Equilibrium Distribution of Charges, Capacities, and Affine Mappings

by

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ABSTRACT

In this dissertation, we will explore the interactions of potential charges within simple geometric domains. The positions of these charges in reaching an extremal position will then be transitioned to a study on the interaction of the charges based on the shape of the domain changing. This will lead into a final study, where a conformal invariant, based on the shape of a particular domain, is examined as the domain is transformed by a complex mapping. The common thread between all of these topics is that a particular energy for a system is studied; the energy of each system, though, is based on the interactions the charges have in their respective domains.

In Chapter 2, we will discuss the historical background of the study of charge placement with regard to minimizing potential energy. Some of the applications of potential energy in other fields will be discussed as well.

In Chapter 3, the planar configurations of charges on simple geometric domains will be discussed. Of interest is the placement of charges that produce an extremal logarithmic potential energy based on the domain containing these charges.

In Chapter 4, we will discuss the break-of-symmetry effect with regard to the position charges on domains as the shape of the domains changes. We will show that n extremal charges will have a convergent limit set of n charges. However, for one particular system the critical point of the function describing the system's logarithmic potential energy will be where a "jump" occurs in the extremal position of the charges.

In Chapter 5, we will discuss the affine capacity of a system of charges. The affine modulus of a quadrilateral is introduced, and the behavior of the affine modulus of two "essentially different" quadrilaterals under an affine transformation will also be discussed.

LIST OF FIGURES

CHAPTER 1
INTRODUCTION

The goal of this work is to study two types of energy of systems of charges and discuss some related questions. The study of distribution of charges within particular domains is a classical subject that has attracted much attention from physicists, mathematicians, and researches from other fields for many years. It originated in the work of J. J. Thomson who, when modeling the distribution of electrons in atoms, asked what the configuration of n electrons on the sphere in $\mathbb{R}^3$, which provides the minimal energy of their interaction, would be.

For $n = 2, 3$, and 4, the solution to this problem has been known for more than 100 years. But only recently (see [23]) was the case of five electrons on a sphere with the potential satisfying the Coulomb law completely resolved.

In this dissertation we deal with the distribution of charges in planar domains. In Chapter 2, we provide an overview of the history of the study of charge placement with regard to minimizing potential energy. We also will discuss some applications of this problem in other fields.

In Chapter 3, the minimal energy problem for planar configurations of charges will be discussed. The main question here is to find exact placements of small numbers of charges which realize the minimal energy when particles freely move within prescribed, rather simple, geometrical configurations. In particular, we give complete solutions to the problem for $3, 4$ and, in some cases, for 5 charges distributed over triangles, rectangles, and circular rings.

In Chapter 4, we discuss the behavior of minimal configurations of charges in varying domains. A particular example of a family of rectangles shrinking to a line segment will be used to demonstrate the so-called *break of symmetry effect* when a small perturbation of configuration leads to an essentially different distribution of charges minimizing the energy of the system. The asymptotic distribution of large (but finite) number of n charges when $n \to \infty$ will be also discussed. It is known for a long time that under appropriate normalization this limit distribution gives the "uniform distribution" of the unit charge with the minimal energy over a domain where the charges are allowed to move.

In Chapter 5, we will discuss a new type of energy of a system of charges, which is called the *affine capacity*, as well as the related notion of the *affine modulus* of a quadrilateral. The affine modulus was introduced and studied in some recent papers by T. Iwaniec, L. Kovalev, and J. Onninen (see [14]). Specifically, in the last section of this chapter we will prove that every affine mapping preserving modules of two "essentially different" quadrilaterals must be conformal. An interesting and much more difficult problem on the maximal number of essentially different rectangles the modules of which can be preserved by a polynomial mapping of a given order n without being conformal also will be discussed.

