. Multi-dimensional warranty policies are fairly recent and consider more than two variables in the model; certain complex equipments, construction, and home warranties are examples of multi-dimensional warranties. Such policies are conveniently modeled using continuous stochastic process. Reliability-Improvement Warranty (RIW) are a special case of multi-dimensional warranty, in which the manufacturer agrees not only to provide repair-replacement for failed products during warranty but also guarantees on Mean Time Between Failures (MTBF). Our research considers only 1-D and 2-D warranty models, but we are confident that suitable extensions to this research can easily span the scope of multi-dimensional warranty models.

1.2.3 Cost of Warranty servicing

Based on cost associated with servicing a warranty, we can classify warranties in three broad groups: Free Replacement policy (FRW), Pro-Rata policy (PRW) and Combination policy . Under FRW policy the manufacturer agrees to repair or replace the failed product under warranty at no cost to the customer who purchased the product. As a part of the quality commitment, usually there is no limit on the number of repair-replacements during the FRW unless pre-specified by the manufacturer. FRW policies are more commonly applied to electronics and consumer durables, namely computers and television sets etc. In PRW policy, the customer pays a portion of the cost towards repair or replacement with the manufacturer. In other words, the manufacturer offers a rebate for repair-replacement of the failed product under warranty. The cost of pro-ration is usually dependent on the functional variables of the model, such as age, time, or both, and the rebate function is usually a decreasing function of the functional variables. For example, the cost to replace a failed product after 5 years is lower than replacing the product upon failure after 10 years. Rebate functions can be linear, non-linear, discrete and continuous. We shall discuss them more elaborately in Section 3.2.3. PRW policies are usually offered for non-repairable products like automobile tires and batteries, where failures occur due to accumulated damage. Combination warranty includes policies where two or more of the FRW and PRW type policies are employed. Combination policies are employed in models when warranty terms change at one or more points during the warranty coverage period. Combination policies are created to balance the costs and risks of the policy between customer and manufacturer, as FRW policy tends to favor the buyer and PRW policy favors the manufacturer. For example, a combination type warranty policy coverage begins with a FRW period, which enhances the marketability of the product and limits the liability of product failure for the manufacturer until a certain period of time, and is subsequently followed by a PRW period. In our research we study combination policies for 1-D model.

1.2.4 Duration of Warranty

Warranty policies come with an important classification regarding their duration of coverage. Most importantly, the question which arises in the event of product failure during the warranty period is what happens to the terms of the warranty, and does repair or replacement during warranty change the original warranty. A renewing warranty policy typically restarts the original warranty which came with the initial purchase of the product. If the original product is covered for "k" units of time at the time of purchase and any failure before time "k" say at "m" ($m < k$), then product replacement during "m" extends the warranty to "m+k" time units. A non-renewing policy is the exact opposite of the policy with a renewing option. The original warranty policy remains unchanged in the event of repair-replacement of product during warranty coverage. We use only the non-renewing option throughout the models we discuss in this research as this is a more realistic option and is the most commonly used option in warranty modeling.

In the next section we describe the research objectives and problem statement followed by chapter summary.

1.3 Research Objectives and Problem Statement

In this research, we analytically model and analyze one and two dimensional warranty models with a selection of repair options available to the manufacturer. We derive the expected costs for warranty servicing to the manufacturer and numerically illustrate the validity of results. In addition, we develop optimal policies for the manufacturer to minimize the cost and develop effective strategies for warranty servicing. Manufacturers perspective on warranties usually deal with how does warranty costs change with parameters of the model, how do different policies compare with each other, what would be the optimal servicing strategy pertaining to decisions of repair-replacement, what kind of policies are dictating the market etc. This research attempts to show the impact of warranty service policies affecting the preferences of manufacturers and justify the policies developed in the warranty models.

This research will address the following aspects:

- We develop a one dimensional warranty model with initial base warranty period (BWP) during which the manufacturer can employ three options for repair-replacement namely, minimal repair, general repair, and replacement upon product failure at no cost to customer. Upon completion of the base warranty period, a pro-rata period (PRP) begins, the customer may purchase a new product at pro-rated cost during PRP period. We derive the expressions for expected number of failures during warranty, expected time between successive purchases, manufacturers expected profit and the sale-volume. We shall establish the link between sales volume, cost of warranty servicing, and number of repairs during warranty period in the model developed. Stationary points and second-order conditions for the manufacturers profit function, optimal purchase price and optimal warranty period duration for each repair-replacement option associated with the model are obtained. We numerically illustrate the optimal product price, pro-rata period, and repair option to maximize the manufacturer's profit, when the lifetime distribution of the product is: Increasing failure rate (IFR) type, Decreasing failure rate/ Increasing failure rate (DFR/IFR) type.
- We develop a two dimensional warranty model based on time and usage of the product with replacement-repair option for product failures, and the cost for

repair-replacement is assumed non-constant and is pro-rated according to the failure time and usage of the product in different regions of the two dimensional axis. We derive the expected cost for the manufacturer under two different cases, namely restricted and unrestricted based on usage rates in different regions. We compare the effects of two different repair strategies based on where the replacement option can be ideally exercised by the manufacturer, and we compare the merits of both the repair strategies. We numerically illustrate the effectiveness of the policies developed with product usage rate following uniform distribution and bi-variate failure time data of traction motor failure.

The analytical models developed in this research can be applied to products with a variety of lifetime distributions. The solution to these models have been numerically illustrated to demonstrate their validity . Expected costs serve as an ideal bench mark in industry for a variety of purposes. Most estimates derived often lack clear interpretations, and this research attempts to synthesize the gap between theory and practice. The vast majority of literature on warranty modeling covers constant repair costs, and deals with specific cases that limit the distributions of the usage, failure rate, etc. instead of modeling a more general approach. Also, such modeling analysis aids in understanding the logic behind the models without the involvement of extensive simulations. Simulated results and field data could always be used to validate the models developed and to gauge the associated parameters to a more precise scale, but often such data is not available due to the confidentiality nature of the data.

1.4 Research Outputs and Outcomes

This research can be useful for a wide variety of purposes particularly, the derived expected measures can be used to accurately forecast warranty servicing costs and product pricing. In many cases, warranty costs can be traced down and recovered from the suppliers of low quality and defective materials. Product manufacturers, in order to insulate themselves from unwanted expenses, depend on these estimated costs for servicing. We demonstrate procedures to obtain accurate estimates of servicing costs to make realistic options. A robust statistical framework to model and predict warranty claims may be developed using the output and strategies developed here in. Manufacturers expected profit and cost distributions are derived and optimal parameters obtained. This research can be used by decision makers, marketing professionals, reliability engineers and managers in developing servicing strategies and comparing a wide variety of choices currently prevailing in market.

1.5 Chapter Summary

Warranty modeling and analysis for products and services is one of the popular and emerging areas in today's competitive markets. We develop one-dimensional and two-dimensional warranty models with different options for repair-replacement upon product failure and develop strategies to minimize the expected cost of warranty servicing. The results of this research will aid manufacturers in decisions pertaining to the warranty terms, reserve management, and explore the benefits of how key variables in the model affect the overall performance of the warranty management solution offered. We numerically illustrate the effects of such policies developed in the study.

In chapter 2 we review the relevant literature on existing methods for modeling and analysis of one-dimensional and two-dimensional warranty policies and discuss the merits and demerits of each approach. In chapter 3 we discuss methodology for addressing our research objectives. Chapter 4 deals with modeling, development, and analysis of one-dimensional warranty model. Two-dimensional warranty model is modeled and analyzed in chapter 5. In chapter 6 we present conclusions and discuss areas for further research.

CHAPTER 2
REVIEW OF LITERATURE

In this chapter we shall review the vast literature available on warranty modeling, the vastness stems due to the interdisciplinary nature and differing perspectives of the research topic. Warranty models and policies have been analyzed by economists, statisticians, management consultants, engineers, mathematical modelers, marketing professionals and people from many other disciplines. Our approach to warranty modeling in this research pertains to development of analytical models and strategies that minimize the expected cost of warranty servicing and we shall confine our attention only to those aspects. In Section 2.1, we focus our attention on analysis pertaining to one-dimensional policies and two-dimensional policies in section 2.2. In section 2.3 we shall briefly review some important literature pertaining to warranty analysis with alternative perspectives, some recent survey articles, and books on warranty costing and modeling methods, and in section 2.4 we summarize the chapter.

2.1 One-Dimensional Warranty Policies

One-dimensional policies are characterized by a single variable such as age or usage of the product since purchase. While 1-D policies have been researched over many years, majority of research in 1-D models have been focused on obtaining expected measures. The major drawback in 1-D warranty modeling is that analytical models often involve cumbersome expressions in terms of convolutions of integrals, transforms, and limited to very few distribution functions. Products which usually characterize the terms warranty in terms of one variable are high mix and volume manufactured goods. Typical policies involving 90 day risk free and covered under one-dimensional policy. The advantage in using single varaible makes the models easier to analyze and permits inclusion of factors realted to risk, customization etc.

Blischke and Scheuer [13] analyzed FRW and PRW policies from the view point of both customer and manufacturer. The manufacturer offers the choice to buy the product with warranty or without warranty to the customer, and they calculated the indifference price for which the customer would buy the product with and without warranty.

Anderson [2] developed a framework for a non-specific profit maximization model to optimally specify product price and warranty period. Glickman and Berger [38] considerably extend [2] by considering a displaced log-linear demand function of product price and warranty period duration, subsequently embedding the demand function in manufacturers profit function to maximize and yield optimal product price and warranty period.

Balachandran et al. [7] developed a Markov model for estimating the cost of repair-replacement of the product which consists of three independent components whose failure times are governed by exponential distribution. They obtain expected warranty costs for the policy where, the first and second failure of the product is rectified by repair and the third failure is corrected by component replacement.

Blischke and Scheuer [14] applied renewal theoretic arguments to obtain the expected number of replacements during the product life cycle. They studied several failure time distributions, and due to the inherent difficulties in analytical modeling, they developed a simulation program to evaluate the effect of different failure time distributions.

Mamer [62] derived the expected warranty cost for products sold with both FRW and PRW warranty, and product replacements for all failures until the product life cycle ends. He derived the expected costs under three cases, namely PRW, FRW, and no warranty, and compared how each case fared with one another.

Many authors develop cost limit policies for replacements and minimal repairs during warranty period, Park [91] obtained cost limits for replacements done under minimal repair option. Kapur et al. [54] optimize the number of repairs, cost limit on the number of minimal repairs done during warranty duration and developed suitable cost limit policies.

Thomas [107] developed a combination policy for non-repairable products in one-dimension, in which he split the warranty coverage period $(0, W_2]$ into two distinct periods $[0, W_1] \cup [W_1, W_2]$. Product failures in $[0, W_1]$ are replaced free of cost and product failures in $[W_1, W_2]$ are replaced at prorated cost. He obtained warranty costs for different failure time distributions and developed a procedure to find the optimal warranty period to minimize the expected warranty costs.

Nguyen and Murthy [84] obtained expressions for expected total warranty costs and confidence interval for a fixed lot size model. They also derive the expected warranty cost in any time interval during the product life cycle under continuous sales and expected number of product returns for repair under FRW policy. The failure time distribution of the product was general and cost of repair was dependent on the number of repairs performed earlier. In [85], they analyze two general warranty policies involving an initial FRW period followed by pro-rata period. They derived long run average cost for consumer, manufacturer's profit, and bounds for expected total costs when the failure time distribution is new better than used (NBU). In [86], they studied a new formulation of warranty policy in which manufacturer agrees to repair-replace all product failures at no cost to the customer. The customer returns the failed product during warranty period to the manufacturer and gets a new product or refurbished product. The manufacturer repairs failed products, if it were not repaired earlier, and if its age was less than α with $0 < \alpha \leq 1$, such repaired products were attributed to belong to refurbished product lot. The manufacturer has three choices when a product fails and warranty claim is exercised, namely to replace with a new product, to replace with a refurbished product, and to perform corrective repair action to contribute to the refurbished product. They obtain a optimal policy towards minimizing the expected cost of warranty servicing.

Frees [35] studied approximation techniques to estimate renewal density function for obtaining expected warranty costs. In [36], Frees and Nam investigated combination warranty in which manufacturer replaced all product failures in (0,W) free of cost to the customer, and all failures in the region $(W, W+T)$ the customer buys new product paying pro-rated cost. They derived the expected costs for warranty servicing using straight line approximation technique during the product lifetime.

Cost moments of warranty costs were studied in Balcer and Sahin [8] They consider PRW and FRW policies, assuming successive failures form a renewal process, and obtain moments of the total replacement cost during the product life cycle. They also analyzed the case with continuous failure time distribution for

PRW policy.

Mamer[63] studied random damage model in which the product fails due to accumulated wear or usage. He assumed the damage process was Poisson distributed, and obtained the discounted profits to the manufacturer and customer under FRW and PRW policies with assigned probabilistic measures for customers switching between manufacturers due to bad experiences encountered earlier, i.e. a customer will not replace the product upon failure from the same manufacturer after the expiry of warranty.

The age dependent failure policy is typically used for products with increasing failure rate (IFR) and the product being replaced upon failure or after a certain interval of time. Dimitrov et al.[32] model age-dependent failure/repair model incorporating the notion of calender age and degree of repair to define virtual age of the product. They discuss virtual failure rate and virtual hazard rate function related to the lifetime of the product and derive expected warranty costs under FRW and linear PRW policies.

Many authors studied preventive maintenance of products under warranty. Chun [29] determines optimal number of periodic preventive maintenance operations during the warranty period. They consider minimal repair option and consider free or modified warranty policy. Chun and Lee [30] determine optimal replacement time for system with imperfect preventive maintenance under a combined warranty policy comprised of free and pro-rata warranty. Yeh and Lo [115] investigate preventive maintenance warranty policy for repairable products sold under warranty. The duration of warranty is pre-specified, number of preventive maintenance operations, degree of maintenance and maintenance schedule are determined. They obtain closed form expression of optimal policy when failure distribution follows Weibull distribution.

Sahin and Polatoglu [101] study two types of replacement policies following the expiration of warranty for products with Increasing Failure Rate (IFR) distribution. A type-1 policy, involves minimal repair for a fixed duration of time followed by a replacement, and in a type-2 policy, the user replaces the product after the first failure following the minimal repair period. They discuss renewing and non-renewing options of the above policy and obtain long run mean cost to the customer. Sahin [102] investigates the impact of quality conformance on manufacturers and users replacement costs for products under free-replacement and pro-rata warranty. He concluded that consumer's cost is severely influenced by quality improvement and value of quality inspection by the manufacturer.

In warranty modeling, a major assumption pertains to product usage being continuous, which is true for machinery in continuous operation in industries. Generally, the failure rate for consumer products is dependent on usage. The more frequently the product is used the more probability for the failure of the product. Murthy [75] studied two models based on failure time for repairable and non-repairable product. In model 1 he assumed failure to be dependent on the age and usage of the product without considering the effect of whether the product was in use at the time of failure. In model 2, a Markov process was used to model failure time distribution with constant failure rate like exponential distribution when the product was idle.

Warranty models for reserve management in 1-D was studied by Amato and Anderson. In [1], they developed a general model for estimation of warranty reserves for PRW policy. The manufacturer can opt to invest the allocated warranty reserve to generate additional revenue. When the general price increases, the rebate amount was discounted. The manufacturer reflected the price variation on the product's sold to increase his market capitalization. They obtain expressions to distribute future warranty costs for specific accounting periods of products sold with pro-rata warranties. Also, they broadly deal with additional costing elements by categorizing the warranty claims pertaining to their validity and suggested approaches towards distributing warranty administration costs with the unit cost of the products.

The failure time distribution of the product is estimated from field data and accelerated burn-in procedures. In warranty modeling, the product manufacturer is aware of failures during warranty period only when claims are exercised, and product failures outside the warranty period are usually unknown entities, making it more difficult to obtain statistical interpretation due to abrupt censoring of data. When the Phase-type (Ph-type) distribution is used for modeling failure times, the computation of expectation and analysis is much simplified to elementary matrix operations and matrix-exponentiation. Ph-type distribution is modeled as continuous time Markov chain with $m + 1$ states of which $1, 2, 3..., m$ are transient states and state $(m + 1)$ is absorbing. Ph-type distributions offer a good approximation for many continuous distribution, thus making the model more flexible. Rao [95] developed algorithms for the cash flow from free replacement warranties for products with Ph-type lifetime distribution. He presented extensive numerical study for the model and demonstrated the sensitiveness of the model parameters. Kao and Smith [53] employed phase-type distributions to approximate the failure time distribution, and they obtain the expected cost and unit revenue for warranty servicing. They examined the behavior of renewal function when variability of product lifetimes is large compared to the life cycle of the product and compared their approximations with straight line approximation obtained in [36].

Chukova and Hayakawa [26] modeled warranty claims and evaluated warranty costs allowing non-zero repair time. They use alternating renewal process in finite horizon and derive warranty costs for finite life cycle of the product and attribute a cost for the duration of repair time.

Mi [71] considers random costs for burn-in and renewable warranty of products and compares mean warranty costs for pure replacement-free warranty, mixing renewable warranty, and pure pro-rata warranty policy. Shey-Huei et.al. [104] investigate optimal burn-in time towards achieving tradeoff between reduction of warranty cost and increase in manufacturing cost and obtain expected total cost per unit sold of a general repairable product sold under warranty.

Matis et.al [67] have obtained optimal price and pro-rata period to maximize manufacturers profit for a 1D combined warranty model with different repair options. Hong-Zhong Huang et.al. [40] jointly consider reliability, price and warranty period to obtain total integrated profit for general repairable product sold with free replacement warranty. Osteras et.al. [90] discuss product performance and specifications in new product development for managerial decision making in deciding essential features for warranty in one-dimension model. In the next section we shall study the literature concerning two-dimensional policies. Though 2-D policies have not received much attention from researchers as 1-D policies, recent research in this area has began due to its applicability across various products and industries.

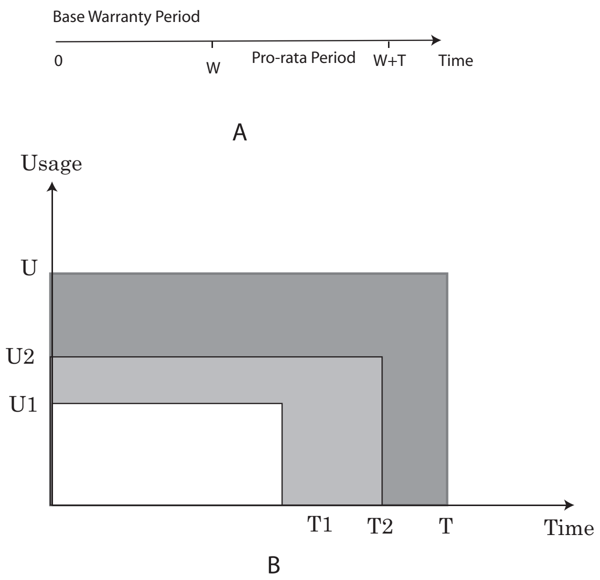


Figure 2.1. A: One - Dimensional Combination Policy and B:Two-Dimensional Policy

2.2 Two-Dimensional Warranty Policies

A two-dimensional warranty policy is characterized by a two-dimensional axis in which usually, the x-axis representing time and y-axis representing usage. Typical examples of two-dimensional policy appear in automobile warranty which is defined with two-attributes say, 3 years and 50000 miles. Another example is in industrial machinery in which the two attributes may be the number of products produced and the age of the machine. 2-D warranty is valid till either of the attributes or both are exceeded. Product failures in different regions of the 2 dimensional region will have different cost implications to the manufacturer, and this presents the need to develop ideal repair-replacement strategies in the case of 2-D warranty models. In other words, to repair product failures during early life is much cheaper to fix than when failures happen during end of product lifetime. Murthy et.al. [83] discusses various modeling process and parameter estimation for two-dimensional failure modeling. Literature pertaining to two-dimensional warranty policies can be classified under two major approaches based on the nature of the underlying point process in the model, namely one-dimensional point process (1-D point process) and two-dimensional point process (2-D point process). We shall study the literature, advantages and disadvantages of each approach in subsequent sections.

2.2.1 One-Dimensional Point Process Approach

One-dimensional point process approach assumes a relationship between the two-variables in the model, namely age and usage. Moskowitz and Chun [74] assume a linear relationship between age and usage of the product and model FRW warranties using conditional poisson process. They develop a Poisson regression model for obtaining the expected warranty costs. Chun and Tang [28] study two-attribute policy using similar approach as [74] and develop decision models to obtain expected total warranty cost and determine effective warranty period.

Jack and Murthy [48] proposed a new repair-replacement strategy for products sold with non-renewing free replacement warranty policy by splitting the rectangular warranty period into three parts, namely region 1,2, and 3. The repair strategy consisted of minimal repair in region-1 and 3 and replacement in intermediate region-2. They developed a control limit sub-optimal strategy with minimal expected cost for servicing. Iskandar and Murthy [45] consider a repairable item under FRW policy and derived expected cost of servicing the warranty, and they compare two strategies in a rectangular region. Iskandar et al. [44] study 2-D combination warranty policy which combines the features of FRW and PRW polcies in 2-D. They discuss renewing and non-renewing options and perform cost analysis of expected servicing costs among combinations of policies developed. Baik et al. [5] extend the concept of minimal repair form 1-D policy to 2-D policy. They compare the strategy of minimal repair to the strategy of replacement upon failure. Iskandar et al. [46] develop a new strategy to replace the failed product for the first time in a specified region of the rectangular warranty and minimally repair all other failures during the warranty period. They derive the expected costs associated with warranty servicing assuming fixed cost for repair under restricted strategy based on product usage. We shall study more about restricted and unrestricted strategies in section 3.2.1. Chukova and Johnston [25] extended [46] and studied unrestricted strategy on usage and compare their results with [46]. Manna et. al. [64] determined optimal warranty region in 2-D policies from customer's perspective where customers utility is measured by length of warranty coverage time.

Gertsbakh and Krodonsky [39] discuss approaches for designing individual warranties for nontypical customers having very low or very high usage rate, and describe a simple way to calculate the warranty limits by minimizing the coefficient of variation. Chen and Popova [22] discuss maintenance policies for products sold with two-dimensional warranties. They derive an iterative procedure to estimate the product failure rate using a likelihood function based on past observations and an optimization algorithm based on Monte Carlo simulation to obtain control limits on warranty expiration. Frickenstein and Whitaker [37] estimate optimal age replacement policy using renewal theoretic arguments when product lifetimes are measured in two time scales and discuss the consistency of the estimators using simulation. Lawless et.al. [60] propose a family of models relating to failure and mileage accumulation addressing the problem of modeling equipment failures in two-dimensions.

In summary, one-dimensional point process approach requires specification of distribution for average usage rate "R", and such a distribution should mimic the variability in usage rate of the customer population and assume a linear relationship between the variables in the model in practice most warranty claims do not contain the necessary information about age and usage of the failed product, and hence only estimates of "r" can be obtained.

2.2.2 Two-Dimensional Point Process Approach

Two-dimensional point process approach involves the use of bi-variate probability distribution function to characterize the failure time distribution of the two variables involved in the model. This method has certain advantages over the 1-D point process approach. In many models using a 1-D point process approach, we may not be able to obtain closed form analytical expressions for associated summary measures, such as expected costs. Also, 2-D point process approach does not deal with exploiting the renewal theoretic properties of the model, and solving the kernel of the renewal function presents a significant challenge in the 1-D point process approach. Yang and Nalchas [113] discuss the structure for bivariate availability, reliability functions, and its associated modeling methods. Yang et al.

[114] discuss issues connected to bivariate reliability models and its applications to maintenance planning, and such models models can be used to warranty modeling with suitable assumptions. They also discuss the structure of the bivariate probability distributions, hazard function, moment generating function and renewal function and numerically illustrate using bi-variate exponential distribution.

Let the failure process be characterized by bi-variate distributions function of the form $F(t, x)$, and let (T_1, X_1) be the time to first failure and product usage at first failure. Let (T_i, X_i), for $i \geq 2$, denote time between ith and $(i+1)st$ failure and the item usage between ith and $(i+1)st$ failures. The bi-variate distribution function for (T_i, X_i) is $F_i(t, x)$, where $i \geq 1$. The sequence $(T_i, X_i), i \geq 1$ are independent and identically distributed random variables. There are some limitations to the choice of these bivariate distribution function namely, the distribution form of $F(t, x)$ should belong to specific classes of distributions, like Beta-Stacy distribution, multivariate Pareto, multivariate Pareto of second kind, bivariate exponential etc. The conditional expectation on usage should be an increasing function of time i.e., greater the time between failures implies greater the usage, and finally the form of the distribution function depends on the type of corrective action taken upon failure i.e., replacement or repair. 2-D point process models are quite popular since a variety of non-parametric models can be analyzed using this approach.

Murthy et. al. [82] studied 2-D FRW policies for non-repairable products. They considered four different policies spanning different warranty regions and derived expressions for the expected unit warranty cost and life cycle cost of products for all the four policies.

Eliashberg et al. [34] considered the problem of calculating the optimal warranty reserve needed for the manufacturer in 2-D warranty model involving FRW policy. They assume the product usage rate follows a logistic function and develop their model using a decision theoretic framework. They estimate model parameters using bayesian inference to obtain a solution yielding optimal value of warranty reserve.

Kim and Rao [57] consider non-repairable product whose failure time are described by bi-variate exponential distribution and derive analytical expressions for the two-dimensional renewal function and the expected cost of warranty servicing. They numerically illustrate the effect of correlation between the variables in the model with the cost.

Iskandar [43] describes a computational procedure and program to solve two-dimensional renewal integral equation and present numerical results for bivariate exponential and Beta Stacy distributions. Recently, Jung and Bai [50] proposed a method of estimating lifetime distribution for products under two-dimensional warranty in which the variables associated are statistically correlated, and they outline procedures for obtaining likelihood estimators and marginal distributions. Manna et al. [65] address the problem of constructing a probability model for product failure instances indexed by two scales.

In the next section, we shall look into literature pertaining to the analysis of warranty policies with alternative perspectives and application standpoint, which include risk sharing, warranty reserve management, and claims modeling, etc. Additionally, we present some excellent survey articles and informative books and case studies in the area of warranty modeling and analysis.

2.3 Warranty Policies with Alternate Perspectives

Warranty policies have been studied with alternate perspectives to obtain the distributions of cost, revenue and profit for both the manufacturer and customer in [100] and [93]. Singpurwalla et al. [103] consider the statistical and game theoretical aspects of warranty analysis by classifying the 2-D regions defined by policies as manufacturer friendly, user friendly, or neutral.

Young and Tang [27] consider producers and customers risk preferences to obtain optimal warranty price assuming constant failure rate and repair costs during warranty period. Ritchken and Tapiero [97] consider optimality of warranty design with the risk aversion aspect of the buyer and seller. Pham and Zhang develop a software reliability cost model to minimize expected software cost with the time to remove error detected in the system. They also derive the risk cost of software failure and warranty cost.

Wortman and Elkins [111] consider time dependent measures of warranty, such as total remaining warranty coverage time and total number of items under warranty at time t, and derive analytical expressions which will aid warranty managers to manage reserves and foster better understanding of the type of policy to be developed and initiated with the products. Ja et al.[47] discuss a non-renewable minimal repair warranty policy with time dependent cost and estimate the warranty cost for setting aside reserve funds.

Lawless [59] explore several issues pertaining to warranty databases and its associated statistical properties they also consider examples involving claim data for automobiles and refrigerators. In [60], Kalbfleisch et al. discuss how warranty claims are used to estimate the expected number of warranty claims per unit in service as a function of the time in service and present forecast methods for delay adjustments and lags corresponding to the delays in warranty claims. Karim and Suzuki in [55] present a comprehensive literature review on the analysis of warranty claim data.

Opp et.al. [89] consider minimizing the cost of outsourcing warranty repairs when product failures are dynamically routed to service vendors. Jack and Murthy [49] model customer satisfaction in terms of likelihood of customer not switching to another manufacturer and develop optimal servicing strategies towards warranty servicing.

The applications of renewal theory is profound in warranty modeling context. We refer the the following seminal papers on renewal theory and its suitable applications to warranty analysis. Lomnicki [61] studied a Weibull renewal process. Hunter [41, 42] obtained basic results and bounds on the renewal functions in two dimensions. Blischke [12] discusses the applications of renewal theory in the analysis of FRW policies. Brown et al. [18] developed a Monte Carlo estimator to obtain expected number of renewals in $[0, t]$. They compare several unbiased estimators to the one they obtained, and numerically illustrate the results.

Bai and Pham [4] develop a full-service warranty (FSW) for a repairable multi-component system, in which the failed component or subsystem will be repaired. Additionally, maintenance operations will be performed to prevent further failures, and under FSW, they derive cost models for complex systems with series, parallel, series-parallel and parallel-series configurations from the manufacturer's point of view. Zuo et al. [117] study a servicing policy for multi-state deteriorating and repairable products. They base the decision of repair-replacement based on the deterioration degree of the item and residual warranty period, and obtain optimal values of the same to minimize warranty servicing cost of the manufacturer.

Warranty policies for used products have been studied by Chattopadhyay and Murthy in [21]. They derive expected warranty cost of the product to the manufacturer, which is dependant on age and /or usage of the product, maintenance history, and terms of the warranty. They discuss both FRW and PRW policies for used products. In [20], they discuss three new cost sharing warranty policies for second-hand products. They analyze the warranty costs for those three policies and discuss some extensions to the model they studied.

We refer the readers to excellent books written in the area of warranty modeling by Blischke and Murthy in [9] and [10]. Thomas[109] integrates principles of product quality, reliability and warranty warranty feedback about product performance. Brennan [17] explores several approaches to using warranties to ensure quality and profitability for supplier and customers. The recent survey paper written by Murthy and Djamaludin [76] presents a detailed literature review concerning all major topics related to warranty policies. Blischke and Murthy have studied product warranty management in three major contexts, namely formulation of taxonomy for warranty policies [16], as a integrated framework [78], and review of mathematical models [81]. Thomas and Rao [106] review the literature on warranty models and suggest some future research directions. The link between product warranty and reliability is discussed by Murthy in [79]. Warranty logistics is another important area of current research, and many of the recent notions, along with cost cutting measures, deal with consumer satisfaction and is studied by Murthy et al. in [77]. Issues pertaining to warranty framework encompassing product life cycle was examined by Murthy and Blischke in [80]. Thomas [108] explains the role of warranty in engineering economic decisions and presents comprehensive methods of integrating warranty data in planning and accounting decisions. Interesting case studies, with emphasis on product warranty, are presented in [11]. In particular, they extract information from field data regarding warranty claims to increase reliability by averting the reoccurrence of quality problems in current designs and decrease development time in highly competitive market. Steve et al. [52] outlines methods to reduce warranty cost using time dependent warranty event rates and discusses how a leading electronics manufacturer, HP, approaches warranty problems. They highlight the need for integrated reliability and financial planning system composed of different functional groups due to the very different interpretation of "Warranty" by each group.

2.4 Chapter Summary

In this chapter, we summarized the available literature pertaining to warranty models. Although, one-dimensional warranty models has been widely researched, there are novel approaches to modeling and analysis as this research aims to accomplish and presents ideal strategy for repair-replacement action in the event of product failure in one-dimension. Two-dimensional policies are recent addition to the warranty community, and literature pertaining to them are limited. This clearly presents availability of vast opportunity for research in this area. We shall study two-dimensional model in this research, compare the cost effectiveness of two strategies based on usage rate of products for the manufacturer and compare it's effectiveness in terms of expected cost for warranty servicing. In section 2.3, we presented the literature on warranty policies with analysis based on alternative perspectives, important books and survey articles in the area of warranty modeling. We confined our attention to warranty literature pertaining to mathematical models and their analysis. The other perspectives regarding product warranties, say marketing, sales, legislative and several other aspects, have not been included in this chapter due to their vastness. In next chapter, we shall study the methodologies we employ towards accomplishing the research objectives and understand the mathematical abstractness and rigor in modeling and analysis of one-dimensional and two-dimensional warranty models.

CHAPTER 3
RESEARCH METHODOLOGY

3.1 Introduction

In this chapter, we discuss the mathematical methods and modeling approaches we have employed to accomplish the objectives of this research. We discuss in detail the available repair-replacement options for the manufacturer in order to minimize the expected costs for warranty servicing and effective strategies are discussed. The primary focus of this chapter addresses questions related to how long should the manufacturer the extend FRW option, the optimal duration of the PRW period in 1-D models, when the manufacturer should extend the replacement option in the rectangular region in 2-D models, and how does one servicing strategy differ from other in terms of costs. In section 3.1, we discuss one-dimensional model and elaborate on assumptions associated with the model, section 3.2 discusses the two-dimensional model, analyzed using one-dimensional point process approach and assumptions of the model, in section 3.3 points to limitations of this research and validation of the results obtained, and section 3.4 discusses the summary of the chapter.

3.2 One-Dimensional Model

In the 1-Dimensional model, we will study a non-renewing, combined warranty policy. The warranty coverage period can be split into two important parts namely, the Base-Warranty Period (BWP) and Pro-Rata Warranty (PRW). BWP can be interpreted as the warranty period which comes default with the product purchase, and alternately the BWP can also be interpreted as manufacturers' basic warranty coverage offered at no cost to the customer. In most cases, the BWP comes in the form of risk-free trial or money back period. Usually the cost for a BWP warranty would already be factored in the product price. Product failures during BWP are rare and unfortunate. The optimal duration of BWP is determined the manufacturer using a variety of methods like burn-in and accelerated life testing. Generally a manufacturer will not extend a warranty for the product which could fail repeatedly during the BWP period, and BWP is much smaller than the product lifetime. In other words, if the lifetime of the product is 3 years, it is unreasonable for the manufacturer to extend BWP of 5 years on the product. The ideal choice for the BWP, and repair options available for the manufacturer during BWP, play a vital role in determining the product price and in evolving an effective repair strategy.

The PRW is often considered as an extended option to the BWP and comes under different names like extended warranty, optional coverage warranty etc. The price for PRW warranty is borne by the customer, and in the event of product failure, the manufacturer will fix the product and bring it to working state or completely replace the product with a new or refurbished product. This stresses the importance of obtaining optimal PRW duration and its dependence on product pricing decisions. The cost of PRW will be a portion of the purchase price of the product to get the product fixed or replaced. Another important myth in the context of refurbished products is the common misconception that refurbished products are used products. The bulk of the refurbished products come from products which were found to have quality defects and market recalls, and it is not uncommon for the manufacturer to fix the above products and supply it to the customer for a lesser price, instead of accruing a total loss on such products.

In 1D warranty model, we consider a repairable item purchased at time $t = 0$ whose time to failure is given by the non-negative random variable X with distribution function $F(\cdot)$. We refer the reader to figure-2.1(A) for a one-dimensional combination warranty in which the duration $(0, W]$ refers to BWP period and $(W, W + T]$ is the PRW period. During the initial BWP period on $(0, W]$, the manufacturer can provide either replacement, general repair, or minimal repair at no cost to the customer upon product failure, and based on the type of repair-replacement option exercised the manufacturer incurs a different cost. Upon completion of the BWP, a PRW period during $(W, W + T]$ begins in which the customer agrees to purchase a new item upon failure at a cost that is linearly proportional to the original purchase price, say P. We will discuss in detail about the repair-replacement options and how they are characterized mathematically in the next section.

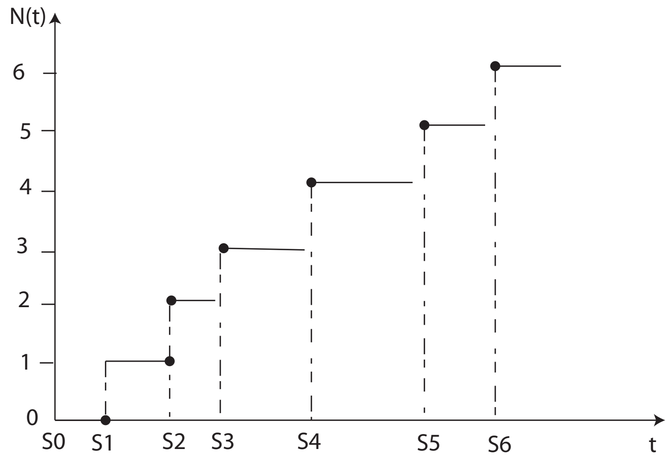


Figure 3.1. Sample path realization of counting process

3.2.1 Replacement Option

Let $N(t)$ denote the number of failures of the product in $(0, t]$. Let Z_i denote the time interval between n^{th} and $(n-1)^{st}$ failure. Let S_n be the time of occurrence of n^{th} event with $S_0 = 0$, $S_n = Z_1 + Z_2 + \cdots + Z_n$, for $n \geq 0$. In the replacement option, the counting process $\{N(t), t \geq 0\}$ forms a renewal process in $(0, W]$, where $N(t) = sup\{S_n \leq t, n \geq 0\}$. During each product failure, a new product is replaced. Though the replacement option is not a economical choice to the manufacturer in the case of non-renewing warranty policy during BWP, it may be in the case for policy with a renewing option. Intuitively, in the case of renewing option, if the product fails before the expiry of BWP then the replaced product must survive another BWP duration for the manufacturer to be devoid of warranty costs. The time between failures are independent and identically distributed with the distribution function for failure being the same as lifetime distribution of the product. The expected time of the between failures will involve n-fold convolution of distribution function, the solution to which can be obtained using Laplace transforms when lifetime distribution is simple. When the lifetime of the product is flexible and robust, then inverting Laplace transforms may be complicated. We will resort to suitable computational procedures approximations as given by Ross [99].

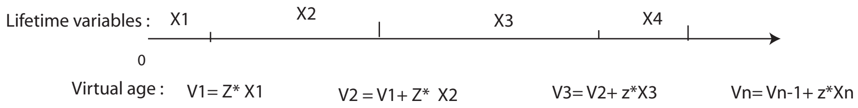


Figure 3.2. Sample path reliazation of the General Repair Option for Model-I

3.2.2 Minimal Repair Option

Under a minimal repair option, the counting process $\{N(t), t \geq 0\}$ on $(0, W]$ becomes a Non-Homogeneous Poisson Process (NHPP) with intensity function given by $\lambda(t)=\frac{f(t)}{\bar{F}(t)}$. NHPP is a non-stationary poisson process with stationary independent increments satisfying the following properties:

(i) $N(0) = 0$
(ii) $\{N(t), t \geq 0\}$ has indepedent increments
(iii) $\Pr\{N(t+h) - N(t) = 1\} = \lambda(t) * h + O(h)$
(iv) $\Pr\{N(s+t) - N(t) = n\} = e^{-[m(s+t)-m(s)]} * \frac{[m(s+t)-m(s)]^n}{n!}$,
$n \geq 0$ with $m(t) = \int\limits_0^t \lambda(x)\, dx$

as shown in Ross [98], where $N(s+t) - N(t)$ is a Poisson random variable with mean $m(s+t) - m(s)$, and $N(t)$ is Poisson with mean $m(t)$.

The NHPP arises in many areas of reliability theory as the process which records the epochs of the number of repairs that a minimally repaired system undergoes (Ascher and Feingold [3]) The rate of occurrence of failures in NHPP is monotone and provides a good justification for use with minimal repair policy. A complex system having many components can fail independently of each other, and replacing a failed component will have a very little or no effect on future system reliability. Also, in many other applications, the NHPP occurs as a process that records the number and the values associated with a sequence of non-negative stochastically independently distributed random lifetime variables.

3.2.3 General Repair Option

The general repair option was originally introduced by Kijima [56] in the study of repairable systems. In the previous sections, we have seen the replacement option which restores the failed product to "as good as new" and the minimal repair option to “as bad as old” type of repair, which restricts the nature of the repair action to two extremes. The general repair can be viewed as an intermediate between the extremes of minimal repair and replacement. The failure process $N(t)$ is specified through the “virtual age”, $V_{n-1} = y$ immediately after the $(n-1)^{th}$ repair. The virtual age V_n of the product immediately following the n^{th} general repair of degree z, $0 \leq z \leq 1$, is given by $V_n = V_{n-1} + zX_n$. Kijima studied two models based on how the repair affects the virtual age process, namely: Model-I given by $V_n = V_{n-1} + zX_n$ where the n^{th} general repair does not remove the damages incurred before the $(n-1)^{th}$ repair but reduces the additional age X_n to zX_n, and Model-II given by $V_n = z(V_{n-1} + X_n)$ where the n^{th} repair affects the accumulated virtual age until n^{th} failure given by $V_{n-1} + X_n$. Let $S_n = \sum_{i=1}^{n} X_i$ for $S_0 = 0$ be the “real age” of the system at the n^{th} failure, which denotes the elapsed time since the product was put in operation, and X_i represent the lifetime between $(i-1)^{th}$ and i^{th} repair. In our 1-D model, we use Model-I for modeling and analysis of general repair action, where $z = 0$ for all $n \geq 1$ corresponds to perfect repair or replacement option and $z = 1$ for all $n \geq 1$ corresponds to minimal repair. We present an example to show the effect of the degree of repair on the real age and virtual age, as given in Table 3.1. Let us assume the first five failure times of a product happen at the following epochs $\{5, 8, 9.5, 11, 15\}$, when $z = 0$ the virtual age v_n of the product will be zero always since $v_0 = 0$ and $v_{n-1} = v_n$. The real age $S_n = 0, n \geq 1$ since at each failure instant the product is replaced. When $z = 1$, the virtual age $v_n = v_{n-1} + X_n$ of the product will be $\{v_0 = 0, v_1 = 5, v_2 = 8, v_3 = 9.5, v_4 = 11, v_5 = 15\}$ and real age of the system $\{s_0 = 0, s_1 = 5, s_2 = 8, s_3 = 9.5, s_4 = 11, s_5 = 15\}$, i.e. the degree of repair applied does not affect the failure rate of the product when minimal repair is performed.

Commonly a general repair type of action can be interpreted as an overhaul type of repair action. Assuming a system has virtual age $V_{n-1} = y$ immediately following the $(n-1)^{th}$ repair, it follows that the n^{th} lifetime variable X_n has the distribution function

$$P(X_n \leq x \mid V_{n-1} = y) = \frac{F(x+y) - F(y)}{\overline{F}(y)},$$

as shown by Kijima in [56]. Figure 3.2 depicts the sample path realization of virtual aging process, where X_i is the time between failures. In model-I, the counting process $\{N(t),\ t \geq 0\}$ becomes a g-renewal process. Since $\{V_n\}_0^{\infty}$ the sequence of

Table 3.1. Impact of degree of repair on the virtual age

		Virtual Age				
Xi	Real Age	z=0.1	z=0.3	z=0.5	z=0.7	z=0.9
	0	0	0	0	0	0
5	5	0.5	1.5	2.5	3.5	4.5
8	8	0.8	2.4	4	5.6	7.2
9.5	9.5	0.95	2.85	4.75	6.65	8.55
11	11	1.1	3.3	5.5	7.7	9.9
15	15	1.5	4.5	7.5	10.5	13.5

virtual age is Markov and preserves stochastic orderings it makes general repair process more interesting. Also, $\{X_n\}_0^\infty$ the sequence of lifetime variables, $\{S_n\}_0^\infty$ sequence of real age can be statistically obtained when the failure distribution is given. In our model, as the virtual age $\{V_i\}$ is increasing with respect to i implying that system failure rate increases to ∞ i.e. the operating times are stochastically decreasing to limiting value of 0.

3.2.4 Expected Costs, Sales Volume and Manufacturers Profit

We have used renewal theory arguments to obtain the expected measures associated with the 1D warranty model. We have obtained the following measures as part of the study: expected number of replacements during the lifetime of the product, expected time between successive purchase, and manufacturer's long run average profit per unit time. To model the expected sales volume we used Glickman and Berger [38] model to obtain the optimal product price and pro-rata period. We discuss the details of sales volume function and manufacturers profit more elaborately in the next chapter. We have used linear pro-rata cost for replacements in PRW period. Model optimization procedure consisted of developing expressions for the stationary points associated with profit function, which in our case is a bi-variate function of the purchase price "P" and pro-rata period duration "T". Also, we have derived the second-order conditions about each stationary point to maximize profit function of the manufacturer and establish the concavity. Finally we have illustrated the applicability of the model using couple of numerical examples where the lifetime distribution of the product follows Gamma Distribution (IFR type) and Extended Weibull model (DFR/IFR type). The detailed derivation procedures and illustrations is given in chapter 4.

3.3 Two-Dimensional Model

We study a two-dimensional warranty policy based on time and usage as classifying variables. Consider a rectangular region characterized by $[0, U) \times [0, T)$, where U represents usage of the product and T represents time period. We develop a combination warranty policy in 2-dimensions by splitting the 2-D region into three discrete areas as given by Figure 2.1(B) The cost for repair is prorated based on time and usage, and product failures in different regions have different cost implications to the manufacturer based on the type of repair-replacement performed. We derive the expected cost for servicing the warranty in the event of product failure for the manufacturer based on two repair strategies, which is discussed in detail in section 3.3.4 for restricted and unrestricted product usage rates. We use a 1-dimensional point process approach to model product failures, which assumes a linear relationship between the age and usage of the product.

3.3.1 Modeling usage rate

Let us assume that customer purchases a repairable product initially at time $t = 0$ with 2D warranty expiring at the end of "T" time units or "U" units of usage or both. The usage and age the product are linearly related as given by Equation 3.1, with $A^*(t)$ representing the age of the product and $U^*(t)$ representing the usage of the product currently in use.

$$A^*(t) = R * (U^*(t)) \tag{3.1}$$

In Equation 3.1, "R" is non-negative random variable, which represents the "usage rate", with density function $r(x)$ and cdf $R(r) = Pr\{R \leq r\}$. There are many choices of distributions for modeling the product usage rate, and since there is a considerable variation in the usage of products among different users, the choice of a suitable distribution should be representative of most users. Since the usage measure is the time varying covariate of the time since the product was put to use, the ideal

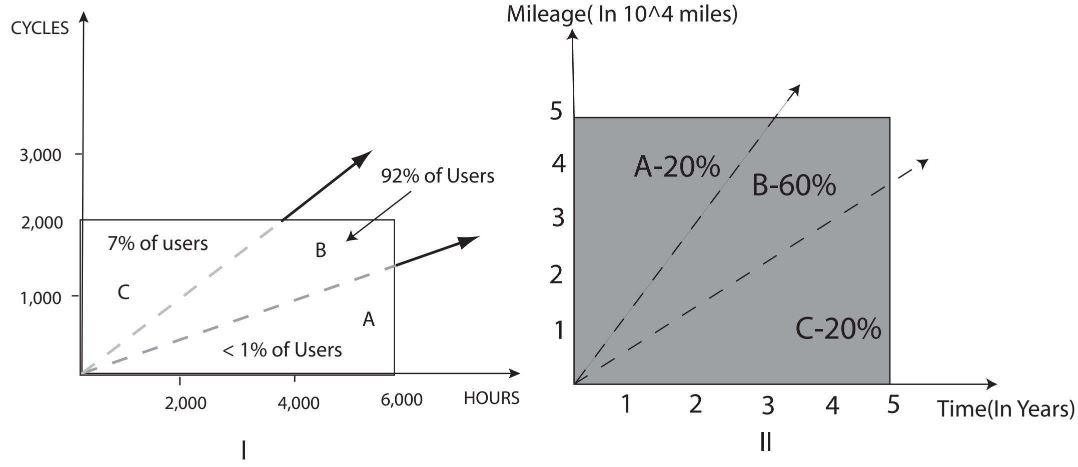


Figure 3.3. I:92 percent of users in 2D rectangular warranty policy offered by manufacturer for engines. II:Three categories of car users. Source: [39]

choices for modeling usage rates is offered by the uniform, and gamma distributions, as given in Figure 3.4. The shaded region represents the limits of product usage, and we note that gamma distribution sweeps the region in x-y plane entirely while the uniform distribution sweeps the region within confined limits r_l and r_u, which looks like a section of pie. We use uniform usage rate whose CDF is given by:

$$R(r) = \frac{1}{r_u - r_l}; \quad r_l \leq r \leq r_u$$

to model the usage rate for the traction engine data given in [34] to obtain the expected costs in our 2D warranty model.

Gertsbakh and Kardonsky [39] discuss the advantages of designing individual warranties based on individual usage rates. Figure 3.2 (I) depicts the usage rate pattern for aircraft engines used by Aeroflot Airlines between $1980 - 1985$ based on the data 543 engines with 2D rectangular warranty coverage having limits of 6000 hours of time and 2000 operational cycles, 92% of engines had usage rate in the area represented by "B".