CHAPTER 2
A SHORT SURVEY OF ELECTROSTATIC MODELS

2.1 Background Information

The study of the placement of charges on a conductor is a problem that stretches back over 100 years. The work of J.J. Thomson first examines the problem in a spacial sense with his "plum pudding model" of the atom and his investigation of the ground state of spherical shells of electrons [27]. Here is a mathematical formulation of the problem. Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$, let P be a set of n distinct points $p_1, \ldots, p_n$ on $\mathbb{S}^2$, and let $E : \mathbb{R} \to \mathbb{R}$ be a non-negative function. The total E-energy of P is defined to be the sum

$$\mathcal{E}(P) = \sum_{i>j} E(\|p_i - p_j\|), \tag{2.1}$$

where $\|\cdot\|$ is the usual norm in $\mathbb{R}^3$.

The problem is then: Describe all possible sets P, which make the energy $\mathcal{E}(P)$ to be as small as possible.

The case $E(r) = r^{-1}$ is specially interesting to physicists since it corresponds to the so-called *Coulomb potential*. In this case the points can be naturally considered as *electrons*. There certain values of n where the minimal configurations are known for a long time. The cases $n = 2$ and $n = 3$ are trivial while in the cases $n = 4$, $n = 6$, and $n = 12$ the minimizing configuration consists of 4, 6, and 12 electrons situated at the vertices of a regular tetrahedron, regular octahedron, and regular icosahedron, respectively. It worth mentioning that the case of 5 electrons was resolved only in 2010 by Richard E. Schwartz [23].

A study in two dimensions of the placement of charges, as opposed to three-dimensional cases, lends itself to easier investigation. The examination of the placement of charges on a line was started by Stieltjes in the late nineteenth century. In the general situation, the two-dimensional case, and its connection to polynomial functions of a complex variable, was studied by Fekete.

A short explanation of how the energy of charges can be studied by using the complex analysis technique is as follows. In the study of electrostatics, a source charge corresponds to a uniformly charged line which is perpendicular to the complex plane $\mathbb{C}$ at the point z_0. If this uniformly charged line L is located at $z_0 = 0$ and carries a charge density of $q/2$ Coulombs per unit length, then the electric field $\mathbf{E}$ is given by $\mathbf{E}(x, y) = q/\bar{z}$. From this the complex potential is given by $F(z) = -q \log z$. If the uniformly charged line L is located at the point $z_0 \neq 0$, then the electric field is given by $\mathbf{E}(x, y) = q/(\bar{z} - \bar{z_0})$. The corresponding complex potential is given by $F(z) = -q \log(z - z_0)$. Since $F(z)$ is a multivalued analytic function, the rich variety of methods of complex analysis can be used to study the energy minimizing configurations.

2.2 Some Applications and Related Problems

The study of logarithmic potentials has many flavors, thanks to the number of fields that are connected to the work. As cited by Marcellan [18], in two-dimensional studies, the use of special polynomials, especially orthogonal polynomials, plays a very important role in the study of logarithmic potential. The zeros of certain Jacobi polynomials are the electrostatic equilibrium positions of weighted charges on a line segment. It is also possible to use the Rodriguez formulation of Jacobi polynomials to study the equilibrium positions of the charges, but another important fact found is that the particular Jacobi polynomial provides the only polynomial solutions of a particular linear differential equation. It is also known (see [2]) that the zeros of the Jacobi polynomial $P_n^{(2p-1,2q-1)}(1 - 2x)$ provide the minimal logarithmic potential energy for a particular electrostatic system.

In other science fields, the hydrogen atom with a logarithmic potential energy function in a two-dimensional space has been explored [9]. Study of the shape of the conductor has been generalized even further, as research has been conducted on the electrostatic field due to electrons located at points on a simple closed analytic curve C [16]. One of the most interesting branches of exploration begins with Smale's Conjecture [25]. While Smale's Conjecture itself does not directly involve logarithmic potentials, a proposition by Dimitrov connects Smale's Conjecture with the study of logarithmic potentials and electrostatic equilibrium positions of charges [7].

The electrostatic equilibria of n discrete charges of size $1/n$ on a two dimensional conductor has also been studied [5]. Further results into this study are part of the focus of this paper, as we will explore what occurs on various symmetric conductor shapes such as circles, ellipses, and squares. Also, our study will look into how the equilibrium position of the charges on these conductors is affected as we deform the previously-listed conductors into a line segment.

Research into logarithmic potential and equilibrium positions has also been conducted in three dimensions and more. As stated earlier, one of the most notable excursions into this realm was done by J.J. Thomson and his research into the plum pudding model of the atom [27]. Thomson was interested in the location of electrons in an atom and in determining the ground state of those electrons. But basic chemistry is not where the research stops. Trying to determine where n points on a unit sphere S^d in $\mathbb{R}^{d+1}$ has application to various fields such as crystallography, viral morphology, molecular modelling, and global positioning [13]. Moreso than finding the position of n points on such a sphere, finding the minimal energy produced by an arrangement of n points on the sphere is of interest as well [17]. Study beyond the unit sphere S^d in $\mathbb{R}^{d+1}$ has also been conducted, where the equilibrium distribution of charges on the surface of shapes such as cubes and cones have been studied [22].

CHAPTER 3
ENERGY MINIMIZING CONFIGURATIONS IN STANDARD GEOMETRIES

While some work has been done studying the configuration of point charges in a two-dimensional conductor (see [5], [7], and [19]), more work has been done examining the three-dimensional model, especially on $\mathbb{S}^n$ spheres in $\mathbb{R}^{n+1}$ dimensions (some examples are [4], [8], [20], and [21]). Thus, the two-dimensional model is still rich in possible exploration and untapped knowledge.

These two-dimensional models will be the focus of this chapter. To begin our study, however, we will introduce some information regarding the equilibrium position of charges on a line segment. We shall also examine what happens to the configuration of point charges in a two-dimensional conductor, and we will find what the equilibrium positions of these charges are. From there, we shall examine the behavior of these charges as we deform them into different shaped conductors (e.g. from a square to a rectangle to a line segment).

3.1 General Minimal Energy Problem

Electrostatics and potential energy have been a constant source of study for over a century. Specifically, the study of the placement of charges on a conductor has inspired a multitude of results with implications in various academic fields. The foundation for this work, done by Stieltjes [26] on orthogonal polynomials, has inspired literally hundreds of authors to conduct research into other areas of science, such as the numerical approximation of integers and digital signal processing.

Our field of study will involve electrostatics and logarithmic potential energy. While much of the work we shall consider evolved from the work of Stieltjes, we shall also focus some attention on the work of Steve Smale. The applications into electrostatics and logarithmic potential energy will come from results derived from the study of complex polynomials and complex rational functions.

To begin our analysis, the following is the general form of the energy equation on the minimal energy.

Definition 3.1. *Let $X = \{\nu_1, \dots, \nu_n\}$ be a system of n charges in the complex plane. Let m_k be the mass of charge ν_k, and let $M = \{m_1, \dots, m_n\}$ be the set of* *masses associated with their respective charges in* X*. We assume that the charge* ν_k *is situated at the point* $z_k \in \mathbb{C}$*. Then the total energy* $E(X, M)$ *of this system, determined by the energy of the mutual interactions of these charges, is defined by*

$$E(X, M) = E(\nu_1, \ldots, , \nu_n; m_1, \ldots, m_n) = -\sum_{1\le j<k\le n} m_j m_k \, \log |z_k - z_j|. \qquad (3.1)$$

For a given compact set $F \subset \mathbb{C}$, and a given set of n masses $M = \{m_1, \ldots, m_n\}$, let $\mathscr{X}_F^n(M)$ denote the set of all systems $X = \{\nu_1, \ldots, \nu_n\}$ of n charges $\nu_1, \ldots, \nu_n$ with masses $m_1, \ldots, m_n$ which are distributed on F. That is, if ν_k is situated at z_k, then $z_k \in F$.