In Figure 3.3 (II) illustrates the usage rate pattern of automobiles based on warranty claims reported for 8394 cars as given by Lawless et al. [60], the median

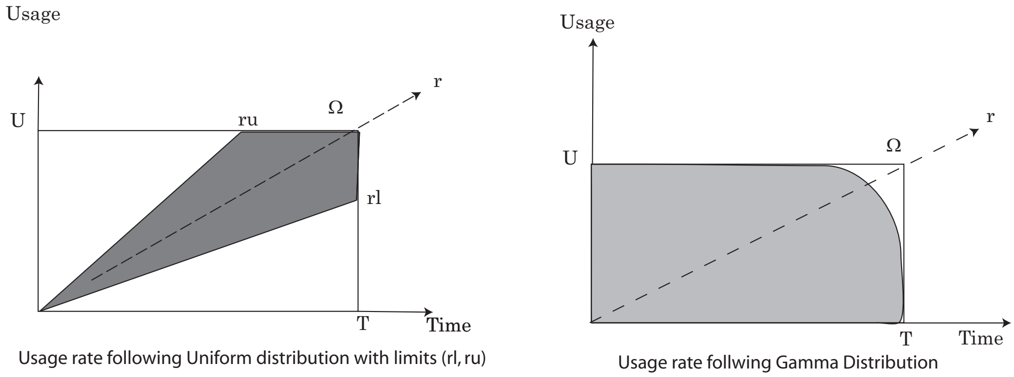


Figure 3.4. Warranty regions covered when usage rate follow Uniform and Gamma distribution

usage rate was around 13000 miles/year, with 60% of the population in the range of $8000 - 17000$ miles/year, 20% having < 8000 miles/year, and remaining 20% having > 17000 miles/year.

A typical example of gamma distributed usage rate would be in modeling the usage of consumer goods, electronics etc. which have a wide span of coverage from low to high usage rates, while machinery that operate under fixed conditions and times of operation may be best modeled using a uniform distribution.

3.3.2 Product failures and failure intensity rate of the product

Product failures depend on several factors, such as product design, quality of the materials used, usage intensity, types of maintenance action performed, operating conditions etc. Failure modeling can be categorized into system level and component level. For example, in automobiles, when critical components fail, say the engine fails to start, such failure can be viewed as a system level failure, and component level failures due to non-critical aspects like the failure of honk etc. Based on modeling principles, there are two major approaches to model failures in 2D models : (i) Black-box approach and the (ii) White-box approach. The black-box approach is a data based empirical method in which the understanding of different failure causes is not very obvious and incorporate random aspects of unforeseeable failures. The white-box approach is used when more detailed component level behavior and failure modes are understood. At the system level, failures of independent components are considered. In our 2D warranty model we employ a two dimensional black box approach, and we shall review the point process, hazard function and failure intensity associated with this approach, and how we used the procedure to the traction motor failure data given in Table 3.2. We refer the readers to [83] for detailed explanation of approaches to two-dimensional failure modeling including issues involving parameter estimation and comparison between one-dimensional and two-dimensional case.

Product failures occur according to Poisson process are modeled using one-dimensional point process. Let $N(t|r)$ represent the number of failures in $[0, t)$ with intensity function $\lambda(t|r)$ conditioned on usage rate r, where, $\lambda(t|r) = \text{Pr.}\ \{\text{currently working item will fail in } (t, t + \Delta t)\}$. We shall discuss the structural form of $\lambda(t|r)$.

Let $S_{1|r}$ be the time to first failure; conditioning on $R = r$ the distribution function of first time to failure conditioned on usage rate r is given by:

$$F_{S_{1|r}}(t|r) = 1 - e^{-\int_0^t \lambda(x|r)dx} \tag{3.2}$$

On unconditioning, the CDF of time to first failure is given by:

$$F(t) = \int_0^\infty \left(1 - e^{-\int_0^t \lambda(x|r)dx}\right) * R(x) \tag{3.3}$$

The structure for $\lambda(t|r)$, the failure intensity function, is obtained by modeling $\lambda(t|r)$ as a function of age, usage and average usage rate, as given by

$$\lambda(t \mid r) = \Psi(r, A^*(t), U^*(t)) \tag{3.4}$$

The function $\Psi(.)$, is an increasing function and $\lambda(t \mid r)$ can be represented using a variety of structural forms, the simplest being the polynomial of order one : $\lambda(t \mid r) = \Theta_0 + \Theta_1 r + \Theta_2 A^*(t) + \Theta_3 U^*(t)$, where non-negative parameters Θ_i

associated with the model with $i = 0, 1, 2, 3$ are to be estimated based on the choice of functional form. The choice of a functional form depends on the availability of data, and Iskandar et al. [46] outline procedure for model selection. Basu and Rigdon [96] give precise details in obtaining the empirical estimates of the parameters in the model. In our 2-D model, we will use the polynomial function of order 2 to model the data of traction motor data given by:

$$\lambda(t \mid r) = \Theta_0 + \Theta_1 r + \Theta_2 U^*(t)^2 + \Theta_3 A^*(t) U^*(t) \tag{3.5}$$

due to its appropriateness with the data. We discuss the fit diagnostics and model summary more detail in chapter-5. We employed class of simple regression model known as “collapsible model” due to its applicability to handle two-dimensional predictions involving time and usage. See section 7.3 in [33] for more details on the use of regression models for reliability given usage history.

3.3.3 Unrestricted and Restricted Product Usage Rates

In the two-dimensional warranty model represented by rectangular region $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$, restrictions limiting the usage and the time duration limits for the expiry of warranty in the subregions defined by Ω_i for $i = 1, 2, 3$ based on usage can be imposed by manufacturers. The average usage rates for each rectangular subregion (Ω_i) is given by r_i for $i = 1, 2, 3$ and is defined as :

$$r1 = \frac{U1}{T1}, r2 = \frac{U2}{T2}, r3 = \frac{U}{T} \tag{3.6}$$

Based on the above definition of average usage rates in the rectangular regions, we define (i) Restricted and (ii) Unrestricted case as follows:

$$Restricted : r1 = \frac{U1}{T1} = \frac{U2}{T2} = r2, r3 = \frac{U}{T} \tag{3.7}$$

$$Unrestricted : r1 = \frac{U1}{T1}, r2 = \frac{U2}{T2}, r3 = \frac{U}{T} \tag{3.8}$$

Iskandar and Murthy [46] derive the expected costs for warranty servicing under the restricted case, and Chukova and Johnston [25] extended their work by deriving

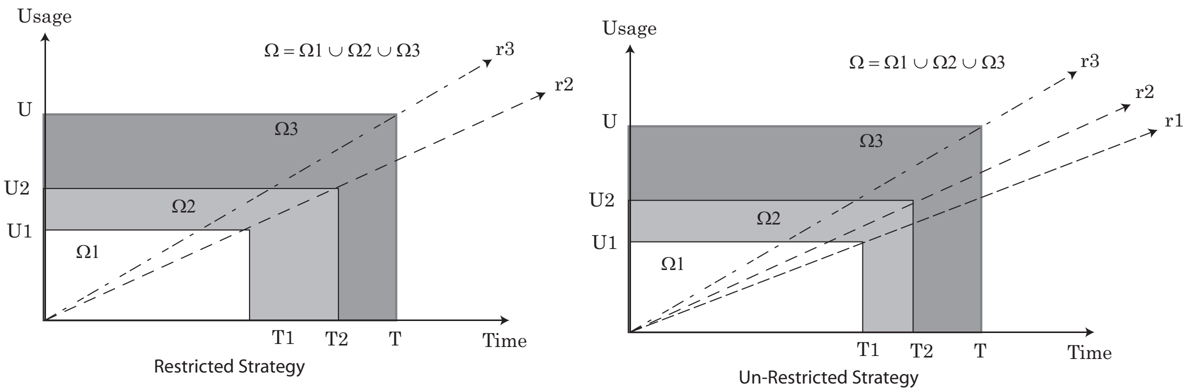


Figure 3.5. Restricted and Unrestricted strategy in 2-Dimensional warranty policy

the expected costs for warranty servicing in the unrestricted case. In our research we derive expected costs for both restricted and unrestricted cases and compare their effects under two repair different strategies which is explained in section 3.3.4. This will enable the manufacturer to suitably design the warranty limits while minimizing the overall costs of servicing.

In the restricted case, we derive the expected costs using the 3 variable set $\pi\{T1, T2, r1\}$ for the two possible realizations with 1)$r_2 < r_3$ and 2)$r_3 < r_2$. In Figure 3.8, we give a sample realization of the different warranty regions in the restricted case. We would require to investigate three possibilities of the usage rate r to reside in each of the two realizations, namely: $r \leq r_2$, $r_2 < r \leq r_3$, and $r_3 < r$ for the realization $r_2 < r_3$ and $r \leq r_3$, $r_3 < r \leq r_1$, $r_2 < r$ for $r_3 < r_2$.

For the unrestricted case, we refer the reader to Figure 3.5 for a sample illustration of the warranty region for the unrestricted case. The restriction on the usage limits in unrestricted strategy leads to the different shaded areas in $\Omega1, \Omega2, \Omega3$ within the rectangular region, as shown in figure 3.9. The parameter set in unrestricted case is given by 4 variables, $\pi\{T1, T2, r1, r2\}$, assuming that U,T are already fixed by the manufacturer and would like to know the implications of the change in expected costs based on increasing or reducing the limits on time and usage. There are six possible realizations as given by figure 3.9 with:

$$\begin{aligned}
&1.\, r_1 \leq r_2 \leq r_3\\
&2.\, r_3 \leq r_2 \leq r_1\\
&3.\, r_2 \leq r_3 \leq r_1\\
&4.\, r_1 \leq r_3 \leq r_2\\
&5.\, r_2 \leq r_1 \leq r_3\\
&6.\, r_3 \leq r_1 \leq r_2
\end{aligned} \tag{3.9}$$

There are four possibilities for average usage rate "r" to reside in each realization, given by:

$$\begin{aligned}
&1.\, r_1 \leq r_2 \leq r_3; (i)r \leq r1, (ii)r_1 \leq r \leq r_2, (iii)r_2 \leq r \leq r_3, (iv)r3 \leq r.\\
&2.\, r_3 \leq r_2 \leq r_1; (i)r \leq r3, (ii)r_3 \leq r \leq r_2, (iii)r_2 \leq r \leq r_1, (iv)r1 \leq r.\\
&3.\, r_2 \leq r_3 \leq r_1; (i)r \leq r2, (ii)r_2 \leq r \leq r_3, (iii)r_3 \leq r \leq r_1, (iv)r1 \leq r.\\
&4.\, r_1 \leq r_3 \leq r_2; (i)r \leq r1, (ii)r_1 \leq r \leq r_3, (iii)r_3 \leq r \leq r_2, (iv)r2 \leq r.\\
&5.\, r_2 \leq r_1 \leq r_3; (i)r \leq r2, (ii)r_2 \leq r \leq r_1, (iii)r_1 \leq r \leq r_3, (iv)r3 \leq r.\\
&6.\, r_3 \leq r_1 \leq r_2; (i)r \leq r3, (ii)r_3 \leq r \leq r_1, (iii)r_1 \leq r \leq r_2, (iv)r2 \leq r.
\end{aligned} \tag{3.10}$$

The unrestricted case is a more flexible assumption and incorporates broader class of models and aids the manufacturer in designing a broad spectrum of warranty policy limits, while the restricted case will provides useful bounds for the expected costs.

3.3.4 Repair-Replacement Decisions and Costs for servicing in 2-Dimensional Model

We describe the types of repair-replacement action which are performed upon failure in different regions of 2-D domain. The manufacturer can adopt to repair-replacement upon product failures and the expected costs will be change based on the decision. Assuming that S be the cost of replacing the product, i.e. cost incurred by the manufacturer towards replacing first failure in Ω_2 for strategy-A and replacing first failure in Ω_3 for strategy- B. We assume manufacturer incurs pro-rata cost for performing minimal repairs based on the region of failure, time and mileage at the instant of failure for warranty servicing. Let S_1, S_2 and S_3 be the cost of proration for all minimal reapirs in Ω_1, Ω_2 and Ω_3 respectievly, with $S_1 \leq S_2 \leq S_3$. The pro-rata functions are defined as follows, where $\phi_i(t, x)$ corresponds to rectangular region Ω_i. In this research we use linear pro-rata functions for all repairs in Ω_i as given in [110], where $0 < \alpha \leq 1$:

$$\phi_1(t, x) = \left\{ \begin{array}{cc} \alpha(\frac{t}{T1})(\frac{x}{U1})S_1, & (t, x) \in \Omega_1 \\ 0, & Otherwise \end{array} \right\} \tag{3.11}$$

$$\phi_2(t, x) = \left\{ \begin{array}{cc} \alpha(\frac{t}{T2})(\frac{x}{U2})S_2, & (t, x) \in \Omega_2 \\ 0, & Otherwise \end{array} \right\} \tag{3.12}$$

$$\phi_3(t, x) = \left\{ \begin{array}{cc} \alpha(\frac{t}{T})(\frac{x}{U})S_3, & (t, x) \in \Omega 3 \\ 0, & Otherwise \end{array} \right\} \tag{3.13}$$

The bi-variate function $\phi_i(t, x)$ for $i = 1, 2, 3$ is a positive, increasing function in both the variables t, x as the cost of performing minimal repair on early failure is cheaper than failure at the end of warranty duration. There are a number of choices for pro-rata functions which linear, non-linear, minimal, maximal, and exponential etc. as given in [110] pp. 369. Many authors consider the pro-rata function in the sense of offering rebate to the customer if the product fails during warranty, so the rebate function would be decreasing function in terms of t and x. But this research does not include option for offering rebate to the customer, rather obtains the expected cost to the manufacturer based on different product usage and warranty expiration limits. While manufacturers may dynamically change their rebate functions according to performance of the product in the market. The vast majority of literature is available assuming repair costs as constant which is convenient from the modeling perspective but does not reflect a realistic estimate of the expected repair costs. The manufacturers can probabilistically adjust the cost of pro-ration using α. Also pro-rata refers to the rebate offered by the manufacturer to the loyalty of customers in cases where the warranty is backed by money-back offer, this also highlights the applicability of warranty policies with pro-rata costs to non-repairable products. Based on the availability of data regarding the warranty claim rates and costs for servicing, such rebate functions can be derived for specific products.

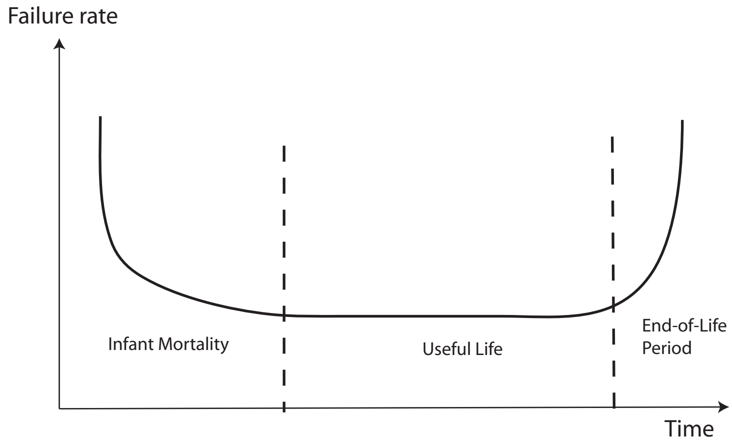


Figure 3.6. Relevance of repair action with regard to bathtub failure function

3.3.5 Repair-Replacement Strategy for Two-Dimensional Model

In this section, we develop two policies for servicing non-renewing 2D warranty based on when the manufacturer can replace the product and explain the rational of the servicing policy.

Strategy-A: All product failures in $\Omega 1$ are minimally repaired at prorated cost given by $\phi_1(t,x)$, where $(t,x)\epsilon\Omega 1$. The first failure in region $\Omega 2$ the manufacturer replaces the product which incurs a cost of S_1 and subsequent product failures are minimally repaired at prorated cost given by $\phi_2(t,x)$ where $(t,x)\epsilon\Omega 2$. All product failures in $\Omega 3$ are minimally repaired at prorated cost given by $\phi_3(t,x)$,where $(t,x)\epsilon\Omega 3$.

Strategy-B: All product failures in $\Omega 1$ are minimally repaired at prorated cost given by $\phi_1(t,x)$, where $(t,x)\epsilon\Omega 1$. All product failures in $\Omega 2$ is minimally repaired at prorated cost given by $\phi_2(t,x)$, where $(t,x)\epsilon\Omega 2$. The first failure in region $\Omega 3$ the manufacturer replaces the product which incurs a cost of S_1 and subsequent product failures are minimally repaired at prorated cost given by $\phi_3(t,x)$ where $(t,x)\epsilon\Omega 3$

The rational for the repair-replacement strategy is obtained from bathtub curve which is a common model for the failure rate of products. The classic bathtub curve has three parts: infant mortality; useful life; and wearout. Strategy-A, The initial phase of the curve displays decreasing failure rate (DFR) of products and it would

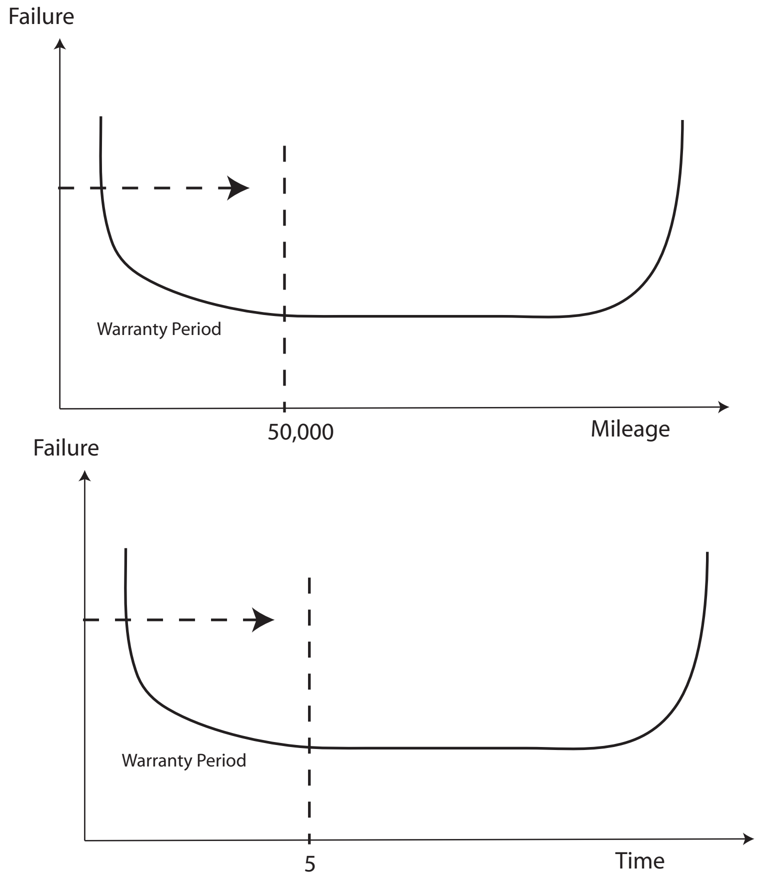


Figure 3.7. Failure rate function realized in terms of time and mileage

be ideal for the manufacturer to minimally repair the product failures in $\Omega1$ so that the product attains useful life phase. Failures during useful life would appear randomly due to unforseen quality defects,so the manufacturer agrees to replace the first failure as a part of quality commitment to manufacture a failure free product and subsequent product failures in $\Omega2$ are minimally repaired. Since most products have a finite useful life time and begin to wearout, it would be more appropriate to perform minimal repairs on failures in the wear out phase to bring it to a functioning state for a short span of time and would cost the minimal cost to the manufacturer satisfying the warranty contract. Beyond $\Omega3$, the manufacturer will no longer service the product. Strategy-B, Certain products begin to be more prone to failures after in use for longer duration of time, in such cases the manufacturer

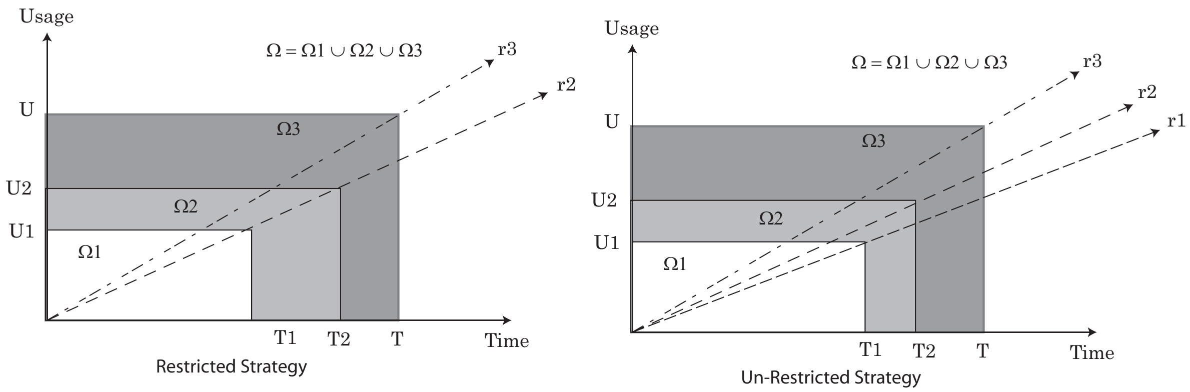


Figure 3.8. Different warranty regions in the restricted case

may consider the replacement in $\Omega 3$ phase. Industrial machinery are ideal products for strategy-B. The design of warranty contracts are product specific and this research demonstrates the ability to develop warranty strategies and obtain the expected costs based on factors pertaining to usage, failure rates and lifetime distribution of the product. We use conditioning arguments to obtain the expected costs of warranty servicing for the manufacturer for restricted and unrestricted cases of strategy-A and strategy-B. We shall compare and numerically illustrate the differences in both the servicing strategies.

3.4 Assumptions and Limitations of the Research

In this section we briefly discuss the assumptions and limitations associated with the research.

Assumptions: Time to replace-repair is assumed to be negligible. All failures result in a warranty claim. Each claim is valid. Both the warranty models (1-D and 2-D) have been developed for warranties with non-renewing option only. We do not consider the effect of routine maintenance operations on the product under warranty. All products have finite lifetime. We assume repeat purchase option in 1D model. All products are assumed to have finite life cycle, which is a common and reasonable. All products are repairable, even tough results for non-repairable products can be obtained easily through this work.Product failures are independent and identically distributed.

Limitations: Warranty research is predominantly data driven, and due to the confidentiality nature of product failure times and warranty claims, this research uses only limited data as was available. This research employs computational approximations, and the numerical results obtained are estimates of the expected costs. A variety of empirical distributions can be used to improve the accuracy of the results derived. Parameter estimation involved in this research can be refined using more data and powerful techniques.

3.5 Chapter Summary

In this chapter we discussed the methods and modeling techniques employed in the study of one-dimensional and two-dimensional warranty models. Although one-dimensional models are simple, they pose tremendous challenges in obtaining the summary measures of the model, especially in obtaining stationary points and evaluating second-order conditions for the different repair-replacement choices to the manufacturer. Two-dimensional polices are more flexible and are widely applicable for a variety of products. Among the two available modeling approaches in 2-D models, we adopt a one-dimensional point process approach to model and analyze the two-dimensional rectangular warranty model. 2-D models have not be studied extensively by researchers. This research will serve as a vital contribution to the existing literature on 2-D models and presents a variety of improvements possible, both theoretically and in applications. In the next chapter, we will discuss the analytical results of one-dimensional warranty model and present the numerical results pertaining to the model.