Let $\mathscr{M} = \{M\}$ be a collection of the sets of masses M associated with $\mathscr{X}$. We can then define the following problem.

Problem 1. *(The Minimum Energy Problem.) For a given compact set* F *and a given collection* $\mathscr{M} = \{M\}$ *of sets of masses* $M = \{m_1, \ldots, m_n\}$*, find*

$$E(F, \mathscr{M}) = \inf E(X, M)$$

where the infimum in (1) is taken over all $M \in \mathscr{M}$ *and all* $x \in \mathscr{X}_F^n$*.*

If $\mathscr{M}$ consists of a single mass distribution M, then the Minimum Energy Problem can be written as

$$E(F, M) = E(F, \mathscr{M}).$$

3.2 The Stieltjes Problem

The motivation for this topic revolves around variations of the Stieltjes Problem. In particular, we will be using the Stieltjes Problem on the unit circle as a foundation to begin our study, as this will be the easiest case to study and use as a test example (that is, this will give us a problem and solutions that we can check against known results).

In order to facilitate our study, it is worth noting the basic example of the Stieltjes problem: the study of charges on a unit interval. The general case of the study on the unit interval will be given, but this will not be investigated. It will provide background for our studies, and this background will be seen to easily translate to the circle.

The Stieltjes Problem on the Unit Interval

This investigation begins a survey [28] of what is known as the Stieltjes problem on a line segment: a two-dimensional electrostatics problem. Suppose that n unit charges are distributed in the open interval (a, b) located at points $x_1, x_2, \ldots, x_n$. The interval can possibly be of infinite length. The expression $D(x_1, x_2, \ldots, x_n)$, defined by

$$\prod_{1\leq j<k\leq n} |x_k - x_j| = D(x_1, x_2, \ldots, x_n)$$

is called the **discriminant of the points** $x_1, x_2, \ldots, x_n$. The charges repel each other, and the energy of the system can be found by the law of logarithmic potential using the expression

$$-\log D(x_1, x_2, \ldots, x_n) = \sum_{1\leq j<k\leq n} \log \frac{1}{|x_k - x_j|}$$

and the energy describes the location of the charges in the interval. The minimum of this energy occurs when there is electrostatic equilibrium.

The classical problem that has been examined is where, in addition to the n unit charges in the interval $[-1, 1]$, we have charges of fixed size $p > 0$ at $z = 1$ and $q > 0$ at $z = -1$. Each of the n unit charges interacts with the charges at 1 and -1 as well as with each other. The electrostatic energy of this system is given by

$$E(x_1, x_2, \ldots, x_n) = \\ -pq\log 2 - p\sum_{j=1}^{n} \log|1 - x_j| - q\sum_{j=1}^{n} \log|1 + x_j| - \sum_{1\leq j<k\leq n} \log|x_k - x_j|. \quad (3.2)$$

The following result has been established in [26].

Theorem 3.2. *The expression* (3.2) *becomes a minimum when* $x_1, x_2, \ldots, x_n$ *are the zeros of the Jacobi polynomial* $P_n^{(2p-1,2q-1)}(x)$.

Let $H(x_1, \ldots, x_n) = -\log E(x_1, \ldots, x_n)$. Then

$$H(x_1, x_2, \ldots, x_n) = 2^{pq} \prod_{i=1}^{n} (1 - x_i)^p (1 + x_i)^q \prod_{1 \le i < j \le n} |x_i - x_j| \tag{3.3}$$

and Theorem 3.2 can be reformulated as follows:

Theorem 3.3. *The expression* (3.3) *becomes a maximum when* $x_1, x_2, \ldots, x_n$ *are the zeros of the Jacobi polynomial* $P_n^{(2p-1,2q-1)}(x)$.

Theorems 3.2 and 3.3 are standard results in the literature (see [2], [26], and [28]), and the mention of these results is meant to provide background for the Stieltjes problem extended to the unit circle, their results will not be proven here, nor will the unit interval problem be examined in this paper. This case will be a limiting case that we will explore as a deformation of other conductors.

The Stieltjes Problem on the Unit Circle

This is a variation of the Stieltjes Problem [2] on the unit interval. Later in this paper, we will consider where unit electrical charges are placed on the unit circle and there is also a unit electrical charge placed at the origin of the circle. This example has been documented, and for our purposes it is an ideal place to begin our examination because we have, in essence, a beginning point provided for our to-be-examined problem. This is the most basic form our extremal problem can attain, and it is the easiest to examine.

To begin our extremal problem study, we are considering the situation where freely moving charges lie on a thin circular conductor of unit radius. Without loss of generality , through reduction or enlarging as necessary, we can transform the study from any circle of radius r, where $r > 0$, into one on the unit circle. Since we are considering the general example, we will look at the energy when n charges are places on the circle. The logarithmic potential energy for this system is given by

$$W = -\sum_{1 \le j < k \le n} \log |e^{i\theta_k} - e^{i\theta_j}| \tag{3.4}$$

and furthermore, we see that W has a minimum value when the charges are in the equilibrium position. The following result can be found in [2].

Theorem 3.4. *The minimum value of equation 3.4 is* $-\frac{n}{2}\log n$*, and this is attained when* $e^{i\theta_k}$ *for* $k = 1, \ldots, n$ *are the roots of the equation* $x^n \pm 1 = 0$.

Proof. We will need to rewrite $|e^{i\theta_k} - e^{i\theta_j}|$, with $\theta_k \geq \theta_j$, as follows:

$$\begin{aligned}
|e^{i\theta_k} - e^{i\theta_j}| &= |(\cos\theta_k + i\sin\theta_k) - (\cos\theta_j + i\sin\theta_j)| \\
&= \sqrt{(\cos\theta_k - \cos\theta_j)^2 + (\sin\theta_k - \sin\theta_j)^2} \\
&= \sqrt{2[1 - \cos(\theta_k - \theta_j)]} \\
&= 2\sin\left(\frac{\theta_k - \theta_j}{2}\right) \\
&= 2\left[\frac{1}{2i}(e^{i(\theta_k-\theta_j)/2} - e^{-i(\theta_k-\theta_j)/2})\right] \\
&= \frac{1}{i}\left(e^{i\theta_k} - e^{i\theta_j}\right)e^{-i\theta_j/2}e^{-i\theta_k/2}
\end{aligned}$$

From this step, the minimum for W is found to be

$$\frac{\partial W}{\partial\theta_k} = -i\left(\frac{n-1}{2} - \sum_{j\neq k}\frac{e^{i\theta_k}}{e^{i\theta_k} - e^{i\theta_j}}\right), \qquad k = 1, \ldots, n.$$

Define $f(x) = \prod_{j=1}^{n}(x - e^{i\theta_j})$, similar to the Stieltjes problem on the unit interval. Using this function f, we can write

$$\frac{f''(e^{i\theta_k})}{2f'(e^{i\theta_k})} = \sum_{j\neq k}\frac{1}{e^{i\theta_k} - e^{i\theta_j}}.$$

Renaming $f = y$ and letting $x = e^{i\theta_k}$, we see that f is a solution of the equation

$xy'' - (n-1)y' = 0$ by the following:

$$
\begin{aligned}
x \cdot \frac{y''(x)}{2y'(x)} &= \sum_{j \neq k} \frac{x}{x - e^{i\theta_j}} = \frac{n-1}{2} \\
\frac{y''}{y'} &= \frac{n-1}{x} \\
xy'' - (n-1)y' &= 0
\end{aligned}
$$