Failure no.	Age(days)	Mileage(In10K)
1	166	0.9766
2	35	0.2041
3	249	1.2392
4	190	0.9889
5	27	0.0974
6	41	0.1594
7	59	0.2128
8	75	0.2158
9	223	1.1187
10	952	4.766
11	335	1.3827
12	164	0.5992
13	145	0.6925
14	170	0.7078
15	140	0.7553
16	498	2.5014
17	571	2.538
18	499	2.6433
19	340	1.6494
20	160	0.7162
21	128	0.5922
22	31	0.1974
23	65	0.203
24	221	1.2532
25	316	1.4796
26	22	0.0979
27	261	1.5062
28	32	0.2062
29	397	1.6888
30	48	0.3099
31	1	0.0028
32	27	0.0095
33	295	1.26
34	140	0.8067
35	827	4.1425
36	2	0.0105
37	209	1.2302
38	29	0.0447
39	166	0.9766
40	1200	5.7304

Table 3.2. First 40 Observed Failure Times and Mileage for the Traction Motor Data. Source: Management Science, Vol. 43, No. 7.-Jul., 1997- pp. 973

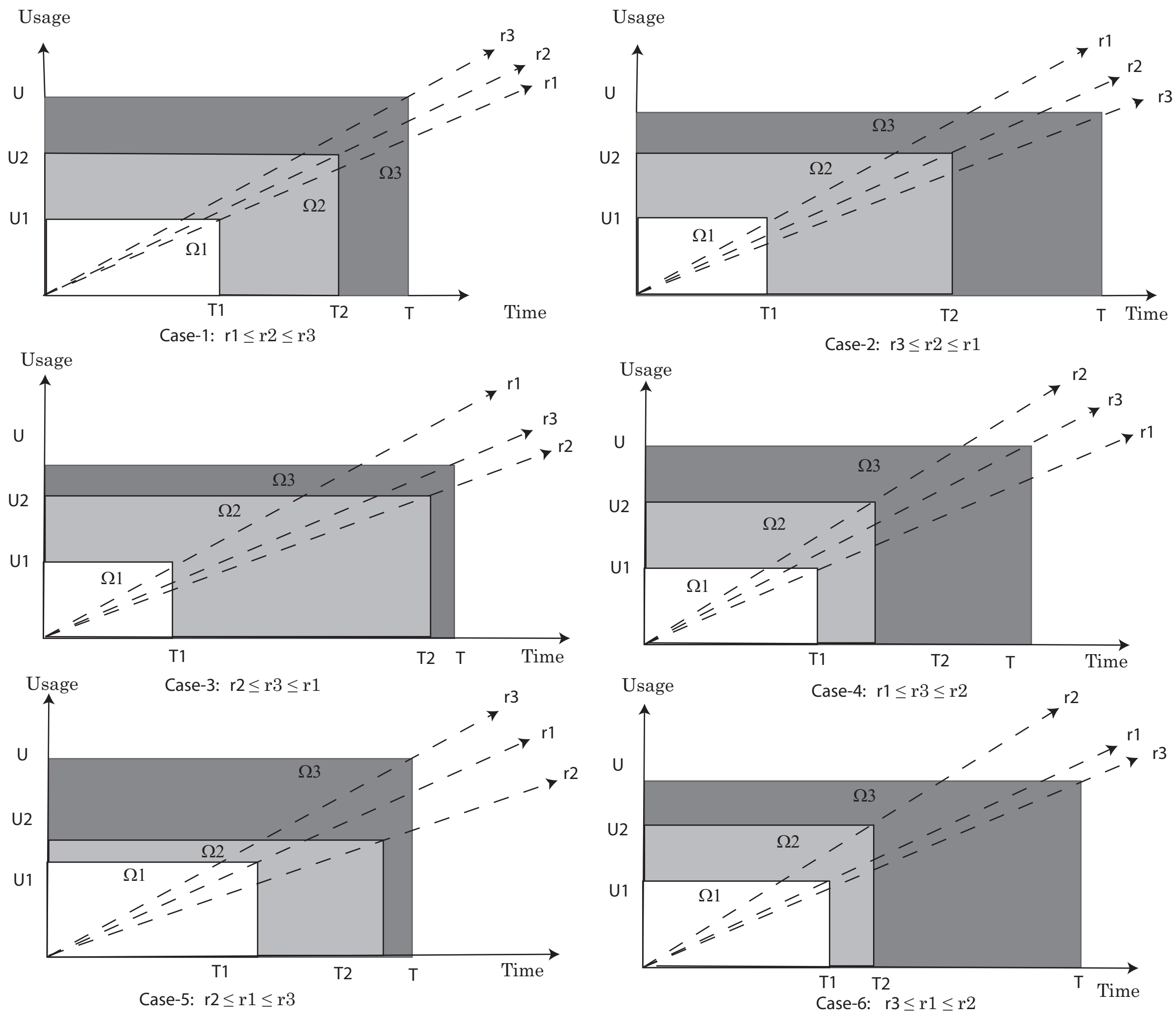


Figure 3.9. Different warranty regions in the unrestricted case

CHAPTER 4
ONE-DIMENSIONAL MODEL

4.1 Introduction

In this chapter, we consider a repairable product under a non-renewing combined warranty policy, which is subject to a displaced log-linear demand function of the product's price and pro-rata period length. In section 4.2, we discuss model development and derive the expressions for the manufacturers long run average profit per unit time under replacement, minimal, and general repair options. Under each repair option, we obtain expressions for expected number of renewals, forward recurrence time distribution and the expected time between two successive purchases. Section 4.3 deals with optimization procedures involved in the model, we obtain the stationary points of the profit function, and the necessary second order conditions, which correspond to a relative maximization of profit, are given. In section 4.4, we provide numerical illustrations when the lifetime distribution of the product is (i) Gamma order-2 (IFR type) and (ii) Weibull extended distribution (DFR/IFR type), which demonstrates the selection of optimal product pricing, pro-rata length determination, and repair option to maximize the manufacturers profit.

Optimizing warranty policies from the perspective of manufacturers is an area of considerable managerial interest. Typically, the manufacturer's objective is to maximize profit through the optimal specification of product price and warranty period length, while the consumer's objective is to minimize cost through the optimal selection of a product and policy. From the perspective of the manufacturer, much of the optimization literature related to price and period determination has been confined to base warranty replacement policies. In particular, classic work by Anderson [2] develops the framework for a non-specific profit maximization model that seeks to optimally specify the product price and warranty period. Glickman and Berger [38] considerably extend this work by considering a displaced log-linear demand function of product price and warranty period length, which is subsequently embedded in a profit function and maximized to yield optimal product price and warranty period specifications. We discuss the sales volume function and manufacturers profit function in section 4.2.

Two common types of policies examined in the literature are base and pro-rata warranty policies. In base warranty policies, the manufacturer agrees to either repair or replace the a product at no cost to the customer throughout the warranty period, and in pro-rata warranty policies, repairs or replacements are done at a cost proportional to the operating time of the product. In general, base warranty policies favor the consumer, and pro-rata policies favor the manufacturer. To balance the risk, a combined warranty policy is a combination of base and pro-rata policies in which a base warranty period is immediately followed by a pro-rata period

4.2 Model Development

We consider a repairable item purchased at time $t = 0$, whose time to failure is given by the non-negative random variable X with distribution function $F(\cdot)$. During an initial base warranty period (BWP) period on $(0, W]$, the manufacturer agrees to provide either replacement, general repair, or minimal repair at no cost to the customer upon failure. Upon completion of the BWP, a pro-rata (PRW) period on $(W, W + T]$ begins in which the customer agrees to purchase a new item upon failure at a cost that is linearly proportional to the original purchase price P. Upon completion of the PRW, it is assumed that the customer purchases a new product at full price upon failure. Thus, letting Y denote the time from purchase until the first failure after W, it follows that the customer's purchase price, $C(Y)$, is given by

$$C(Y) = \begin{cases} P\frac{Y-W}{T} & \text{if } W \leq Y < W + T \\ P & Y \geq W + T \end{cases} . \tag{4.1}$$

The time between two successive purchases forms a renewal cycle, and the purchase price depends on the length of the cycle. Hence, letting $F_\gamma(t)$ be the distribution function of the forward recurrence time $\gamma(W)$ measured from W, it follows from (4.1) that the expected purchase price of the customer in any cycle is given by

$$\begin{aligned} E[C] &= \frac{P}{T}\int_0^T t dF_\gamma(t) + P\int_T^\infty dF_\gamma(t) \\ &= \frac{P}{T}\int_0^T \overline{F}_\gamma(t)dt . \end{aligned} \tag{4.2}$$

The expected sales volume, q, for this product is a bivariate function of P and T, and assumed to follow the displaced log-linear function given by:

$$q(P,T) = k_1 P^{-a} \left(T + k_2\right)^b \ , \tag{4.3}$$

as shown in [38], where $k_1 > 0$ is a demand amplitude factor, $k_2 \geq 0$ is a constant of PRP period displacement, $a > 1$ is price elasticity, and $0 < b < 1$ is warranty period elasticity.

In general, log-linear demand functions are widely used in econometric modeling when the percentage contribution of associated variables are exponential in nature, yet widely varying in degree. It is classically evident that price has an exponential impact on demand in many consumer choice applications as argued in [31], and plausible that the pro-rata period length follows similarly. In addition to the obvious importance of product price, the rationale for forecasting sales volume on the PRP length T rather than the BWP length W is that the latter is typically correlated with a measure of product burn-in or infant mortality and is assumed to be fixed accordingly across competitive products. The PRP length, however, is not directly related to product measures, such as the hazard or mean residual life of a product, and is often considered to be competitive measure between manufacturer's products. Following this rationale, we assume that the length of the PRP will be influential in forecasting sales volume while the BWP is not.

Note that (4.3) permits non-zero demand when $T = 0$ for $k_2 > 0$, does not permit $P = 0$, and is disproportionately impacted by unit changes in P over T as price elasticity may be infinitely scaled upward. As such, the characteristics of this demand model allow for a parametrization which captures the underlying demand dynamics of consumer markets in the manner in which they are commonly related to these variables. In practice, this model may be fit to historical data using 2-parameter regression to estimate the elasticities a and b, assuming that both k_1 and k_2 may be reasonably hypothesized for a particular product scenario. If the error structure is assumed to be of a product form, a linearization though the log transformation of (4.3) may be fit using standard least-squares multiple regression, else a nonlinear regression method should be used to estimate the parameters, a treatment of which is given in [70, Section 3].

Letting C_p be the production cost, $N(t)$ be the number of failures in $(0, t]$, and $\nu = C_r, C_m, C_g$ be the manufacturer's cost of BWP repair based on the choice employed by the manufacturer towards servicing product failure, it follows from (4.2) and (4.3) that the manufacturers long run average profit per unit time, π, of product selling price P and PRW period of length T is given by

$$\pi(P,T) = \frac{P^{-a}k_1\left(T+k_2\right)^b\left(\frac{P}{T}\int_0^T \overline{F}_\gamma(t)dt - (\nu E[N(W)] + C_p)\right)}{E[Y]} \tag{4.4}$$

In the following subsections, we develop expressions for $E[N(W)]$, $F_\gamma(t)$, and $E[Y]$ under replacement, minimal, and general repair options, which are then inserted into (4.4).

4.2.1 Replacement Repair

When the manufacturer agrees to replace the product on failure during the BWP, the counting process $\{N(t), t \geq 0\}$ is a renewal process on $(0, W]$. Hence, for $t \leq W$ and letting $M(t) = E[N(t)]$, it follows that the renewal function $M(t)$ is obtained as a solution to

$$M(t) = \sum_{n=1}^{\infty} F_n(t) \ , \tag{4.5}$$

where $F_n(t)$ is the n-fold convolution of $F(t)$ with itself. Note that a solution to (4.5) may often be conveniently obtained by using Laplace transforms, as shown in Kao [53, Section 3.1]. When the life time distribution is complex or mixture of simple distributions, a solution to (4.5) may be difficult to obtain. In such cases, we can adopt to approximation techniques as highlighted in [99]. It follows that the distribution function $F_\gamma(t)$ of the forward recurrence time $\gamma(W)$ is defined as

$$F_\gamma(t) = P(\gamma(W) < t) = F(W+t) - \int_0^W \overline{F}(W+t-x)dM(x) \tag{4.6}$$

as shown in Nguyen and Murthy [85], from whence $\overline{F}_\gamma(t) = 1 - F_\gamma(t)$.

Let $\mu = E[X]$ be the mean time to failure, the expected time between two successive purchases is given by:

$$E[Y] = \mu\,[1 + M(W)] \tag{4.7}$$

4.2.2 Minimal Repair

When the manufacturer opts for minimal repair policy, the number of failures $\{N(t), t \geq 0\}$ on $(0, W]$ forms a Non-Homogeneous Poisson Process (NHPP) with intensity function $\lambda(t)=\frac{f(t)}{\overline{F}(t)}$ as shown in Ross [98]. Let $R(t) = E[N(t)]$, it follows that

$$R(W) = \int_0^W \lambda(u)du \ , \tag{4.8}$$

and that the complementary distribution function of the forward recurrence time $\gamma(W)$ is given by

$$\overline{F}_\gamma(W) = P(\gamma(W) > t) = \frac{\overline{F}(W+t)}{\overline{F}(W)} \ . \tag{4.9}$$

Using (4.9), it follows that the time between successive purchases is

$$E[Y] = W + \frac{\int_W^\infty \overline{F}(u)du}{\overline{F}(W)} \tag{4.10}$$

4.2.3 General Repair

In section 3.2.3, briefly introduced the concept of virtual aging, properties of a general repair policy and its relationship with the real age of the product.

Letting $H(t) = E[N(t)]$ denote the expected number of failures on $(0, W]$, and $r(t) = \frac{f(t)}{\overline{F}(t)}$ be the failure rate of the distribution function $F(t)$, we note that

$$H(W) = E[N(W)] = \int_0^W E[r(t - (1-z)S_{N(t)})]dt \ . \tag{4.11}$$

Define the g-renewal density function $h(t) = E[r(t - (1-z)S_{N(t)})]$, which gives the mean number of general repairs to be expected in $(t, t+dt)$, as

$$h(t) = f(t) + \int_0^t h(x)\frac{f(t - x + zx)}{\overline{F}(zx)}dx, \quad t \geq 0 \ . \tag{4.12}$$

Note that (4.12) is obtained by arguing that the failure at t could be either the first failure or a subsequent failure, whose probabilities are respectively given by the two terms on the right-hand side of the equation, and may be combined with (4.11) to obtain $H(W)$. Note that a solution to the integral equation for $h(t)$ may be obtained through numerical methods, such as an iterative scheme shown in Masujima [66, Section 3.1]. The complementary distribution function of the forward recurrence time $\gamma(W)$ is given by

$$\overline{F}_\gamma(t) = P(\gamma(W) > t) = \overline{F}(W + t) + \int_0^W h(x)\frac{\overline{F}(W + t + x(b-1))}{\overline{F}(bx)}dx \qquad (4.13)$$

where the g-renewal density $h(x)$ is specified in (4.12). The expected time between purchases is thus obtained as

$$E[Y] = W + \int_0^\infty \overline{F}_\gamma(t)dt \qquad (4.14)$$

4.3 Model Optimization

The profit function of (4.4) may be optimized with respect to price and pro-rata length under minimal, general, and replacement repair options to maximize the manufacturers profit. A comparison of these optimal objectives may further provide for the optimal selection between repair options. We develop expressions for the stationary points of the profit function and for the second-order conditions about each point to determine concavity.

4.3.1 Stationary Points

The first partial derivatives of the profit function (4.4) with respect to price P and pro-rata length T are given by

$$\frac{\partial\pi(P,T)}{\partial P} = \frac{P^{-a-1}\left(a\left(C_p + \nu E[N(W)]\right)T + (P - aP)\int_0^T \overline{F}_\gamma(t)\,dt\right)k_1\left(T + k_2\right)^b}{E[Y]T} \qquad (4.15)$$

$$\frac{\partial \pi(P,T)}{\partial T} = -P^{-a}k_1(T+k_2)^{b-1}\left\{\begin{array}{c} \frac{P(\int_0^T \overline{F}_\gamma(t)\,dt)(-bT+T+k_2)}{E[Y]\,T^2} + \\ \frac{bT^2(C_P+\nu E[N(W)]}{E[Y]\,T^2} - \frac{PT(T+k_2)\overline{F}_\gamma(t)}{E[Y]\,T^2} \end{array}\right\} \tag{4.16}$$

which are subsequently set equal to zero and solved simultaneously to find the roots P and T. Letting $P^\star$ and $T^\star$ denote the optimal price and pro-rata length respectively, the root of (4.15) with respect to P yields

$$P^\star = \frac{a\,(C_p+\nu E[N(W)])\,T^\star}{(a-1)\int_0^{T^\star}\overline{F}_\gamma(t)\,dt}\ . \tag{4.17}$$

Using (4.17), the root of (4.16) with respect to T yields

$$T^* = \frac{(a-b)\int_0^{T^*}\bar{F}_\gamma(t)\,dt - ak_2\bar{F}_\gamma(T^*)}{2a\bar{F}_\gamma(T^*)} \pm \frac{\sqrt{4a^2k_2\left(\int_0^{T^*}\bar{F}_\gamma(t)\,dt\right)\bar{F}_\gamma(T^*) + \left((b-a)+\int_0^{T^*}\bar{F}_\gamma(t)dt + ak_2\bar{F}_\gamma(T^*)\right)}}{2a\bar{F}_\gamma(T^*)} \tag{4.18}$$

There is no closed-form representation of (4.18), yet the expression may be solved using a simple iterative scheme, thereby yielding two values for the roots $T^\star$. These may then be substituted into (4.17) to obtain the corresponding $P^\star$ values, and hence the two stationary points of the bi-variate profit function $\pi(P,T)$. Note that only those stationary points in which $P^\star, T^\star \geq 0$ are of interest and treat the other point as spurious.

4.3.2 Second-Order Conditions

The concavity about each stationary point may be assessed through the associated principal determinants of the Hessian matrix. Letting $\pi_{pp} = \frac{\partial^2\pi(P,T)}{\partial P^2}$ be the first principal determinant of the Hessian with respect to P, it follows that

$$\pi^*_{PP} = -\frac{a(C_p+\nu E[N(W)])\left(a(C_p+\nu E[N(W)])T^*/(a-1)\int_0^{T^*}\bar{F}_\gamma(t)dt\right)^{-(a+2)}k_1(T^*+k_2)^b}{E[Y]} \tag{4.19}$$

from whence $\pi^{\star}_{pp} \leq 0$ for all $T^{\star} \geq 0$. Letting $\pi_{TT} = \frac{\partial^2 \pi(P,T)}{\partial T^2}$ be the first principal determinant of the Hessian matrix with respect to T, it follows that

$$\begin{aligned}\pi_{TT} \quad = \quad & \frac{k_1 \left(T + k_2\right)^{b-2}}{aE[Y]T^3} \left(\frac{a \left(C_p + \nu E[N(W)]\right) T}{(a-1) \int_0^T \overline{F}_{\gamma}(t)\, dt} \right)^{1-a} \left(\left(\int_0^T \overline{F}_{\gamma}(t)\, dt \right) \right. \\ & \left((b-1)(b-2a)T^2 - 2a(b-2)k_2 T + 2ak_2^2\right) + aT\left(T + k_2\right) \\ & \left. \left(2\left((b-1)T - k_2\right) y(T) + T\left(T + k_2\right) \overline{F}'_{\gamma}(T)\right) \right) , \end{aligned} \tag{4.20}$$

from whence $\pi^{\star}_{TT} \leq 0$ iff

$$\begin{aligned} & \left(\int_0^{T^{\star}} \overline{F}_{\gamma}(t)\, dt \right) \left((b-1)(b-2a)(T^{\star})^2 - 2a(b-2)k_2 T^{\star} + 2ak_2^2\right) \\ & \quad + aT^{\star}\left(T^{\star} + k_2\right) \left(2\left((b-1)T^{\star} - k_2\right) \overline{F}_{\gamma}(T^{\star}) + T^{\star}\left(T^{\star} + k_2\right) \overline{F}'_{\gamma}(T^{\star})\right) \leq 0 \, . \end{aligned} \tag{4.21}$$

Letting $\pi_{PT} = \left(\frac{\partial^2 \pi(P,T)}{\partial T^2}\right)\left(\frac{\partial^2 \pi(P,T)}{\partial P^2}\right) - \left(\frac{\partial^2 \pi(P,T)}{\partial P \partial T}\right)^2$ be the second principal determinant of the Hessian matrix, it follows that

$$\begin{aligned}\pi_{PT} \quad = \quad & \frac{a \left(C_p + \nu E[N(W)]\right)^2 T^2}{\left(\int_0^T \overline{F}_{\gamma}(t)\, dt\right)^2} \left(- a\left(T + k_2\right)^2 \left(\int_0^T \overline{F}_{\gamma}(t)\, dt - T\overline{F}_{\gamma}(T) \right)^2 - \frac{\left(\int_0^T \overline{F}_{\gamma}(t)\, dt\right)}{a-1} \right. \\ & \left(\left(\int_0^T \overline{F}_{\gamma}(t)\, dt \right) \left((b-1)(b-2a)T^2 - 2a(b-2)k_2 T + 2ak_2^2\right) + aT\left(T + k_2\right) \right. \\ & \left. \left. \left(2\left((b-1)T - k_2\right) \overline{F}_{\gamma}(T) + T\left(T + k_2\right) \overline{F}'_{\gamma}(T)\right) \right) \right) , \end{aligned} \tag{4.22}$$

from whence $\pi^{\star}_{PT} \geq 0$ iff

$$-a\left(T^{\star}+k_{2}\right)^{2}\left(\int_{0}^{T^{\star}}\overline{F}_{\gamma}(t)\,dt-T^{\star}\overline{F}_{\gamma}(T^{\star})\right)^{2}-\frac{\left(\int_{0}^{T^{\star}}\overline{F}_{\gamma}(t)\,dt\right)}{a-1}\Bigg(\left(\int_{0}^{T^{\star}}\overline{F}_{\gamma}(t)\,dt\right)$$
$$\left((b-1)(b-2a)(T^{\star})^{2}+2ak_{2}\left((2-b)T^{\star}+k_{2}\right)\right)+aT^{\star}\left(T^{\star}+k_{2}\right)$$
$$\left(2\left((b-1)T^{\star}-k_{2}\right)\overline{F}_{\gamma}(T)+T^{\star}\left(T^{\star}+k_{2}\right)\overline{F}'_{\gamma}(T^{\star})\right)\Bigg)\;\geq\;0 \tag{4.23}$$

A necessary condition for $(P^{\star}, T^{\star})$ to be a relative maximum is for $\pi^{\star}_{PP}, \pi^{\star}_{TT} < 0$ and $\pi^{\star}_{PT} > 0$, and note that only one of the two stationary points may yield a relative maximum. In practice, often only one of the stationary points is on the domain of $P, T \geq 0$, and this point typically satisfies the second-order conditions.

4.4 Numerical Illustration

A pair of numerical examples are presented in this section that illustrate the procedure through which the manufacturers long run average profit may be optimized under different BWP repair options. In particular, the optimal price P and pro-rata length T are determined for each option, and then used as a basis for selection amongst available alternatives.

4.4.1 IFR Illustration

Consider a product whose time to failure X is governed by a gamma distribution of order two,

$$F(x) = 1 - e^{-\lambda x}(1 + \lambda x)\ , \tag{4.24}$$

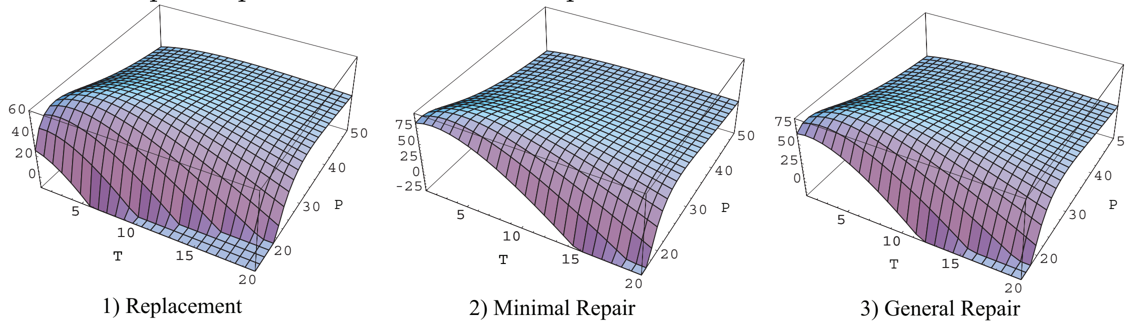
from whence the failure rate function is increasing (IFR). The expected time to failure of this product is known to be $E[X] = 20$ time units, and hence $\lambda = 0.1$. The cost to produce the product is $C_p = 10$ and the base warranty period is $W = 10$ time units. Demand for this product has an price elasticity of $a = 3$, warranty period elasticity of $b = 0.5$, amplitude factor of $k_1 = 10^6$ units, and a period displacement of $k_2 = 2$ time units. The cost of the different repair options in the BWP are $C_r = 12$ for replacement, which is the cost of production plus an additional amount for shipping and set-up, $C_m = 2$ for a minimal repair, and $C_g = 5$

for a general repair that reduces the 'age' of the product by $z = 0.5$. Optimized policies for the different repair options are shown in Table 4.1, and plots of the expected profit function for the different repair options are given in Figure 4.1. Note that for this illustration, the expected profit functions are concave on $P^\star, T^\star \geq 0$ and the minimal repair option at the optimum $\pi(P^\star, T^\star) = 94.317$ yields the largest per unit time profit to the manufacturer. Hence, parameterized minimal repair selection should be offered by the manufacturer to the consumer.

Table 4.1. Optimal Price and Pro-Rata Measures for the Different Repair Options in the IFR Example

	$M(W)$	$E[Y]$	$(P^\star, T^\star)$	$\pi^\star_{PP}$	$\pi^\star_{TT}$	$\pi^\star_{PT}$	$\pi(P^\star, T^\star)$
Replacement	0.284	25.677	(22.460,4.729)	-0.711	-0.961	0.527	59.775
Min Repair	0.307	25.000	(17.760,4.195)	-44.852	-43.113	1474.440	94.317
Gen Repair	0.303	25.428	(19.237,4.401)	-1.305	-1.398	1.397	80.487

Figure 4.1. Expected Profit for the 1) Replacement, 2) Minimal Repair, and 3) General Repair Options in the IFR Example



4.4.2 DFR/IFR Illustration

Consider a product whose failure rate function is decreasing (DFR) for an initial period followed by IFR for the remaining life. Towards this end, let the time to failure X be governed by the Weibull extension model

$$F(X) = 1 - \exp\left\{-\lambda\alpha\left[e^{t/\alpha^\beta} - 1\right]\right\} \tag{4.25}$$

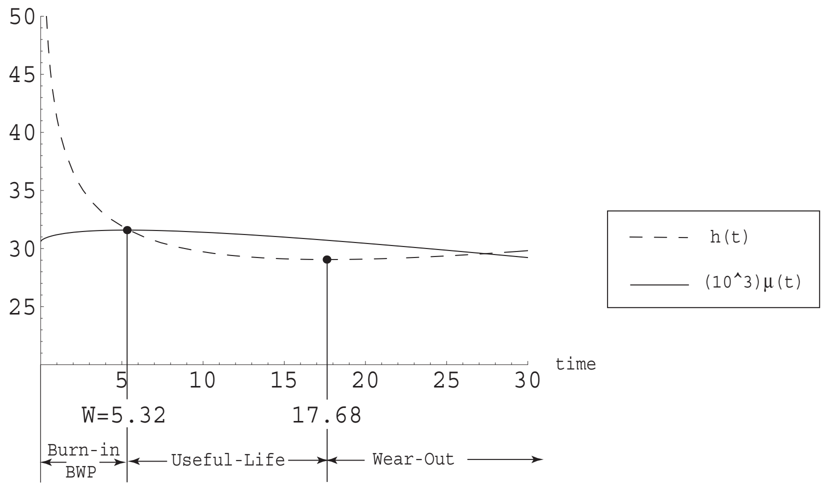


Figure 4.2. Reliability Phases and Base Warranty Period Determination for the DFR/IFR Example

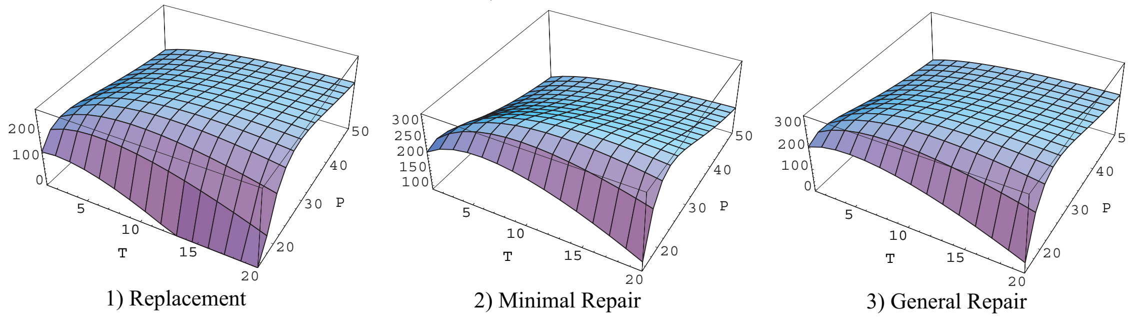
as proposed in [112] and [105]. Letting $\beta = 0.8$, $\alpha = 100$, and $\lambda = 0.02$, it follows that the expected time to failure for this product is a known quantity of $E[X] = 30.615$ time units. The BWP for this product will be set at that point where the mean residual life (MRL) of X reaches a maximum, which is numerically found to be $W = 5.319$. Note that the MRL is defined as $\mu(t) = E[X - t \mid X \geq t]$ and describes the expected remaining lifetime of the product given that it has survived to time t. Hence, we assume the BWP is specified by the manufacturer so as to provide consumer protection through the burn-in phase of reliability, as shown in Figure 4.2. For reference, the useful-life phase of this product will be defined as that period between the maximum of $\mu(t)$ and minimum of the failure rate function $h(t) = \lambda\beta(t/\alpha)^{\beta-1} \exp\left[(t/\alpha)^{\beta}\right]$, i.e. $(5.32, 17.68]$, and the wear-out phase encompasses all time beyond that.

As with the previous illustration, the costs of this model are given by $C_p = 10$, $C_r = 12$, $C_m = 2$, and $C_g = 5$ with $z = 0.5$. The demand for this product has a price elasticity of $a = 2.5$, warranty period elasticity of $b = 0.5$, amplitude factor of $k_1 = 10^6$ units, and a period displacement of $k_2 = 2$ time units. Optimized policies for the different repair options are shown in Table 4.2, and plots of the expected profit per unit time for the different repair options are given in Figure 4.3. Note that for this illustration, the expected profit functions are concave on $P^\star, T^\star \geq 0$ and the general repair option at the optimum $\pi(P^\star, T^\star) = 331.810$ yields the largest per unit time profit to the manufacturer. Hence, the parameterized general repair selection should be offered by the manufacturer to the consumer.

Table 4.2. Optimal Price and Pro-Rata Measures for the Different Repair Options in the DFR/IFR Example

	$M(W)$	$E[Y]$	$(P^\star, T^\star)$	$\pi^\star_{PP}$	$\pi^\star_{TT}$	$\pi^\star_{PT}$	$\pi(P^\star, T^\star)$
Replacement	0.204	36.871	(24.528,12.468)	-1.663	-0.79023	0.573	282.718
Min Repair	0.201	41.5872	(20.808,12.5144)	-2.786	-0.919	0.799	321.639
Gen Repair	0.202	36.902	(22.026,12.498)	-2.571	-0.952	0.781	331.810

Figure 4.3. Expected Profit for the 1) Replacement, 2) Minimal Repair, and 3) General Repair Options in the DFR/IFR Example



4.5 Conclusion

In chapter we developed a procedure was optimally specify the price, pro-rata length, and type of repair to maximize the manufacturers long run average profit per unit time for products under non-renewing, combined warranty policy. There were significant challenges associated in obtaining optimal values of the model and we resorted to a number of numerical and computational approximation procedures. We presented couple of numerical examples to demonstrate the application of this procedure to a practical problem and highlights it's usefulness in a managerial context. Several extensions to the 1D model developed herein are discussed in Chapter-6.

CHAPTER 5
TWO-DIMENSIONAL MODEL

5.1 Introduction

In this chapter we study a non-renewing two-dimensional warranty policy defined by rectangular region based on time and usage with pro-rata costs for repair option during warranty coverage period. Product failures in different warranty regions have different cost implications based on repair strategy employed. We derive expected cost of the manufacturer for servicing the warranty in the event of product failure. The costs for repair are prorated based on time and usage of the product, and we derive expressions for the expected cost of warranty servicing in both restricted and unrestricted cases. We develop two repair strategies based on when the manufacturer can exercise replacement option and numerically compare them. In section 5.2, we describe the procedures in model development, which include regression modeling of the failure and usage rate. In section 5.3, we derive analytical expressions for expected costs associated with servicing in the unrestricted case and restricted case. In section 5.4, we present the results obtained through numerical calculations, and in section 5.5 we discuss the conclusions, scope for further work and modeling challenges associated with the model.

The main objective of the manufacturer is to minimize the expected cost for warranty servicing 2-D warranty policies are defined by regions of different geometrical shapes that have different implications on cost for manufacturer, and are specific to the failure data, type of product etc. Warranty limits set by the manufacturer is the driving force for the costs, which is a strictly increasing random variable. In other words, failures associated with products during early life are cheaper to fix than the failures which happen at the end phase. The motivating idea for our 2D model came from the constant cost assumption in [46] and [25] which under/over estimate the costs, and our model significantly extends this work by incorporating pro-rated costs for warranty servicing, since most maintenance and service activities accrue costs over time. In the current model, we apply linear pro-rata costs for servicing as described in section 3.3.4 by equations (3.11), (3.12), (3.13).

5.2 Model Development

In the 2D model, we choose the Uniform distribution to describe the usage pattern of the traction motor failure data. The Gamma, bi-variate exponential, Weibull, log-normal distribution etc. are other popular choices to model usage distribution. A variety of empirical distributions can be characterized based on the availability and suitability to model usage patterns. Based on linear relationship between the product usage $U(t)$ and age $A(t)$, as given by equation 3.1, $A(t) = R * U(t)$. We now set the threshold values of the uniform distribution, to describe the usage pattern under three categories namely, Light Usage, Medium Usage and Heavy Usage. We obtain the expected values of age and usage at first failure. Taking expectations on both sides of equation 3.1, we obtain

$$E[A] = E[R]\,E[U] \tag{5.1}$$

$$E[R] = \frac{E[A]}{E[U]} \tag{5.2}$$

The mean age at first failure is given by:

$$E[A] = \int_{r_l}^{r_u} E[A|r] * R(r), \tag{5.3}$$

where $E[A|r]$ is the expected value of the usage conditioned on r and is obtained from,

$$\begin{aligned} E[A|r] &= \int_0^{\infty} (1 - F[t|r])dt \\ Where\ \ F[t|r] &= 1 - e^{-\int_0^t \lambda(x|r)dx} \end{aligned} \tag{5.4}$$

The mean usage rate given by $E[R]$ is obtained from $\left(\frac{r_l + r_u}{2}\right)$. Numerically evaluating $E[A]$ and using $E[R]$ in equation (5.2), we obtain mean usage at first failure $E[U]$.

The failure intensity function given by $\lambda(t|r)$ is obtained using a multiple regression modeling with the following terms as independent regressors: r, the average usage rate, $U(t)$, the usage rate and $A(t) * U(t)$, from whence the structural form as given in equation (3.5). We assume that the average mileage accumulation

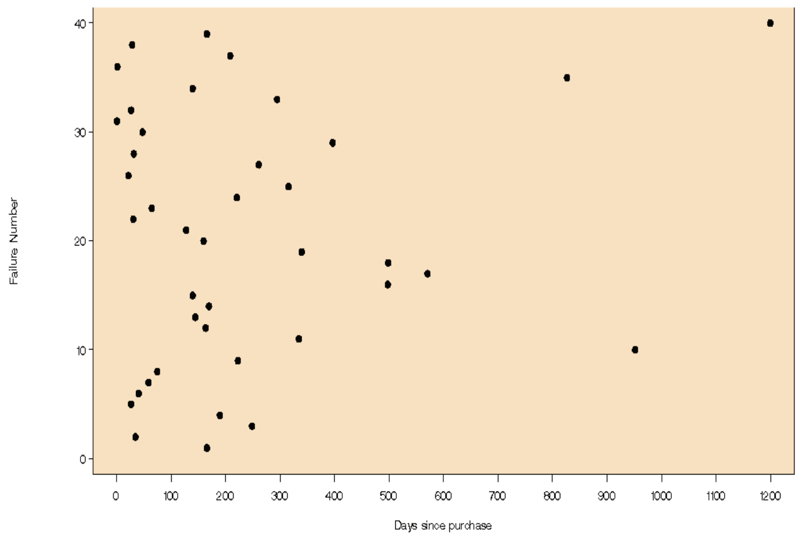


Figure 5.1. Scatter Plot of Failure number Versus Days in Service

rate is linear with $r = \frac{E[A]}{E[U]}$ due to its wide applicability for most populations as it averages out the high and low accumulators. In section 4.2, pp. 99 of [96] gives method to obtain natural estimates for the failure intensity of repairable system and works well when there is large amounts of data. We restrict our attention to regression modeling approach due to the limitations on the failure data and the procedure yields fairly reasonable estimates for the coefficient of the regressors. Figure 5.1 gives the scatter plot of failure number versus days since purchase, we can observe that most traction engines failed before reaching 400 days, and in figure 5.2, a plot of failure number versus mileage at time of failure suggests most reported failures occur prior to 20000 miles. In this respect, the manufacturer should make appropriate choice on the duration of warranty based on the sample estimates.

We used Proc Reg procedure using SAS system to obtain the coefficients of the structural form of the failure intensity as given by equation (3.5). From Table 5.1, we present the summary output with the parameter estimates, 95% confidence interval, and model fit statistic $R^2 = 0.8876$. We strongly believe that this approximation could be further refined given more data points on failure times. Equation 5.5 represents the estimated failure intensity function which is used to

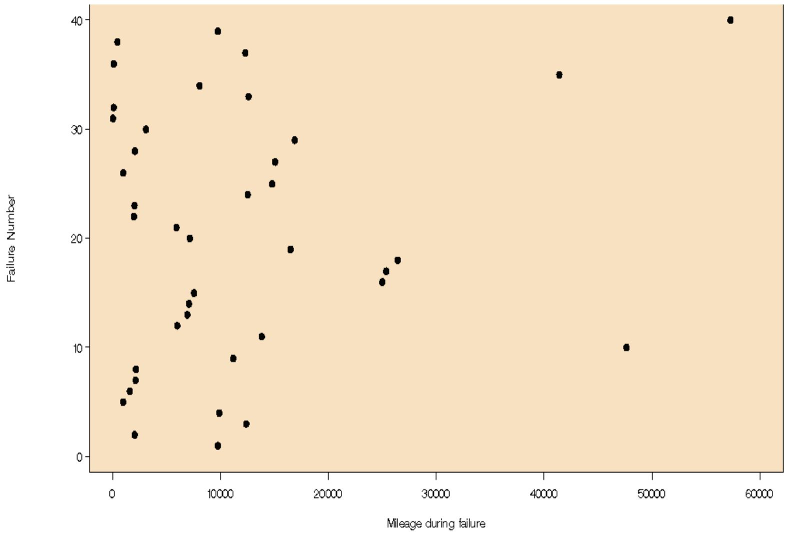


Figure 5.2. Scatter Plot of Failure number Versus Mileage at time of failure

```
                          The REG Procedure
                            Model: MODEL1
                  Dependent Variable: F4 failureintensity

          Number of Observations Read                        41
          Number of Observations Used                        40
          Number of Observations with Missing Values          1


                          Analysis of Variance

                                       Sum of           Mean
  Source                   DF         Squares         Square    F Value    Pr > F

  Model                     3        18.22135        6.07378      94.71    <.0001
  Error                    36         2.30858        0.06413
  Corrected Total          39        20.52993


           Root MSE              0.25323    R-Square     0.8876
           Dependent Mean        0.64767    Adj R-Sq     0.8782
           Coeff Var            39.09906


                         Parameter Estimates

                                   Parameter       Standard
Variable      Label         DF      Estimate          Error    t Value    Pr > |t|

Intercept     Intercept      1       0.16151        0.15782       1.02      0.3130
Comp2         afrate         1       0.10946        0.09113       1.20      0.2375
Comp1         usage_2        1       0.00584        0.16821       0.03      0.9725
Comp3         atusage        1       0.17109        0.29844       0.57      0.5700


                        Parameter Estimates

          Variable    Label              DF      95% Confidence Limits

          Intercept   Intercept           1      -0.15856        0.48158
          Comp2       afrate              1      -0.07536        0.29428
          Comp1       usage_2             1      -0.33532        0.34699
          Comp3       atusage             1      -0.43419        0.77636
                            The REG Procedure
                              Model: MODEL1
                  Dependent Variable: F4 failureintensity

              Durbin-Watson D                1.799
              Number of Observations            40
              1st Order Autocorrelation      0.051
```

Table 5.1. Estimates of failure intensity coefficient - SAS Output

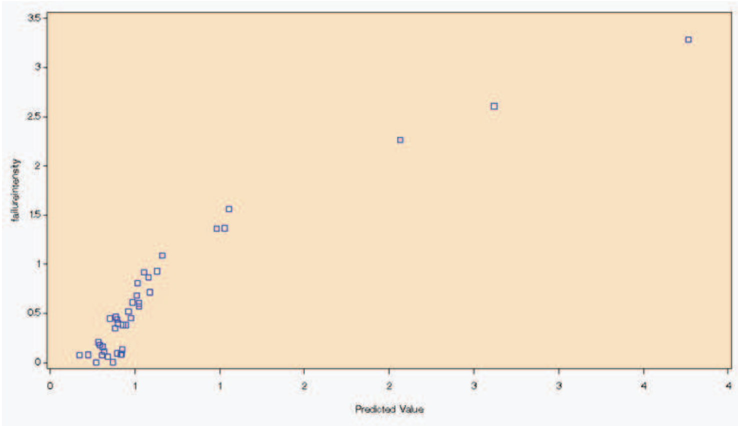


Figure 5.3. Predicted value versus failure intensity

obtain the expected costs.

$$\lambda(t \mid r) = 0.16151 + 0.10946r + 0.00584U^*(t)^2 + 0.17109A^*(t)U^*(t) \tag{5.5}$$

Figure 5.3 presents the plot of predicted values versus failure intensity function, and we do not observe any abnormalities suggesting that the model is a good fit. Figure 5.4 gives the residual plot of the predicted values and we see there is no pattern or denseness of data, and the normal probability plot of the residuals in the model given by figure 5.5 suggests no nonlinearities, thus satisfying the assumptions of the regression model. In the next section we derive the analytical expressions for expected costs using the failure intensity function obtained above and a uniform usage rate distribution.

The 2D warranty region is sub-divided into three regions namely Ω_1, Ω_2, and Ω_3 such that $\Omega_1 \cup \Omega_2 \cup \Omega_3 = \Omega$ and $\Omega_1 \cap \Omega_2 \cap \Omega_1 = \phi$, where ϕ denotes null set. We consider two servicing strategies based on ideal placement of replacement option in either Ω_2 or Ω_3 and minimizing the expected cost for servicing. In the subsequent sections we shall derive expected costs for the 2D model for restricted and unrestricted usage rate.

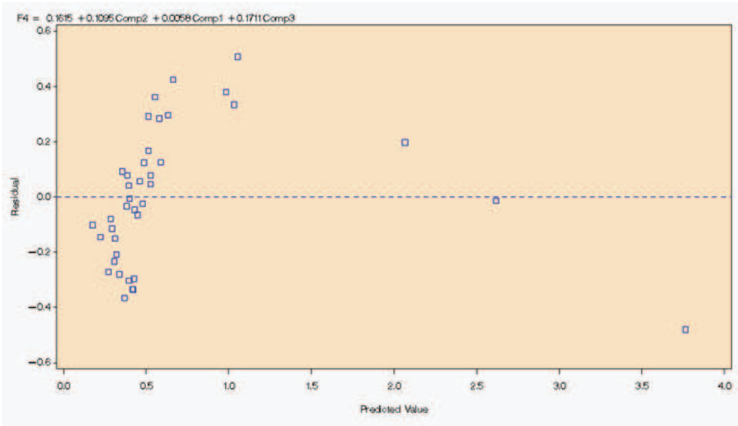


Figure 5.4. Residuals versus predicted values

5.3 Expected costs for servicing warranty for Unrestricted Case

The parameter set in the unrestricted case involves four variables denoted by $\pi\{T1, T2, r1, r2\}$. There are six possible realizations involved in the unrestricted case as discussed in section 3.3.3. In each of the 6 realization we need to consider 4 subsumed possibilities for usage rate r for both strategy-A and strategy-B.

5.3.1 Strategy-A

All product failures in $\Omega1$ is minimally repaired at prorated cost given by $\phi_1(t, x)$, where $(t, x)\epsilon\,\Omega1$. The first failure in region $\Omega2$ the manufacturer replaces the product which incurs a cost of S_1 and subsequent product failures are minimally repaired at prorated cost given by $\phi_2(t, x)$ where $(t, x)\epsilon\,\Omega2$. All product failures in $\Omega3$ are minimally repaired at prorated cost given by $\phi_3(t, x)$,where $(t, x)\epsilon\,\Omega3$.

5.3.1.1 Case-1: $r_1 \leq r_2 \leq r_3$

There are four possibilities for average usage rate r in the realization $r_1 \leq r_2 \leq r_3$, namely $(i)r \leq r_1$, $(ii)r_1 \leq r \leq r_2$, $(iii)r_2 \leq r \leq r_3$, $(iv)r_3 \leq r$.

We begin to derive expected costs for subcase $(i)r \leq r_1$ and extend the analysis for all subcases subsequently. Let$E^{\Omega_1}[\pi|r]$ denote the conditional expected cost for failures in region Ω_1, which depends on the usage rate in the region Ω_1 and is given

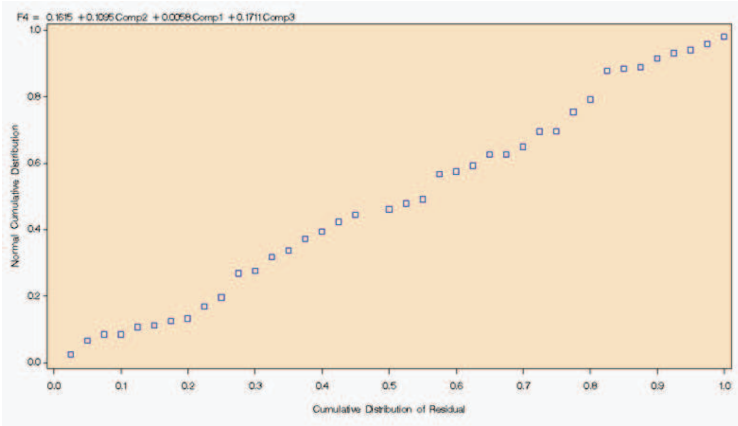


Figure 5.5. Normal Probability Plot of the residuals

by,

$$E^{\Omega_1}[\pi|r] = \begin{cases} \int_0^{T1}\int_0^{U1} \phi_1(t,x)\lambda(x|r)dx\,dt & ,\forall\,(t,x)\ \in \Omega_1 \\ 0, \quad otherwise \end{cases} \tag{5.1}$$

The conditional expected cost in region Ω_2 depends on whether the first failure since purchase happens in the Ω_2 region, i.e. beyond $T1$. Define $S_{T1|r}$ as the r.v. of first failure after T_1 and $F_{S_{T1|r}}$ the conditional CDF. Then the distribution of first failure after $T1$ is given by,

$$\begin{aligned} F_{S_{T1|r}}(t) &= \int_{T1}^{t} \lambda(x|r)\, e^{[\Lambda(T1|r)-\Lambda(x|r)]}dx \\ f_{S_{T1|r}}(t) &= \lambda(t|r)\, e^{[\Lambda(T1|r)-\Lambda(t|r)]} \end{aligned} \tag{5.2}$$

where $\Lambda(t|r) = \int_0^t \lambda(x|r)\,dx$ denotes the cumulative failure function between $(0,t)$.