We can further refine the equation for $f(x)$. By letting $v = y'$, we rewrite our equation into $xv' - (n-1)v = 0$. This gives us $v' - \left(\frac{n-1}{x}\right) v = 0$. Letting $P(x) = -\frac{n-1}{x}$ and $Q(x) = 0$, we see that the integrating factor is

$$I(x) = e^{\int -\frac{n-1}{x}\, dx} = e^{-(n-1)\ln x} = x^{-(n-1)}.$$

Then we have the following for v:

$$v = \frac{1}{x^{-(n-1)}} \left[\int 0 \, dx + C \right] = Cx^{n-1}$$

Thus we can find y by integration on the above equation:

$$y = \int v \, dx = \int Cx^{n-1} \, dx = Cx^n + D$$

Thus f satisfies the equation $f(x) = Cx^n + D$. From our original definition of $f(x)$, we know that $C = 1$, since the coefficient of x^n is 1, and we also know that $D = \pm 1$, since the roots of $f(x)$ lie on the unit circle. We thus have

$$f(x) = x^n \pm 1.$$

The smallest value of W is $-\frac{1}{2}\log \Delta$, where Δ is the discriminant of $f(x)$. Because

$$\Delta = \left| \prod_{k=1}^{n} f'(e^{i\theta_k}) \right| = \left| \prod_{k=1}^{n} ne^{i(n-1)\theta_k} \right| = n^n,$$

then the minimum potential energy is given by $-\frac{n}{2}\log n$. □

The reason for looking at the Stieltjes problem is because this is the easiest scenario for the general cases we want to investigate. This gives us a groundwork: begin examining the basic situations and expanding out from the simplest example (i.e. the appropriate Stieltjes problem for the number of points in our system). The Stieltjes problem on the unit circle can be generalized even more, looking at cases with charges of size p at $\theta = 0$ and size q at $\theta = \pi$ (see [10]). We will not be examining these more specific cases.

Our situations are going to focus on what happens when we divide the points between two concentric circles. We will begin by looking at the Stieltjes problem for three points on a circle. We will examine the scenarios when we have two points on a circle where the radius of the larger circle is growing to infinity and one point on a unit circle, and the other scenario involves two points on a circle where the radius of the larger circle is growing to infinity and one point on a unit circle. The case of having a system with four points will be more involved. After examining these systems, we shall look at the general case and derived results for the general case.

3.3 Concentric Circles

The general scenario (see [7]) for our examined problems involves a system of two concentric circles with $n \geq 2$ charges: one fixed negative charge with force of $|\frac{n-2}{2}|$ located at the origin and $n-1$ free unit charges located at $z_1, \ldots, z_{n-1}$ in the complex plane $\mathbb{C}$. We will suppose the electrostatic field generated by these charges obey the law of logarithmic potential, i.e. all charges are uniformly distributed along infinite stright lines perpendicular to the complex plane. Our concern is when the points $z_1, \ldots, z_{n-1}$ are located in an annulus $R(a, b) = \{z : a \leq |z| \leq b\}$, where $0 < a < b < \infty$.

We can immediately simplify the problem by using the Maximum Principle [12], given by

Theorem 3.5. *(The Maximum Principle.) Let $f(z)$ be a nonconstant analytic function in a bounded domain D, and suppose $f(z)$ is continuous on $\overline{D}$. Then $f(z)$ attains its maximum modulus on the boundary ∂D of D.*

This allows us to consider cases where the points $z_1, \ldots, z_{n-1}$ lie on the boundary circles of the annulus $R(a, b)$.

Three Charges on Two Circles

This discussion focuses on looking at the behavoir of three points that are placed on two concentric circles and examining the behavior of those points as we increase the distance between these two circles. There is a fourth electrical charge point built into the model we are using, and it is placed at the origin of the circles. The effect of that fourth point is considered when we find the total energy of the system, but (as we shall show) since we are looking at the behavior of the points in a pairwise system, this point at the origin gives us the effect of having to consider the effect of the individual points themselves on the circles.

Before we begin looking at the possibilities this problem presents us, it is worth noting that the energy equation we will use is given by

$$E(z_1, z_2, z_3) = \left(\sum_{j=1}^{3} \log |z_j| \right) - \left(\sum_{1<j,k<3} \log |z_k - z_j| \right). \tag{3.5}$$

Two Points on the Outer Circle

We begin by considering a pair of concenric circles where the outer circle has two points that lie on it, the inner circle has one point on it, and there is also one point that is located at the origin.