Based on the conditional failure densities given above we have two possible cases i.e.$S_{T1|r} \leq T_2$ and $S_{T1|r} > T_2$. The conditional expected cost for failures in region Ω_2

is given by:

$$E^{\Omega_2}[\pi|r] = S_1 \int_{T1}^{T2} \lambda(t|r)\, e^{[\Lambda(T1|r)-\Lambda(t|r)]}dt + \int_{T1}^{T2} \left(\int_{t}^{U2} \phi_2(t,x)\lambda(x-t|r)\,dx \right) \lambda(t|r)\, e^{[\Lambda(T1|r)-\Lambda(t|r)]}dt, \quad \forall (t,x) \in \Omega_2$$

$$0\,, \quad otherwise \tag{5.3}$$

The conditional expected cost in the region Ω_3 depends on two possibilities: whether the first failure beyond $T1$ happens in Ω_2 i.e. $T1 \leq S_{T1|r} \leq T2$ and $S_{T1|r} > T2$.

$$\begin{aligned} E^{\Omega_3}[\pi|r] = &\left(\int_{T2}^{T} \lambda(x-t|r)\,dx \right) \lambda(t|r)\, e^{[\Lambda(T1|r)-\Lambda(t|r)]}dt, \;\; T1 \;\leq\; S_{T1|r} \;\leq T2 \\ &\int_{T2}^{T} \phi_3(t,x)\,\lambda(x|r)e^{[\Lambda(T2|r)-\Lambda(T1|r)]}dx, \;\; S_{T2|r} > T2 \end{aligned} \tag{5.4}$$

Using equations (5.2), (5.3) and (5.4) we obtain the conditional expected costs for $r \leq r_1$:

$$\begin{aligned} E^{(i)}[\pi|r] = &\int_{0}^{T1}\int_{0}^{U1} \phi_1(t,x)\lambda(x|r)dtdx + \int_{T2}^{T}\int_{U2}^{U} \phi_3(t,x)\lambda(x|r)e^{[\Lambda(T2|r)-\Lambda(T1|r)]}dtdx + \\ &\int_{T1}^{T2} \lambda(x|r)e^{[\Lambda(T1|r)-\Lambda(t|r)]}dt \left[S_1 + \int_{t}^{U2} \phi_2(t,x)\lambda(x-t|r)\,dx \right] \end{aligned} \tag{5.5}$$

For subcase (ii) defined by $r_1 \leq r \leq r_2$, the warranty duration in Ω_1 expires due to excess usage i.e. the warranty expires due to exceeding usage limit $U1$ rather than the age limit $T1$. Therefore we replace $T1$ the limit on time duration for warranty expiry in Ω_1 with $\tau_1 = \frac{U1}{r}$ in equation 5.5 to obtain:

$$\begin{aligned} E^{(ii)}[\pi|r] = &\int_{0}^{\tau 1}\int_{0}^{U1} \phi_1(t,x)\lambda(x|r)dtdx + \int_{T2}^{T}\int_{U2}^{U} \phi_3(t,x)\lambda(x|r)e^{[\Lambda(T2|r)-\Lambda(\tau 1|r)]}dtdx + \\ &\int_{\tau 1}^{T2} \lambda(x|r)e^{[\Lambda(\tau 1|r)-\Lambda(t|r)]}dt \left[S1 + \int_{t}^{U2} \phi_2(t,x)\lambda(x-t|r)dx \right] \end{aligned} \tag{5.6}$$

In subcase (iii), defined by $r_2 \leq r \leq r_3$, the warranty duration expires due to

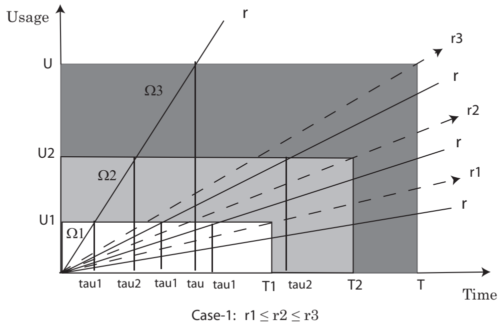


Figure 5.6. Realization of Unrestricted Strategy case-1

excess usage in region Ω_1 and Ω_2 . Therefore we replace $T2$, the limit on time duration for warranty expiry in Ω_2, with $\tau_2 = \frac{U2}{r}$ in equation 5.6 to obtain:

$$\begin{aligned} E^{(iii)}[\pi|r] = &\int_0^{\tau 1}\int_0^{U1} \phi_1(t,x)\lambda(x|r)dtdx + \int_{\tau 2}^{T}\int_{U2}^{U} \phi_3(t,x)\lambda(x|r)e^{[\Lambda(\tau 2|r)-\Lambda(\tau 1|r)]}dtdx + \\ &\int_{\tau 1}^{\tau 2} \lambda(x|r)e^{[\Lambda(\tau 1|r)-\Lambda(t|r)]}dt\left[S1 + \int_t^{U2}\phi_2(t,x)\lambda(x-t|r)dx\right] \end{aligned} \tag{5.7}$$

For subcase (iv), defined by $r_3 \leq r$, the warranty duration expires due to excess usage in all the three regions Ω_1, Ω_2 and Ω_3. Therefore we replace T, the limit on time duration for warranty expiry in Ω_3 with $\tau = \frac{U}{r}$ in equation 5.7:

$$\begin{aligned} E^{(iv)}[\pi|r] = &\int_0^{\tau 1}\int_0^{U1} \phi_1(t,x)\lambda(x|r)dtdx + \int_{\tau 2}^{\tau}\int_{U2}^{U} \phi_3(t,x)\lambda(x|r)e^{[\Lambda(\tau 2|r)-\Lambda(\tau 1|r)]}dtdx + \\ &\int_{\tau 1}^{\tau 2} \lambda(x|r)e^{[\Lambda(\tau 1|r)-\Lambda(t|r)]}dt\left[S1 + \int_t^{U2}\phi_2(t,x)\lambda(x-t|r)dx\right] \end{aligned} \tag{5.8}$$

Upon removing the condition on usage rate in equations (5.5),(5.6), (5.7), and (5.8), we obtain the expected cost for warranty servicing given by,

$$\begin{aligned} E[\pi] = &\int_0^{r_1} E^{(i)}[\pi|r] * R(r)dr + \int_{r_1}^{r_2} E^{(ii)}[\pi|r] * R(r)dr \\ &+ \int_{r_2}^{r_3} E^{(iii)}[\pi|r] * R(r)dr + \int_{r_3}^{\infty} E^{(iv)}[\pi|r] * R(r)dr \end{aligned} \tag{5.9}$$

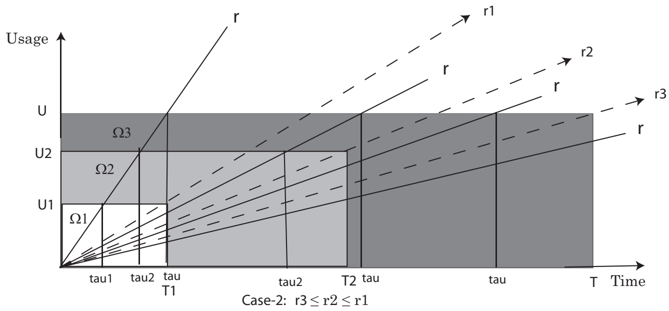


Figure 5.7. Realization of Unrestricted Strategy case-2

In general, the conditional expected cost for each subcase is given by,

$$E^{(\gamma)}_{\{a,b,c\}}[\pi|r] = \int_0^a \int_0^{U1} \phi_1(t,x)\lambda(x|r)dtdx + \int_b^c \int_{U2}^{U} \phi_3(t,x)\lambda(x|r)e^{[\Lambda(b|r)-\Lambda(a|r)]}dtdx + \int_a^b \lambda(x|r)e^{[\Lambda(a|r)-\Lambda(t|r)]}dt\left[S1 + \int_t^{U2} \phi_2(t,x)\lambda(x-t|r)dx\right] \tag{5.10}$$

where γ corresponds to $\{(i),(ii),(iii),(iv)\}$ and set $\{a,b,c\}$ refers to $\{T1,T2,T\}$. We shall use the general equation given by (5.10) to obtain the conditional expected cost for all the subcases in the following cases.

5.3.1.2 Case-2: $r_3 \leq r_2 \leq r_1$

The average usage rate r in this realization can be in the following subcases: $(i)r \leq r_3$, $(ii)r_3 \leq r \leq r_2$, $(iii)r_2 \leq r \leq r_1$, $(iv)r_1 \leq r$.

The equation for the first subcase (i) in all the 6 cases will be the same and is obtained from equation (5.10), given by $E^{(\gamma=i)}_{\{a=T1,b=T2,c=T\}}[\pi|r]$

The expected cost for subcase (ii), defined by $r_3 \leq r \leq r_2$, is obtained from equation (5.10) and given by $E^{(\gamma=ii)}_{\{a,b,c=\tau\}}[\pi|r]$ as the warranty in region Ω_3 expires due to exceeding usage limit U.

The expected cost for subcase (iii), defined by $r_2 \leq r \leq r_1$ is obtained from equation (5.10) and given by $E^{(\gamma=iii)}_{\{a,b=\tau_2,c=\tau\}}[\pi|r]$ as the warranty in region Ω_2 and Ω_3 expires due to exceeding usage limits $U2$ and U respectively.

The expected cost for subcase (iv) will be the same as equation (5.8) given by

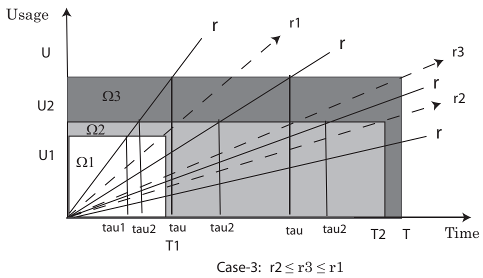


Figure 5.8. Realization of Unrestricted Strategy case-3

$E^{(\gamma=iv)}_{\{a=\tau_1,b=\tau_2,c=\tau\}}[\pi|r]$. Subcase (iv) for all 6 cases will be the same as warranty expires due to average usage rate exceeding limits $U1$,$U2$ and U in the regions defined by Ω_1,Ω_2 and Ω_3 .

On removing the condition on usage rate the expected cost for warranty servicing given by,

$$\begin{aligned} E[\pi] &= \int_0^{r_3} E^{(i)}[\pi|r] * R(r)dr + \int_{r_3}^{r_2} E^{(ii)}[\pi|r] * R(r)dr \\ &+ \int_{r_2}^{r_1} E^{(iii)}[\pi|r] * R(r)dr + \int_{r_1}^{\infty} E^{(iv)}[\pi|r] * R(r)dr \end{aligned} \tag{5.11}$$

5.3.1.3 Case-3: $r_2 \leq r_3 \leq r_1$

There are four possibilities for average usage rate r in the realization $r_2 \leq r3 \leq r_1$, namely $(i)r \leq r_2$, $(ii)r_2 \leq r \leq r_3$, $(iii)r_3 \leq r \leq r_1$, $(iv)r_1 \leq r$.

The conditional expected cost for subcase (i) defined by $r \leq r_2$ is obtained from equation (5.10) given by $E^{(\gamma=i)}_{\{a=T1,b=T2,c=T\}}[\pi|r]$.

For subcase (ii), defined by $r_2 \leq r \leq r_3$, the warranty duration in Ω_2 expires due to excess usage, i.e. the warranty expires due to exceeding usage limit rather than the age. Therefore we replace $T2$, the limit on time duration for warranty expiry in Ω_2, with $\tau_2 = \frac{U2}{r}$. The conditional expectation is obtained from equation (5.10) and given by $E^{(\gamma=ii)}_{\{a,b=\tau_2,c\}}[\pi|r]$.

In subcase (iii) defined by $r_3 \leq r \leq r_1$, the warranty duration expires due to excess usage and in region Ω_2 and Ω_3 . Therefore, we replace $T2$, the limit on time duration for warranty expiry in Ω_2, with $\tau_2 = \frac{U2}{r}$, and T, the limit on time duration for warranty expiry in Ω_3, with $\tau = \frac{U}{r}$. The conditional expectation is obtained

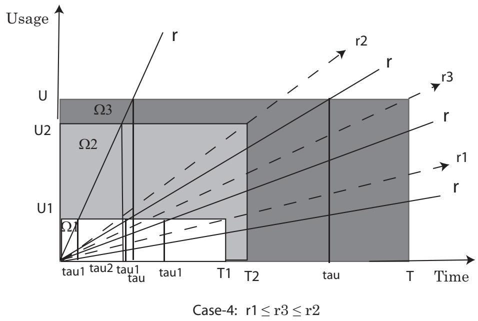


Figure 5.9. Realization of Unrestricted Strategy case-4

from equation (5.10) and given by $E^{(\gamma=iii)}_{\{a,b=\tau_2,c=\tau\}}[\pi|r]$.

For subcase (iv), defined by $r_1 \leq r$, the warranty duration expires due to excess usage in all the three regions Ω_1, Ω_2 and Ω_3. The conditional expectation is obtained from equation (5.10) and given by $E^{(\gamma=iv)}_{\{a=\tau_1,b=\tau_2,c=\tau\}}[\pi|r]$ (same as equation (5.8)).

Upon removing the condition on usage rate the expected cost for warranty servicing given by,

$$
\begin{aligned}
E[\pi] = &\int_0^{r_2} E^{(i)}[\pi|r] * R(r)dr + \int_{r_2}^{r_3} E^{(ii)}[\pi|r] * R(r)dr \\
&+ \int_{r_3}^{r_1} E^{(iii)}[\pi|r] * R(r)dr + \int_{r_1}^{\infty} E^{(iv)}[\pi|r] * R(r)dr
\end{aligned}
\tag{5.12}
$$

5.3.1.4 Case-4: $r_1 \leq r_3 \leq r_2$

There are four possibilities for average usage rate r in the realization $r_1 \leq r_3 \leq r_2$, namely $(i) r \leq r_1$, $(ii) r_1 \leq r \leq r_3$, $(iii) r_3 \leq r \leq r_2$, $(iv) r_2 \leq r$.

The conditional expected cost for subcase (i) defined by $r \leq r_1$ is obtained from equation (5.10) given by $E^{(\gamma=i)}_{\{a=T1,b=T2,c=T\}}[\pi|r]$.

For subcase (ii), defined by $r_1 \leq r \leq r_3$, the warranty duration in Ω_1 expires due to excess usage, i.e. the warranty expires due to exceeding usage limit rather than the age. Therefore we replace $T1$, the limit on time duration for warranty expiry in Ω_1, with $\tau_1 = \frac{U1}{r}$. The conditional expectation is obtained from equation (5.10) and is given by $E^{(\gamma=ii)}_{\{a=\tau_1,b,c\}}[\pi|r]$.

In subcase (iii), defined by $r_3 \leq r \leq r_2$, the warranty duration expires due to excess usage in region Ω_2 and Ω_3. Therefore we replace T, the limit on time

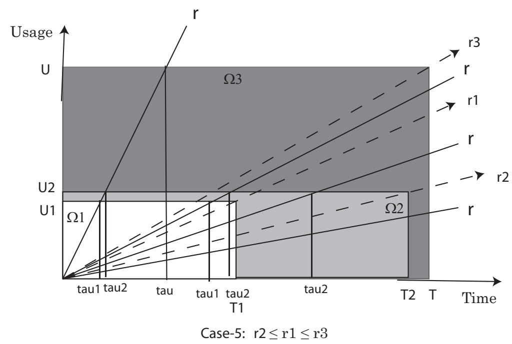


Figure 5.10. Realization of Unrestricted Strategy case-5

duration for warranty expiry in Ω_3, with $\tau = \frac{U}{r}$, and replace $T1$, the limit on time duration for warranty expiry in Ω_1, with $\tau_1 = \frac{U1}{r}$. The conditional expectation is obtained from equation (5.10) and given by $E^{(\gamma=iii)}_{\{a=\tau_1,b,c=\tau\}}[\pi|r]$.

For subcase (iv), defined by $r_2 \leq r$, the warranty duration expires due to excess usage in all the three regions Ω_1, Ω_2 and Ω_3. The conditional expectation is obtained from equation (5.10) and given by $E^{(\gamma=iv)}_{\{a=\tau_1,b=\tau_2,c=\tau\}}[\pi|r]$ (same as equation (5.8)).

Upon removing the condition on usage rate the expected cost for warranty servicing given by,

$$\begin{aligned} E[\pi] = &\int_0^{r_1} E^{(i)}[\pi|r] * R(r)dr + \int_{r_1}^{r_3} E^{(ii)}[\pi|r] * R(r)dr \\ &+ \int_{r_3}^{r_2} E^{(iii)}[\pi|r] * R(r)dr + \int_{r_2}^{\infty} E^{(iv)}[\pi|r] * R(r)dr \end{aligned} \tag{5.13}$$

5.3.1.5 Case-5: $r_2 \leq r_1 \leq r_3$

There are four possibilities for average usage rate r in the realization $r_2 \leq r_1 \leq r_3$, namely $(i) r \leq r_2$, $(ii) r_2 \leq r \leq r_1$, $(iii) r_1 \leq r \leq r_3$, $(iv) r_3 \leq r$.

The conditional expected cost for subcase (i), defined by $r \leq r_2$, is obtained from equation (5.10) given by $E^{(\gamma=i)}_{\{a=T1,b=T2,c=T\}}[\pi|r]$.

For subcase (ii), defined by $r_2 \leq r \leq r_1$, the warranty duration in Ω_2 expires due to excess usage, i.e. the warranty expires due to exceeding usage limit rather than the age. Therefore we replace $T2$, the limit on time duration for warranty expiry in Ω_2, with $\tau_2 = \frac{U2}{r}$. The conditional expectation is obtained from equation (5.10) and given by $E^{(\gamma=ii)}_{\{a,b=\tau_2,c\}}[\pi|r]$.

In subcase (iii), defined by $r_1 \leq r \leq r_3$, the warranty duration expires due to excess usage in region Ω_1 and Ω_2. Therefore, we replace $T1$, the limit on time duration for warranty expiry in Ω_1, with $\tau_1 = \frac{U1}{r}$, and replace $T2$, the limit on time duration for warranty expiry in Ω_2, with $\tau_2 = \frac{U2}{r}$. The conditional expectation is obtained from equation (5.10) and given by $E^{(\gamma=iii)}_{\{a=\tau_1,b=\tau_2,c\}}[\pi|r]$.

For subcase (iv), defined by $r_3 \leq r$, the warranty duration expires due to excess usage in all the three regions Ω_1, Ω_2 and Ω_3. The conditional expectation is obtained from equation (5.10) given by $E^{(\gamma=iv)}_{\{a=\tau_1,b=\tau_2,c=\tau\}}[\pi|r]$ (same as equation (5.8)).

Upon removing the condition on usage rate, the expected cost for warranty servicing given by,

$$\begin{aligned} E[\pi] = &\int_0^{r_2} E^{(i)}[\pi|r] * R(r)dr + \int_{r_2}^{r_1} E^{(ii)}[\pi|r] * R(r)dr \\ &+ \int_{r_1}^{r_3} E^{(iii)}[\pi|r] * R(r)dr + \int_{r_3}^{\infty} E^{(iv)}[\pi|r] * R(r)dr \end{aligned} \tag{5.14}$$

5.3.1.6 Case-6: $r_3 \leq r_1 \leq r_2$

There are four possibilities for average usage rate r in the realization $r_3 \leq r_1 \leq r_2$, namely $(i) r \leq r_3$, $(ii) r_3 \leq r \leq r_1$, $(iii) r_1 \leq r \leq r_2$, $(iv) r_2 \leq r$.

The conditional expected cost for subcase (i), defined by $r \leq r_3$, is obtained from equation (5.10) given by $E^{(\gamma=i)}_{\{a=T1,b=T2,c=T\}}[\pi|r]$.

For subcase (ii), defined by $r_3 \leq r \leq r_1$, the warranty duration in Ω_3 expires due to excess usage, i.e. the warranty expires due to exceeding usage limit rather than the age. Therefore, we replace T, the limit on time duration for warranty expiry in Ω_3, with $\tau = \frac{U}{r}$. The conditional expectation is obtained from equation (5.10) and given by $E^{(\gamma=ii)}_{\{a,b,c=\tau\}}[\pi|r]$.

In subcase (iii), defined by $r_1 \leq r \leq r_2$, the warranty duration expires due to excess usage in region Ω_1 and Ω_3. Therefore, we replace $T1$, the limit on time duration for warranty expiry in Ω_1, with $\tau_1 = \frac{U1}{r}$, and replace T, the limit on time duration for warranty expiry in Ω_3, with $\tau = \frac{U}{r}$. The conditional expectation is obtained from equation (5.10) and given by $E^{(\gamma=iii)}_{\{a=\tau_1,b,c=\tau\}}[\pi|r]$.

For subcase (iv), defined by $r_2 \leq r$, the warranty duration expires due to excess usage in all the three regions Ω_1, Ω_2 and Ω_3. The conditional expectation is

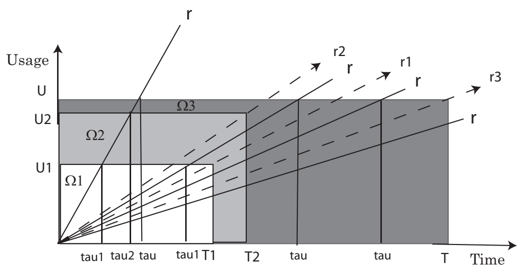


Figure 5.11. Realization of Unrestricted Strategy case-6

obtained from equation (5.10) and given by equation $E^{(\gamma=iv)}_{\{a=\tau_1,b=\tau_2,c=\tau\}}[\pi|r]$ (same as equation (5.8)).

Upon removing the condition on usage rate, the expected cost for warranty servicing given by,

$$\begin{aligned} E[\pi] = \int_0^{r_3} E^{(i)}[\pi|r] * R(r)dr + \int_{r_3}^{r_1} E^{(ii)}[\pi|r] * R(r)dr \\ + \int_{r_1}^{r_2} E^{(iii)}[\pi|r] * R(r)dr + \int_{r_2}^{\infty} E^{(iv)}[\pi|r] * R(r)dr \end{aligned} \tag{5.15}$$

5.3.2 Strategy-B

All product failures in Ω1 are minimally repaired at prorated cost given by $\phi_1(t,x)$, where $(t,x)\epsilon\Omega 1$. All product failures in Ω2 is minimally repaired at prorated cost given by $\phi_2(t,x)$, where $(t,x)\epsilon\Omega 2$. The first failure in region Ω3 is replaced by the manufacturer a cost of S_1, and subsequent product failures are minimally repaired at prorated cost given by $\phi_3(t,x)$, where $(t,x)\epsilon\Omega 3$.

5.3.2.1 Case-1: $r_1 \le r_2 \le r_3$

There are four possibilities for average usage rate r in the realization $r_1 \le r_2 \le r_3$, is $(i) r \le r_1$, $(ii) r_1 \le r \le r_2$, $(iii) r_2 \le r \le r_3$, $(iv) r_3 \le r$.

We shall derive expected costs for subcase $(i) r \le r_1$ and extend the analysis for all subcases previously done for strategy-A. Since the manufacturer exercises replacement option only in Ω_3 region, the conditional expectations in Ω_1 and Ω_2 are straight forward.

Let $E^{\Omega_1}[\pi|r]$ denote the conditional expected cost for failures in region Ω_1, which depends on the usage rate in the region Ω_1, and is given by,

$$E^{\Omega_1}[\pi|r] = \begin{array}{l} \int\limits_0^{T1}\int\limits_0^{U1} \phi_1(t,x)\lambda(x|r)dx\,dt \ \ ,\forall\,(t,x) \ \in \Omega_1 \\ 0, \quad otherwise \end{array} \tag{5.16}$$

The conditional expected costs in Ω_2 is given by,

$$E^{\Omega_2}[\pi|r] = \begin{array}{l} \int\limits_{T1}^{T2}\int\limits_{U1}^{U2} \phi_2(t,x)\lambda(x|r)dx\,dt \ \ ,\forall\,(t,x) \ \in \Omega_2 \\ 0, \quad otherwise \end{array} \tag{5.17}$$

The conditional expected costs in Ω_3 depends on whether the first failure in Ω_3 occurs or not, which results in replacement at a cost $S1$. The conditional expected costs in Ω_3 is then given by,

$$E^{\Omega_3}[\pi|r] = \begin{array}{l} S_1 \int\limits_{T2}^{T} \lambda(t|r)\, e^{[\Lambda(T2|r)-\Lambda(t|r)]}dt + \int\limits_{T2}^{T}\left(\int\limits_{t}^{U}\phi_3(t,x)\lambda(x-t|r)\,dx\right)\lambda(t|r)\,e^{[\Lambda(T2|r)-\Lambda(t|r)]}dt, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall(t,x) \ \in \Omega_2 \\ 0\,, \quad otherwise \end{array} \tag{5.18}$$

Using equation (5.16), (5.17), (5.18), we obtain the conditional expected cost for subcase $(i) r \le r_1$:

$$\begin{array}{l} E^{(i)}[\pi|r] = \int\limits_0^{T1}\int\limits_0^{U1} \phi_1(t,x)\lambda(x|r)dx\,dt + \int\limits_{T1}^{T2}\int\limits_{U1}^{U2} \phi_2(t,x)\lambda(x|r)dx\,dt + \\ \int\limits_{T2}^{T} \lambda(x|r)e^{[\Lambda(T2|r)-\Lambda(t|r)]}dt\left[S1 + \int\limits_{t}^{U}\phi_3(t,x)\lambda(x-t|r)dx\right] \end{array} \tag{5.19}$$

For subcase (ii), defined by $r_1 \le r \le r_2$, the warranty duration in Ω_1 expires due to excess usage i.e. the warranty expires due to exceeding usage limit $U1$ rather than the age limit $T1$. Therefore we replace $T1$, the limit on time duration for warranty expiry in Ω_1, with $\tau_1 = \frac{U1}{r}$ in equation (5.19) to obtain:

$$\begin{aligned} E^{(ii)}[\pi|r] = &\int_0^{\tau 1}\int_0^{U1} \phi_1(t,x)\lambda(x|r)dx\,dt + \int_{\tau 1}^{T2}\int_{U1}^{U2} \phi_2(t,x)\lambda(x|r)dx\,dt + \\ &\int_{T2}^{T} \lambda(x|r)e^{[\Lambda(T2|r)-\Lambda(t|r)]}dt\left[S1 + \int_t^U \phi_3(t,x)\lambda(x-t|r)dx\right] \end{aligned} \tag{5.20}$$

In subcase (iii), defined by $r_2 \leq r \leq r_3$, the warranty duration expires due to excess usage in region Ω_1 and Ω_2 . Therefore, we replace $T2$, the limit on time duration for warranty expiry in Ω_2, with $\tau_2 = \frac{U2}{r}$ in equation (5.20) to obtain:

$$\begin{aligned} E^{(iii)}[\pi|r] = &\int_0^{\tau 1}\int_0^{U1} \phi_1(t,x)\lambda(x|r)dx\,dt + \int_{\tau 1}^{\tau 2}\int_{U1}^{U2} \phi_2(t,x)\lambda(x|r)dx\,dt + \\ &\int_{\tau 2}^{T} \lambda(x|r)e^{[\Lambda(\tau 2|r)-\Lambda(t|r)]}dt\left[S1 + \int_t^U \phi_3(t,x)\lambda(x-t|r)dx\right] \end{aligned} \tag{5.21}$$

For subcase (iv), defined by $r_3 \leq r$, the warranty duration expires due to excess usage in all the three regions Ω_1, Ω_2 and Ω_3. Therefore, we replace T, the limit on time duration for warranty expiry in Ω_3, with $\tau = \frac{U}{r}$ in equation 5.7:

$$\begin{aligned} E^{(iv)}[\pi|r] = &\int_0^{\tau 1}\int_0^{U1} \phi_1(t,x)\lambda(x|r)dx\,dt + \int_{\tau 1}^{\tau 2}\int_{U1}^{U2} \phi_2(t,x)\lambda(x|r)dx\,dt + \\ &\int_{\tau 2}^{\tau} \lambda(x|r)e^{[\Lambda(\tau 2|r)-\Lambda(t|r)]}dt\left[S1 + \int_t^U \phi_3(t,x)\lambda(x-t|r)dx\right] \end{aligned} \tag{5.22}$$

Upon removing the condition on usage rate in equations (5.19),(5.20), (5.21), and (5.22), we obtain the expected cost for warranty servicing given by,

$$\begin{aligned} E[\pi] = &\int_0^{r_1} E^{(i)}[\pi|r] * R(r)dr + \int_{r_1}^{r_2} E^{(ii)}[\pi|r] * R(r)dr \\ &+ \int_{r_2}^{r_3} E^{(iii)}[\pi|r] * R(r)dr + \int_{r_3}^{\infty} E^{(iv)}[\pi|r] * R(r)dr \end{aligned} \tag{5.23}$$

In general, the conditional expected cost for each subcase is given by,

$$\begin{aligned} E^{(\gamma)}_{\{a,b,c\}}[\pi|r] = &\int_0^{a}\int_0^{U1} \phi_1(t,x)\lambda(x|r)dx\,dt + \int_{a}^{b}\int_{U1}^{U2} \phi_2(t,x)\lambda(x|r)dx\,dt + \\ &\int_{b}^{c} \lambda(x|r)e^{[\Lambda(b|r)-\Lambda(t|r)]}dt\left[S1 + \int_t^U \phi_3(t,x)\lambda(x-t|r)dx\right] \end{aligned} \tag{5.24}$$

where γ corresponds to $\{(i),(ii),(iii),(iv)\}$ and set $\{a,b,c\}$ refers to $\{T1,T2,T\}$. We shall use the general equation given by (5.24) to obtain the conditional expected cost for all subcases in the following cases.

5.3.2.2 Case-2: $r_3 \le r_2 \le r_1$

The average usage rate r in this realization can be in the following subcases: $(i)r \le r_3$, $(ii)r_3 \le r \le r_2$, $(iii)r_2 \le r \le r_1$, $(iv)r_1 \le r$.

The equation for the first subcase (i) in all the 6 cases will be the same and is given by $E^{(\gamma=i)}_{\{a=T1,b=T2,c=T\}}[\pi|r]$ (same as equation (5.19))

The expected cost for subcase (ii), defined by $r_3 \le r \le r_2$ is obtained from equation (5.24) and is given by $E^{(\gamma=ii)}_{\{a,b,c=\tau\}}[\pi|r]$ as the warranty in region Ω_3 expires due to exceeding usage limit U.

The expected cost for subcase (iii), defined by $r_2 \le r \le r_1$ is obtained from equation (5.24) and is given by $E^{(\gamma=iii)}_{\{a,b=\tau_2,c=\tau\}}[\pi|r]$ as the warranty in region Ω_2 and Ω_3 expires due to exceeding usage limits $U2$ and U respectively.

The expected cost for subcase (iv), defined by $r_1 \le r$, the warranty duration expires due to excess usage in all the three regions Ω_1, Ω_2 and Ω_3. The conditional expectation is obtained from equation (5.24) and given by $E^{(\gamma=iv)}_{\{a=\tau_1,b=\tau_2,c=\tau\}}[\pi|r]$ (same as equation (5.22)).

On removing the condition on usage rate, the expected cost for warranty servicing is given by,

$$\begin{aligned} E[\pi] = &\int_0^{r_3} E^{(i)}[\pi|r] * R(r)dr + \int_{r_3}^{r_2} E^{(ii)}[\pi|r] * R(r)dr \\ &+ \int_{r_2}^{r_1} E^{(iii)}[\pi|r] * R(r)dr + \int_{r_1}^{\infty} E^{(iv)}[\pi|r] * R(r)dr \end{aligned} \tag{5.25}$$

5.3.2.3 Case-3: $r_2 \le r_3 \le r_1$

There are four possibilities for average usage rate r in the realization $r_2 \le r_3 \le r_1$, namely $(i)r \le r_2$, $(ii)r_2 \le r \le r_3$, $(iii)r_3 \le r \le r_1$, $(iv)r_1 \le r$.

The conditional expected cost for subcase (i), defined by $r \le r_2$, is given by $E^{(\gamma=i)}_{\{a=T1,b=T2,c=T\}}[\pi|r]$ (same as equation (5.19)).

For subcase (ii), defined by $r_2 \le r \le r_3$, the warranty duration in Ω_2 expires due to excess usage, i.e. the warranty expires due to exceeding usage limit rather than the age. Therefore, we replace $T2$, the limit on time duration for warranty expiry in Ω_2, with $\tau_2 = \frac{U2}{r}$. The conditional expectation is obtained from equation (5.24) and given by $E^{(\gamma=ii)}_{\{a,b=\tau_2,c\}}[\pi|r]$.

In subcase (iii), defined by $r_3 \le r \le r_1$, the warranty duration expires due to excess usage in region Ω_2 and Ω_3. Therefore, we replace $T2$, the limit on time duration for warranty expiry in Ω_2 with $\tau_2 = \frac{U2}{r}$, and T, the limit on time duration for warranty expiry in Ω_3, with $\tau = \frac{U}{r}$. The conditional expectation is obtained from equation (5.24) and given by $E^{(\gamma=iii)}_{\{a,b=\tau_2,c=\tau\}}[\pi|r]$.

For subcase (iv), defined by $r_1 \le r$, the warranty duration expires due to excess usage in all the three regions Ω_1, Ω_2 and Ω_3. The conditional expectation is obtained from equation (5.24) and given by $E^{(\gamma=iv)}_{\{a=\tau_1,b=\tau_2,c=\tau\}}[\pi|r]$ (same as equation (5.22)).

Upon removing the condition on usage rate the expected cost for warranty servicing is given by,

$$\begin{aligned} E[\pi] = & \int_0^{r_2} E^{(i)}[\pi|r] * R(r)dr + \int_{r_2}^{r_3} E^{(ii)}[\pi|r] * R(r)dr \\ & + \int_{r_3}^{r_1} E^{(iii)}[\pi|r] * R(r)dr + \int_{r_1}^{\infty} E^{(iv)}[\pi|r] * R(r)dr \end{aligned} \tag{5.26}$$

5.3.2.4 Case-4: $r_1 \le r_3 \le r_2$

There are four possibilities for average usage rate r in the realization $r_1 \le r_3 \le r_2$, namely $(i) r \le r_1$, $(ii) r_1 \le r \le r_3$, $(iii) r_3 \le r \le r_2$, $(iv) r_2 \le r$.

The conditional expected cost for subcase (i), defined by $r \le r_1$, is given by $E^{(\gamma=i)}_{\{a=T1,b=T2,c=T\}}[\pi|r]$ (same as equation (5.19)).

For subcase (ii), defined by $r_1 \le r \le r_3$, the warranty duration in Ω_1 expires due to excess usage, i.e. the warranty expires due to exceeding usage limit rather than the age. Therefore, we replace $T1$, the limit on time duration for warranty expiry in Ω_1, with $\tau_1 = \frac{U1}{r}$. The conditional expectation is obtained from equation (5.24) and is given by $E^{(\gamma=ii)}_{\{a=\tau_1,b,c\}}[\pi|r]$.

In subcase (iii), defined by $r_3 \le r \le r_2$, the warranty duration expires due to excess usage in region Ω_1 and Ω_3. Therefore, we replace T, the limit on time duration for warranty expiry in Ω_3, with $\tau = \frac{U}{r}$, and replace $T1$, the limit on time duration for warranty expiry in Ω_1, with $\tau_1 = \frac{U1}{r}$. The conditional expectation is obtained from equation (5.24) and given by $E^{(\gamma=iii)}_{\{a=\tau_1,b,c=\tau\}}[\pi|r]$.

For subcase (iv), defined by $r_1 \le r$, the warranty duration expires due to excess usage in all the three regions Ω_1, Ω_2 and Ω_3. The conditional expectation is given by equation $E^{(\gamma=iv)}_{\{a=\tau_1,b=\tau_2,c=\tau\}}[\pi|r]$ (same as equation (5.22)).

Upon removing the condition on usage rate, the expected cost for warranty servicing is given by,

$$\begin{aligned} E[\pi] &= \int_0^{r_1} E^{(i)}[\pi|r] * R(r)dr + \int_{r_1}^{r_3} E^{(ii)}[\pi|r] * R(r)dr \\ &+ \int_{r_3}^{r_2} E^{(iii)}[\pi|r] * R(r)dr + \int_{r_2}^{\infty} E^{(iv)}[\pi|r] * R(r)dr \end{aligned} \tag{5.27}$$

5.3.2.5 Case-5: $r_2 \le r_1 \le r_3$

There are four possibilities for the average usage rate r in the realization $r_2 \le r_1 \le r_3$, namely $(i) r \le r_2$, $(ii) r_2 \le r \le r_1$, $(iii) r_1 \le r \le r_3$, $(iv) r_3 \le r$.

The conditional expected cost for subcase (i), defined by $r \le r_2$, is given by $E^{(\gamma=i)}_{\{a,b,c\}}[\pi|r]$ (same as equation (5.19)).

For subcase (ii), defined by $r_2 \le r \le r_1$, the warranty duration in Ω_2 expires due to excess usage, i.e. the warranty expires due to exceeding usage limit rather than the age. Therefore, we replace $T2$, the limit on time duration for warranty expiry in Ω_2, with $\tau_2 = \frac{U2}{r}$. The conditional expectation is given by $E^{(\gamma=ii)}_{\{a,b=\tau_2,c\}}[\pi|r]$.

In subcase (iii), defined by $r_1 \le r \le r_3$, the warranty duration expires due to excess usage in region Ω_1 and Ω_2. Therefore, we replace $T1$, the limit on time duration for warranty expiry in Ω_1, with $\tau_1 = \frac{U1}{r}$, and replace $T2$, the limit on time duration for warranty expiry in Ω_2, with $\tau_2 = \frac{U2}{r}$. The conditional expectation is given by $E^{(\gamma=iii)}_{\{a=\tau_1,b=\tau_2,c\}}[\pi|r]$.

For subcase (iv), defined by $r_3 \le r$, the warranty duration expires due to excess usage in all the three regions Ω_1, Ω_2 and Ω_3. The conditional expectation is by equation $E^{(\gamma=iv)}_{\{a=\tau_1,b=\tau_2,c=\tau\}}[\pi|r]$ (same as equation (5.22)).

Upon removing the condition on usage rate, the expected cost for warranty servicing is given by,

$$\begin{aligned} E[\pi] &= \int_0^{r_2} E^{(i)}[\pi|r] * R(r)dr + \int_{r_2}^{r_1} E^{(ii)}[\pi|r] * R(r)dr \\ &+ \int_{r_1}^{r_3} E^{(iii)}[\pi|r] * R(r)dr + \int_{r_3}^{\infty} E^{(iv)}[\pi|r] * R(r)dr \end{aligned} \tag{5.28}$$

5.3.2.6 Case-6: $r_3 \leq r_1 \leq r_2$

There are four possibilities for average usage rate r in the realization $r_3 \leq r_1 \leq r_2$, namely $(i) r \leq r_3$, $(ii) r_3 \leq r \leq r_1$, $(iii) r_1 \leq r \leq r_2$, $(iv) r_2 \leq r$.

The conditional expected cost for subcase (i), defined by $r \leq r_3$, is given by $E^{(\gamma=i)}_{\{a=T1,b=T2,c=T\}}[\pi|r]$ (same as equation (5.19)).

For subcase (ii), defined by $r_3 \leq r \leq r_1$, the warranty duration in Ω_3 expires due to excess usage, i.e. the warranty expires due to exceeding usage limit rather than the age. Therefore, we replace T, the limit on time duration for warranty expiry in Ω_3, with $\tau = \frac{U}{r}$. The conditional expectation is given by $E^{(\gamma=ii)}_{\{a,b,c=\tau\}}[\pi|r]$.

In subcase (iii), defined by $r_1 \leq r \leq r_2$, the warranty duration expires due to excess usage in region Ω_1 and Ω_3. Therefore, we replace $T1$, the limit on time duration for warranty expiry in Ω_1, with $\tau_1 = \frac{U1}{r}$, and replace T, the limit on time duration for warranty expiry in Ω_3, with $\tau = \frac{U}{r}$. The conditional expectation is given by $E^{(\gamma=iii)}_{\{a=\tau_1,b,c=\tau\}}[\pi|r]$.

For subcase (iv), defined by $r_2 \leq r$, the warranty duration expires due to excess usage in all the three regions Ω_1, Ω_2 and Ω_3. The conditional expectation is by equation $E^{(\gamma=iv)}_{\{a=\tau_1,b=\tau_2,c=\tau\}}[\pi|r]$ (same as equation (5.22)).

Upon removing the condition on usage rate, the expected cost for warranty servicing is given by,

$$\begin{aligned} E[\pi] = \int_0^{r_3} E^{(i)}[\pi|r] * R(r)dr + \int_{r_3}^{r_1} E^{(ii)}[\pi|r] * R(r)dr \\ + \int_{r_1}^{r_2} E^{(iii)}[\pi|r] * R(r)dr + \int_{r_2}^{\infty} E^{(iv)}[\pi|r] * R(r)dr \end{aligned} \tag{5.29}$$

5.4 Expected costs for servicing warranty for Restricted Case

The parameter set in the restricted case is represented by $\pi\{T1, T2, r2\}$ and involves three variables. There are two possible realizations in the restricted case given by, $r_2(= r_1) < r_3$ and $r_2(= r_1) > r_3$. For each realization we need to consider three possibilities for the average usage rate r. In the next section we derive the expected costs for both strategy-A and strategy-B.

5.4.1 Strategy-A

5.4.1.1 Case-1: $r_2 (= r_1) < r_3$

When we consider the realization $r_2(= r_1) < r_3$, the three possibilities for average usage rate r is given by: $(i) r \leq r_2$, $(ii) r_2 < r < r_3$ and $(iii) r > r_3$. We begin deriving the expected costs for subcase $(i) r \leq r_2$ and extend the analysis to other subcases. Denoting $E^{\Omega_1}[\pi|r]$ as the conditional expected cost for servicing product failures in Ω_1 which depends on the average usage rate r, as in equation (5.1) and is given by:

$$E^{\Omega_1}[\pi|r] = \begin{array}{l} \int_0^{T1}\int_0^{U1} \phi_1(t,x)\lambda(x|r)dx\,dt \ \ ,\forall\,(t,x)\ \in \Omega_1 \\ 0, \ \ otherwise \end{array} \tag{5.30}$$

The conditional expected cost in Ω_2 is the same as equation (5.3) and is given by:

$$E^{\Omega_2}[\pi|r] = \begin{array}{l} S_1 \int_{T1}^{T2} \lambda(t|r)\, e^{[\Lambda(T1|r)-\Lambda(t|r)]}dt + \int_{T1}^{T2}\left(\int_t^{U2} \phi_2(t,x)\lambda(x-t|r)\,dx\right)\lambda(t|r)\, e^{[\Lambda(T1|r)-\Lambda(t|r)]}dt, \\ \hfill \forall (t,x)\ \in \Omega_2 \\ 0\,, \ \ otherwise \end{array} \tag{5.31}$$

Finally, the conditional expectation in Ω_3 is same as in equation (5.3), and is given by:

$$E^{\Omega_3}[\pi|r] = \begin{array}{l} \left(\int_{T2}^{T} \lambda(x-t|r)\,dx\right)\lambda(t|r)\, e^{[\Lambda(T1|r)-\Lambda(t|r)]}dt, \ T1\ \leq\ S_{T1|r}\ \leq T2 \\ \int_{T2}^{T} \phi_3(t,x)\,\lambda(x|r)e^{[\Lambda(T2|r)-\Lambda(T1|r)]}dx, \ \ S_{T2|r} > T2 \end{array} \tag{5.32}$$

Using equations (5.30), (5.31) and (5.32) we obtain the conditional expectation for the subcase $r \leq r_2$ as:

$$\begin{aligned} E^{(i)}[\pi|r] = & \int_0^{T1}\int_0^{U1} \phi_1(t,x)\lambda(x|r)dtdx + \int_{T2}^{T}\int_{U2}^{U} \phi_3(t,x)\lambda(x|r)e^{[\Lambda(T2|r)-\Lambda(T1|r)]}dtdx + \\ & \int_{T1}^{T2} \lambda(x|r)e^{[\Lambda(T1|r)-\Lambda(t|r)]}dt\left[S_1 + \int_t^{U2} \phi_2(t,x)\lambda(x-t|r)\,dx\right] \end{aligned} \tag{5.33}$$

In figure (5.12), we observe that warranty expires in Ω_1 and Ω_2 due to excessive

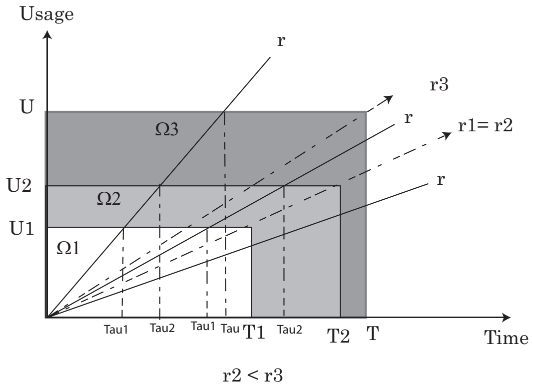


Figure 5.12. Realization of Restricted Strategy case-1

usage. To obtain the expected costs for the subcase $r_2 < r < r_3$, denoted by $E^{(ii)}[\pi|r]$, substitute $T1$ with $\tau_1 = \frac{U1}{r}$ and $T2$ with $\tau_2 = \frac{U2}{r}$ in equation (5.33)

$$\begin{aligned} E^{(ii)}[\pi|r] = \int_0^{\tau_1}\int_0^{U1} \phi_1(t,x)\lambda(x|r)dtdx + \int_{\tau_2}^{T}\int_{U2}^{U} \phi_3(t,x)\lambda(x|r)e^{[\Lambda(\tau_2|r)-\Lambda(\tau_1|r)]}dtdx+ \\ \int_{\tau_1}^{\tau_2} \lambda(x|r)e^{[\Lambda(\tau_1|r)-\Lambda(t|r)]}dt\left[S_1 + \int_t^{U2}\phi_2(t,x)\lambda(x-t|r)\,dx\right] \end{aligned} \tag{5.34}$$

We note in figure (5.12) the warranty duration in Ω_3 expires due to usage. We obtain the conditional expected costs for the subcase $r > r_3$, denoted by $E^{(iii)}[\pi|r]$, by substituting T with $\tau = \frac{U}{r}$ in equation (5.34)

$$\begin{aligned} E^{(iii)}[\pi|r] = \int_0^{\tau_1}\int_0^{U1} \phi_1(t,x)\lambda(x|r)dtdx + \int_{\tau_2}^{\tau}\int_{U2}^{U} \phi_3(t,x)\lambda(x|r)e^{[\Lambda(\tau_2|r)-\Lambda(\tau_1|r)]}dtdx+ \\ \int_{\tau_1}^{\tau_2} \lambda(x|r)e^{[\Lambda(\tau_1|r)-\Lambda(t|r)]}dt\left[S_1 + \int_t^{U2}\phi_2(t,x)\lambda(x-t|r)\,dx\right] \end{aligned} \tag{5.35}$$

Removing the condition on usage rate r in equation (5.33), (5.34) and (5.35), the expected costs for the warranty servicing to the manufacturer is given by,

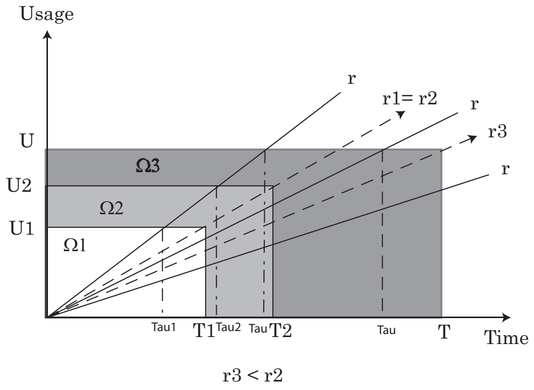


Figure 5.13. Realization of Restricted Strategy case-2

$$
\begin{aligned}
E[\pi] = &\int_0^{r_2} E^{(i)}[\pi|r] * R(r)dr + \int_{r_2}^{r_3} E^{(ii)}[\pi|r] * R(r)dr \\
&+ \int_{r_3}^{\infty} E^{(iii)}[\pi|r] * R(r)dr
\end{aligned} \tag{5.36}
$$

5.4.1.2 Case-2: $r_2(= r_1) > r_3$

As discussed in case-1, the realization given by $r_2(= r_1) > r_3$ has three possibilities for average usage rate r, given by: $(i)r \leq r_3$, $(ii)r_3 < r < r_2$ and $(iii)r > r_2$.

The conditional expected cost for sub case, $(i)r \leq r_3$, is the same as equation (5.33) and is given by,

$$
\begin{aligned}
E^{(i)}[\pi|r] = &\int_0^{T1}\int_0^{U1} \phi_1(t,x)\lambda(x|r)dtdx + \int_{T2}^{T}\int_{U2}^{U} \phi_3(t,x)\lambda(x|r)e^{[\Lambda(T2|r)-\Lambda(T1|r)]}dtdx + \\
&\int_{T1}^{T2} \lambda(x|r)e^{[\Lambda(T1|r)-\Lambda(t|r)]}dt \left[S_1 + \int_t^{U2} \phi_2(t,x)\lambda(x-t|r)\, dx\right]
\end{aligned} \tag{5.37}
$$

Figure (5.13) gives the realization of $r_2(= r_1) > r_3$ and the warranty duration in Ω_3 expires due to exceeding the usage limits. We obtain the conditional expected cost for subcase $(ii)r_3 < r < r_2$ substituting T by τ in equation (5.38), which is given by,

$$E^{(ii)}[\pi|r] = \int_0^{T1}\int_0^{U1} \phi_1(t,x)\lambda(x|r)dtdx + \int_{T2}^{\tau}\int_{U2}^{U} \phi_3(t,x)\lambda(x|r)e^{[\Lambda(T2|r)-\Lambda(T1|r)]}dtdx +$$
$$\int_{T1}^{T2} \lambda(x|r)e^{[\Lambda(T1|r)-\Lambda(t|r)]}dt \left[S_1 + \int_t^{U2} \phi_2(t,x)\lambda(x-t|r)\, dx\right] \tag{5.38}$$

Finally, we obtain the conditional expected cost for subcase $(iii)r > r_2$ by substituting $T1$ by τ_1, and $T2$ by τ_2 in equation (5.39), which is the same as equation (5.35)

$$E^{(iii)}[\pi|r] = \int_0^{\tau_1}\int_0^{U1} \phi_1(t,x)\lambda(x|r)dtdx + \int_{\tau_2}^{\tau}\int_{U2}^{U} \phi_3(t,x)\lambda(x|r)e^{[\Lambda(\tau_2|r)-\Lambda(\tau_1|r)]}dtdx +$$
$$\int_{\tau_1}^{\tau_2} \lambda(x|r)e^{[\Lambda(\tau_1|r)-\Lambda(t|r)]}dt \left[S_1 + \int_t^{U2} \phi_2(t,x)\lambda(x-t|r)\, dx\right] \tag{5.39}$$

Removing the condition on usage rate r in equation (5.37), (5.38) and (5.39), the expected costs for warranty servicing for the manufacturer is given by,

$$\begin{aligned} E[\pi] &= \int_0^{r_3} E^{(i)}[\pi|r] * R(r)dr + \int_{r_3}^{r_2} E^{(ii)}[\pi|r] * R(r)dr \\ &+ \int_{r_2}^{\infty} E^{(iii)}[\pi|r] * R(r)dr \end{aligned} \tag{5.40}$$

5.4.2 Strategy-B

5.4.2.1 Case-1: $r_2(= r_1) < r_3$

There are three possibilities for average usage rate r in the realization $r_2(= r_1) < r_3$, are $(i)r \le r_2$, $(ii)r_2 \le r \le r_3$, $(iii)r > r_3$. We shall derive expected costs for subcase $(i)r \le r_2$ and extend the analysis for all subcases previously done for strategy-A. The expected cost for manufacturer using strategy-B is straight forward since the manufacturer exercises replacement option only in Ω_3 region.

The conditional expectations in Ω_1 and Ω_2 are given by,

$$E^{\Omega_1}[\pi|r] = \begin{array}{l} \int\limits_0^{T1}\int\limits_0^{U1} \phi_1(t,x)\lambda(x|r)dx\,dt \;\; ,\forall\,(t,x) \;\in \Omega_1 \\ 0, \;\; otherwise \end{array} \tag{5.41}$$

The conditional expected costs in Ω_2 is given by,

$$E^{\Omega_2}[\pi|r] = \begin{array}{l} \int\limits_{T1}^{T2}\int\limits_{U1}^{U2} \phi_2(t,x)\lambda(x|r)dx\,dt \;\; ,\forall\,(t,x) \;\in \Omega_2 \\ 0, \;\; otherwise \end{array} \tag{5.42}$$

The conditional expected costs in Ω_3 is given by,

$$E^{\Omega_3}[\pi|r] = S_1\int\limits_{T2}^{T} \lambda(t|r)\, e^{[\Lambda(T2|r)-\Lambda(t|r)]}dt + \int\limits_{T2}^{T}\left(\int\limits_{t}^{U} \phi_3(t,x)\lambda(x-t|r)\,dx\right)\lambda(t|r)\, e^{[\Lambda(T2|r)-\Lambda(t|r)]}dt, \quad \forall(t,x) \;\in\; \Omega_2$$
$$0\,, \;\; otherwise \tag{5.43}$$

Using equation (5.41), (5.42), (5.43), we obtain the conditional expected cost for subcase $(i) r \leq r_2$:

$$\begin{array}{l} E^{(i)}[\pi|r] = \int\limits_0^{T1}\int\limits_0^{U1} \phi_1(t,x)\lambda(x|r)dx\,dt + \int\limits_{T1}^{T2}\int\limits_{U1}^{U2} \phi_2(t,x)\lambda(x|r)dx\,dt + \\ \int\limits_{T2}^{T} \lambda(x|r)e^{[\Lambda(T2|r)-\Lambda(t|r)]}dt\left[S1 + \int\limits_t^U \phi_3(t,x)\lambda(x-t|r)dx\right] \end{array} \tag{5.44}$$

For subcase (ii), defined by $r_2 \leq r \leq r_3$, the warranty duration in Ω_1 and Ω_2 expires due to excess usage i.e. the warranty expires due to exceeding usage limit $U1$ and $U2$ rather than the age limit $T1$ and $T2$. Therefore we replace $T1$, the limit on time duration for warranty expiry in Ω_1, with $\tau_1 = \frac{U1}{r}$, and $T2$, the limit on time duration for warranty expiry in Ω_2, with $\tau_2 = \frac{U2}{r}$in equation (5.44) :

$$\begin{array}{l} E^{(ii)}[\pi|r] = \int\limits_0^{\tau_1}\int\limits_0^{U1} \phi_1(t,x)\lambda(x|r)dx\,dt + \int\limits_{\tau_1}^{\tau_2}\int\limits_{U1}^{U2} \phi_2(t,x)\lambda(x|r)dx\,dt + \\ \int\limits_{\tau_2}^{T} \lambda(x|r)e^{[\Lambda(\tau_2|r)-\Lambda(t|r)]}dt\left[S1 + \int\limits_t^U \phi_3(t,x)\lambda(x-t|r)dx\right] \end{array} \tag{5.45}$$

In subcase (iii), defined by $r > r_3$, the warranty duration expires due to excess usage in region Ω_3, and hence we replace T the limit on time duration for warranty expiry in Ω_3 with $\tau = \frac{U}{r}$ in equation (5.46) to obtain:

$$\begin{aligned} E^{(iii)}[\pi|r] = &\int_0^{\tau 1}\int_0^{U1} \phi_1(t,x)\lambda(x|r)dx\,dt + \int_{\tau 1}^{\tau 2}\int_{U1}^{U2} \phi_2(t,x)\lambda(x|r)dx\,dt + \\ &\int_{\tau 2}^{\tau} \lambda(x|r)e^{[\Lambda(\tau 2|r)-\Lambda(t|r)]}dt \left[S1 + \int_t^U \phi_3(t,x)\lambda(x-t|r)dx\right] \end{aligned} \tag{5.46}$$

Upon removing the condition on usage rate in equations (5.44),(5.45), (5.46), we obtain the expected cost for warranty servicing given by,

$$\begin{aligned} E[\pi] = &\int_0^{r_2} E^{(i)}[\pi|r] * R(r)dr + \int_{r_2}^{r_3} E^{(ii)}[\pi|r] * R(r)dr \\ &+ \int_{r_3}^{\infty} E^{(iii)}[\pi|r] * R(r)dr \end{aligned} \tag{5.47}$$

5.4.2.2 Case-2: $r_2 (= r_1) > r_3$

We derive the expected cost for the realization $r_2(= r_1) > r_3$. The three subcases that the average usage rate r can take is given by: $(i) r \leq r_3$, $(ii) r_3 \leq r \leq r_2$ and $(iii) r \geq r_2$.

The conditional expected cost for subcase $(i) r \leq r_3$ is given by, (same as equation (5.44))

$$\begin{aligned} E^{(i)}[\pi|r] = &\int_0^{T1}\int_0^{U1} \phi_1(t,x)\lambda(x|r)dx\,dt + \int_{T1}^{T2}\int_{U1}^{U2} \phi_2(t,x)\lambda(x|r)dx\,dt + \\ &\int_{T2}^{T} \lambda(x|r)e^{[\Lambda(T2|r)-\Lambda(t|r)]}dt \left[S1 + \int_t^U \phi_3(t,x)\lambda(x-t|r)dx\right] \end{aligned} \tag{5.48}$$

Warranty duration in Ω_3 is exceeded due to usage. Hence, substituting T by τ in equation (5.48), we obtain the conditional expected cost for subcase $(ii) r_3 \leq r \leq r_2$ as:

$$\begin{aligned} E^{(ii)}[\pi|r] = &\int_0^{T1}\int_0^{U1} \phi_1(t,x)\lambda(x|r)dx\,dt + \int_{T1}^{T2}\int_{U1}^{U2} \phi_2(t,x)\lambda(x|r)dx\,dt + \\ &\int_{T2}^{\tau} \lambda(x|r)e^{[\Lambda(T2|r)-\Lambda(t|r)]}dt \left[S1 + \int_t^U \phi_3(t,x)\lambda(x-t|r)dx\right] \end{aligned} \tag{5.49}$$

The conditional expected cost for subcase $(iii) r \geq r_2$ is obtained by Substituting $T1$ by τ_1, and $T2$ by τ_2 in equation (5.49), as the warranty duration in Ω_1 and Ω_2 is exceeded due to usage, and is given by,

$$\begin{aligned} E^{(iii)}[\pi|r] = & \int_0^{\tau_1}\int_0^{U1} \phi_1(t,x)\lambda(x|r)dx\,dt + \int_{\tau_1}^{\tau_2}\int_{U1}^{U2} \phi_2(t,x)\lambda(x|r)dx\,dt + \\ & \int_{\tau_2}^{\tau} \lambda(x|r)e^{[\Lambda(\tau_2|r)-\Lambda(t|r)]}dt\left[S1 + \int_t^U \phi_3(t,x)\lambda(x-t|r)dx\right] \end{aligned} \tag{5.50}$$

Upon removing the condition on usage rate in equations (5.48),(5.49), (5.50), we obtain the expected cost for warranty servicing given by,

$$\begin{aligned} E[\pi] = & \int_0^{r_3} E^{(i)}[\pi|r] * R(r)dr + \int_{r_3}^{r_2} E^{(ii)}[\pi|r] * R(r)dr \\ & + \int_{r_2}^{\infty} E^{(iii)}[\pi|r] * R(r)dr \end{aligned} \tag{5.51}$$

5.5 Numerical Results

We present the numerical results for expected costs to the manufacturer assuming that the cost for replacement $S = 15$ during warranty period (first failure in Ω_2 for strategy-A and Ω_3 for strategy-B), the pro-rata costs for performing minimal repair in the region Ω_i is given by $S_1 = 5$, $S_2 = 10$, $S_3 = 15$. Let us assume that $U = 3$ and $T = 3$, i.e. the maximum limits for the warranty expiry is set at $30,000$ miles and 3 years or whichever happen earlier. Based on usage we discuss the results under three types of product usage classification, namely light, medium and heavy users. Table 5.2 gives the limits for user classification the mean usage rate $E[R]$, mean age at first failure $E[A]$ in years, and mean usage at first failure $E[U]$ in 10^4 miles.

We consider two values for proportionality rate $\alpha = 0.5, 0.75$ to compare the differences in expected cost between strategy-A and strategy-B. It is intuitive that larger α will incur more expense for the manufacturer. Table 5.3 and 5.4 give the expected cost for the manufacturer towards servicing under unrestricted case with $\alpha = 0.5$, The minimum cost for the manufacturer is given in bold face in each case. We also note that the heavy users incur less expense to the manufacturer as their warranty duration expires early as opposed to the case of medium users who incur maximum expense to the manufacturer in each case. The expected costs for light

Table 5.2. Expected values under uniform usage distribution

	ru	rl	E[R]-Mean Usage rate	E[A]-Mean Age at First failure	E[U]-Mean Usage at First failure
Light	0.9	0.1	0.5	2.06968	1.03484
Medium	1.3	0.7	1	1.65792	1.65792
Heavy	2.9	1.1	2	1.27678	2.55357

Table 5.3. Expected values Unrestricted Case under strategy-A with $\alpha = 0.5$

Case	T1	T2	r1	r2	Light	Medium	Heavy
1	1	1.4794	0.5	0.50694	28.3328	37.7771	12.5924
2	0.082192	0.12329	1.825	1.6222	28.0737	37.4316	12.4772
3	0.123287	0.49315	1.622	0.60833	28.6702	38.2269	12.7423
4	0.739726	2	0.67593	1.25	43.1321	57.5094	19.1698
5	1.2328	2	0.81116	0.75	**26.9185**	**35.8913**	**11.9638**
6	0.24657	1	2.02782	2.5	38.459	51.2787	17.0929

users are intermediate between heavy and medium users. Based on this comparison among the six realizations in the unrestricted case, case-5 ($r_2 \leq r_1 \leq r_3$) with parameters $\pi\{T1 = 1.233, T2 = 2, r1 = 0.81116, r2 = 0.75\}$ yields the minimum cost for warranty servicing to the manufacturer for all the three user categories. A warranty with the following $\{U1 = 10,000, U2 = 15,000, U = 30,000\}$ miles and

Table 5.4. Expected values Unrestricted Case under strategy-B with $\alpha = 0.5$

Case	T1	T2	r1	r2	Light	Medium	Heavy
1	1	1.4794	0.5	0.50694	56.2961	75.0615	25.0205
2	0.082192	0.12329	1.825	1.6222	61.2825	81.71	27.2367
3	0.123287	0.49315	1.622	0.60833	61.8808	82.5078	27.5026
4	0.739726	2	0.67593	1.25	60.5984	80.7978	26.9326
5	1.2328	2	0.81116	0.75	**48.7337**	**64.9783**	**21.6594**
6	0.24657	1	2.02782	2.5	58.5136	78.0181	26.006

Table 5.5. Expected values Unrestricted Case under strategy-A with $\alpha = 0.75$

Case	T1	T2	r1	r2	Light	Medium	Heavy
1	1	1.4794	0.5	0.50694	39.6966	52.9288	17.6429
2	0.082192	0.12329	1.825	1.6222	41.6922	55.5896	18.5299
3	0.123287	0.49315	1.622	0.60833	41.6518	55.5358	18.5119
4	0.739726	2	0.67593	1.25	46.4082	61.8776	20.6259
5	1.2328	2	0.81116	0.75	**33.8746**	**45.1661**	**15.0554**
6	0.24657	1	2.02782	2.5	44.3338	59.1118	19.7039

Table 5.6. Expected values Unrestricted Case under strategy-B with $\alpha = 0.75$

Case	T1	T2	r1	r2	Light	Medium	Heavy
1	1	1.4794	0.5	0.50694	60.7533	81.0044	27.0015
2	0.082192	0.12329	1.825	1.6222	65.5761	87.4348	29.1449
3	0.123287	0.49315	1.622	0.60833	66.2411	88.3215	29.4405
4	0.739726	2	0.67593	1.25	78.6802	104.907	34.969
5	1.2328	2	0.81116	0.75	**53.5462**	**71.395**	**23.7983**
6	0.24657	1	2.02782	2.5	71.623	95.4974	31.8325

$\{T1 = 450, T2 = 730, T = 1095\}$ days is the most profitable limits for the manufacturer to extend the warranty based on the traction motor failure data.

In Table 5.5 and 5.6, the expected cost under an unrestricted case with $\alpha = 0.75$ follows similar results as obtained when $\alpha = 0.5$, Realization $r_2 \leq r_1 \leq r_3$ yields the minimum cost of warranty servicing to the manufacturer in all the three user

Table 5.7. Expected values Restricted Case for strategy-A with $r_2 < r_3$, $\alpha = 0.5$

T1	T2	r2(=r1)	Light	Medium	Heavy
1.5	2	0.5	41.2337	54.9782	18.3261
0.5	1	0.5	30.3081	40.4108	13.4703
1	1.5	0.6	31.4913	41.9884	13.9961
1	2	0.75	**27.1552**	**36.2069**	**12.069**
0.2465	0.4931	0.8	28.7994	38.3991	12.7997

Table 5.8. Expected values Restricted Case for strategy-B with $r_2 < r_3$, $\alpha = 0.5$

T1	T2	r2(=r1)	Light	Medium	Heavy
1.5	2	0.5	53.0701	70.7601	23.5867
0.5	1	0.5	58.7939	78.3919	26.1306
1	1.5	0.6	55.0907	73.4543	24.4848
1	2	0.75	**50.2651**	**67.0202**	**22.3401**
0.2465	0.4931	0.8	59.9124	79.8832	26.6277

Table 5.9. Expected values Restricted Case for strategy-A with $r_2 > r_3$, $\alpha = 0.5$

T1	T2	r2(=r1)	Light	Medium	Heavy
0.5	1.25	1.25	32.6804	43.5739	14.5246
1	2	1.4	35.6815	47.5753	15.8584
1.5	2	1.45	**26.8577**	**37.477**	**11.8257**
0.5	1	1.5	33.6018	44.8024	14.9341
1	1.5	1.5	28.5359	38.0479	12.6826

categories.

The restricted case has two realizations, namely $(i) r_2 < r_3$ and $(ii) r_2 > r_3$. In Table 5.7 and 5.8, we present the results for case $(i) r_2 < r_3$ with $\alpha = 0.5$. The parameter set $\pi\{T1 = 1, T2 = 2, r_2 = 0.75\}$ yields the minimum value for both the strategies, whose warranty duration limits are given by
$\{U1 = 7,500, U2 = 15,000, U = 30,000\}$ miles and $\{T1 = 365, T2 = 730, T = 1095\}$ days.

In Tables 5.9 and 5.10, we present the results for case $(ii) r_2 > r_3$ with $\alpha = 0.5$. The parameter set $\pi\{T1 = 1.5, T2 = 2, r_2 = 1.45\}$ yields the minimum value for both strategy A and B, whose warranty duration limits are given by
$\{U1 = 21,750, U2 = 29,000, U = 30,000\}$ miles and
$\{T1 = 548, T2 = 730, T = 1095\}$ days. In Tables 5.11 and 5.12, we present the results for case $(i) r_2 < r_3$ with $\alpha = 0.75$. The parameter set
$\pi\{T1 = 1, T2 = 2, r_2 = 0.75\}$ yields the minimum value for both strategy A and B, whose warranty duration limits are given by

Table 5.10. Expected values Restricted Case for strategy-B with $r_2 > r_3$, $\alpha = 0.5$

T1	T2	r2(=r1)	Light	Medium	Heavy
0.5	1.25	1.25	52.5757	70.101	23.367
1	2	1.4	44.6626	59.5502	19.8501
1.5	2	1.45	**31.0024**	**41.3365**	**13.7788**
0.5	1	1.5	52.3566	69.8088	23.2696
1	1.5	1.5	43.4094	57.8792	19.2931

Table 5.11. Expected values Restricted Case for strategy-A with $r_2 < r_3$, $\alpha = 0.75$

T1	T2	r2(=r1)	Light	Medium	Heavy
1.5	2	0.5	59.7013	79.6018	26.5339
0.5	1	0.5	43.7575	58.3434	19.4478
1	1.5	0.6	44.9895	59.9861	19.9954
1	2	0.75	**34.9753**	**46.6338**	**15.5446**
0.2465	0.4931	0.8	42.0054	56.0072	18.6691

$\{U1 = 7,500, U2 = 15,000, U = 30,000\}$ miles and $\{T1 = 365, T2 = 730, T = 1095\}$ days.

In Tables 5.13 and 5.14, we present the results for case $(ii)r_2 > r_3$ with $\alpha = 0.75$ follows similar results as obtained from $\alpha = 0.5$ with the parameter set $\pi\{T1 = 1.5, T2 = 2, r_2 = 1.45\}$ yielding the minimum value for both strategy A and B.

Table 5.12. Expected values Restricted Case for strategy-B with $r_2 < r_3$, $\alpha = 0.75$

T1	T2	r2(=r1)	Light	Medium	Heavy
1.5	2	0.5	57.4666	76.6214	25.5405
0.5	1	0.5	63.2601	84.3468	28.1156
1	1.5	0.6	59.5812	79.4416	26.4805
1	2	0.75	**55.8433**	**74.4578**	**24.8193**
0.2465	0.4931	0.8	64.3246	85.7662	28.5887

Table 5.13. Expected values Restricted Case for strategy-A with $r_2 > r_3$,$\alpha = 0.75$

T1	**T2**	**r2(=r1)**	**Light**	**Medium**	**Heavy**
0.5	1.25	1.25	40.3198	53.7597	17.9199
1	2	1.4	37.0837	49.4449	16.4816
1.5	2	1.45	**32.2676**	**43.0234**	**14.3411**
0.5	1	1.5	43.5679	58.0905	19.3635
1	1.5	1.5	33.942	45.256	15.0853

Table 5.14. Expected values Restricted Case for strategy-B with $r_2 > r_3$, $\alpha = 0.75$

T1	**T2**	**r2(=r1)**	**Light**	**Medium**	**Heavy**
0.5	1.25	1.25	58.5626	78.0835	26.0278
1	2	1.4	57.8701	77.1601	25.72
1.5	2	1.45	**38.5192**	**51.3589**	**17.1196**
0.5	1	1.5	57.5887	76.785	25.595
1	1.5	1.5	49.3822	65.8429	21.9476

5.6 Conclusions and Future work

We studied a 2-D rectangular warranty model with product failures in different regions, which have different cost implications to the manufacturer. We incorporated pro-rated costs for warranty servicing and studied two strategies based on when the manufacturer can use replacement option. We derived expected servicing costs for both restricted and unrestricted cases. The results derived can be used in product pricing, developing service strategies based on prevailing market conditions and to forecast warranty reserves. Based on the information obtained from this research warranties for specific group of customers can be designed without resorting to extensive simulation. We can extend the current work to obtain the distribution of cost and profit for manufacturers during product life cycle. Expected costs for warranty servicing for regions defined by non-rectangular geometries including triangular, infinite strips can also be developed.

CHAPTER 6
CONCLUSIONS AND FURTHER RESEARCH DIRECTIONS

In this research we studied one-dimensional and two-dimensional warranty models and developed optimal servicing strategies to minimize the expected cost for the manufacturer. We developed suitable analytical models and studied their benefits. Warranty modeling is a very complex topic that affects many aspects of the organization. Analytical study aids in better understanding of the warranty modeling process, unlike the vendor developed software tools which use a black-box approach. We discuss the advantages of the modeling approach we took and would like to extended it to a more general context. The repair-replacement option employed by the manufacturer influences the cost of servicing and the time between failures. Warranty policies serve as a mechanism to mitigate the risks between the manufacturer and the consumer. Our research will help in developing suitable policies without much compromise on profits for the manufacturer.

There are several ways to extend the current research. First, warranty models with risk aversion aspects can be developed to minimize the possibility of severe loss to the manufacturer by probabilistically incorporating product recalls. Important factors in a product recall depend on the manufacturer's relationship with the society, quality, legal penalties and the value of the product. We can enhance the modeling ideas by adding risk as a third dimension to warranty modeling. Much of the current research in warranty modeling focuses on the present worth of warranty reserves, typically the cost analysis for the models developed should consider future worth.

The two-dimensional warranty region could be a defined by different geometries apart from rectangles need to be considered, example include, infinite strips, finite strips, or combination of rectangular and triangular regions, each has its own special implication in terms of modeling and analysis. We would like to extend the scope of the two-dimensional model to non-rectangular regions as shown in Figure-7, tough we are unable to identify suitable applications for those models at this time we are confident that several product classes would belong to those categories.

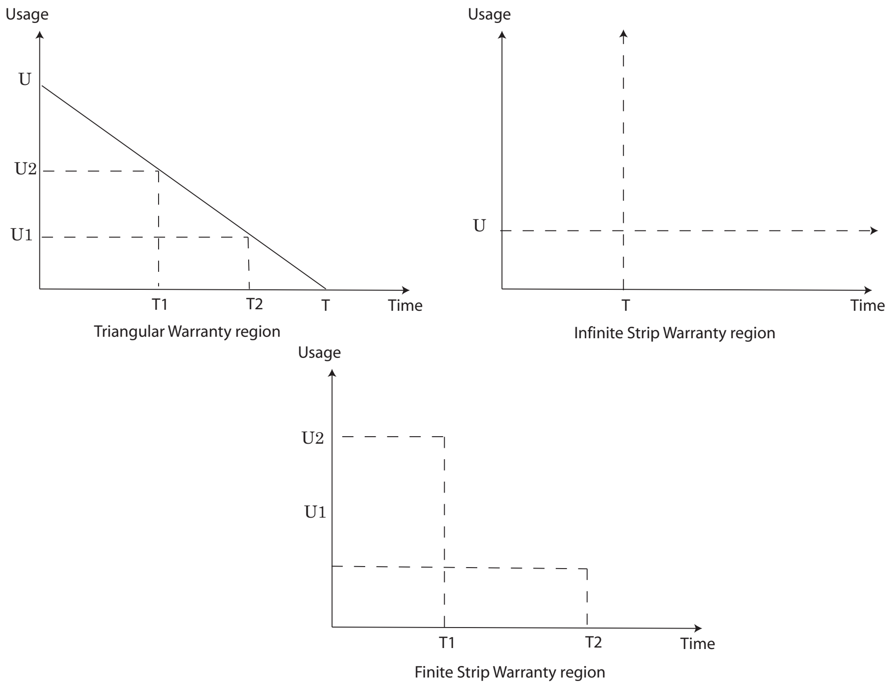


Figure 6.1. Extensions of Warranty models to different regions

Incorporating different failure modes of the product with different forms of failure intensity function for each form of failure can be developed to yield accuracy and precision. The results from this research will also prompt us to investigate 2-D warranty policies with iso-cost regions that give more flexibility of choice for the customers and influence the manufacturer to develop customized warranty policies.

Warranty markets is pegged to be over $26 billion a year with a rapid growth. This clearly gives enough reasons for significant improvements in modeling, analysis and obtaining inferential statistics related to warranty models. The limitations of this research relies heavily on the availability of warranty and failure data which is usually confidential. In terms of parameter optimization in the models developed, real data will assist in identifying suitable techniques for modeling and will give us more flexibility and choice. We wish to develop a robust statistical framework for modeling warranty claims and predict the transient characteristics of the product failures and their associated costs. With the latest improvements of the technology to record warranty failure data, which include methods like Radio Frequency

IDentification (RFID) tags and intelligent information retrieval systems placed in the products, whereby warranty data would not suffer from the disadvantages of censoring and inaccuracies, using such data will aid in obtaining more meaningful information. Statistical analysis of warranty databases is another area we would like to expand this research, incorporating factor variables like fraudulent claims, customer satisfaction index in the models gives valuable insights in developing and administering warranty policies. Extending the analysis to supply chains and to develop dynamic models will yield promising results.

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