Lemma 3.6. *Consider four electric charges with charge $1/n$: one located at the origin, one charge on a circle of radius a, and two charges located on a circle of radius b, where $b > a$. A minimum potential energy among the four points is achieved when the points on the circle of radius b are separated by no more than π radians.*

Proof. By using rotations and stretching/shrinking on the two circles at the same time, without loss of generality we can consider the charge on the circle of radius a to be located at 1, and we can also consider the points on the circle of radius b to be located on a circle of radius r where $r > 1$. Let the points on the circle of radius b be represented by $z_1 = re^{i\theta_1}$ and $z_2 = re^{i\theta_2}$. We want to maximize the logarithmic potential energy.

We start by considering z_1 and z_2 with $0 < \theta_1, \theta_2 < \frac{\pi}{2}$, respectively. Reflect z_1 over the imaginary axis, thus giving us $z_1^* = re^{i(\pi-\theta_1)}$. Note that $|1 - z_1| < |1 - z_1^*|$ and

$|z_2 - z_1| < |z_2 - z_1^*|$. This increases the logarithmic potential energy, and thus we have $\frac{\pi}{2} < \theta_1^* < \pi$ for z_1^*.

Looking at z_2, we can reflect z_2 across the real axis. Label this point z_2^* where $\frac{3\pi}{2} < \theta_2^* < 2\pi$. In doing so, we have $|z_2 - z_1^*| < |z_2^* - z_1^*|$ and $|1 - z_2| = |1 - z_2^*|$. Thus for z_2^*, we have $\frac{3\pi}{2} < \theta_2^* < 2\pi$, as this also increases the logarithmic potential energy.

Given our established conditions, note that $\max|z_2^* - z_1^*| = 2r$. Consider where a diameter passes through z_1^* and 0. Let z_2^* vary from the diameter by angle ε, and consider the two points $re^{i(\theta_2^*+\varepsilon)}$ and $re^{i(\theta_2^*-\varepsilon)}$. Note that both of these points are the same distance from z_1^*, but $|1 - re^{i(\theta_2^*+\varepsilon)}| < |1 - re^{i(\theta_2^*-\varepsilon)}|$.

Thus for maximum logarithmic potential energy, we want $\frac{\pi}{2} < \theta_1^* < \pi$ and $\theta_1^* < \theta_2^* < \pi + \theta_1^*$. □

Theorem 3.7. *Consider four electric charges with charge* $1/n$*: one located at the origin, one charge* $z = ae^{i\alpha}$ *located on a circle of radius* a*, and two charges* $z_1 = be^{i(\theta_1+\alpha)}$ *and* $z_2 = be^{i(\theta_2+\alpha)}$ *located on a circle of radius* b*, where* $b > a$*. Then the minimum potential energy among the four points is achieved when* $\theta_1 = -\theta_2$*.*

Proof. In order to standardize these results with Lemma 3.6, we consider rotating and stretching/shrinking the circles in the same fashion as we did previously and we will work with the points 1, $z_1 = re^{i\theta_1}$, and $z_2 = re^{i\theta_2}$, where $r > 1$.

The electrostatic field generated by the given charges obey the law of logarithmic potential. We can represent the total energy of the field by

$$E(1, z_1, z_2) = \\ -\frac{1}{2}\left(\log\frac{1}{|1|} + \log\frac{1}{|z_1|} + \log\frac{1}{|z_2|}\right) - (\log|1 - z_1| + \log|1 - z_2| + \log|z_2 - z_1|)$$

In order to minimize the energy field, we want to look at the derivative of E with respect to both θ_1 and θ_2.

In order to simplify our work, we shall use the equation

$$E^*(1, z_1, z_2) = -[\log(1 - re^{i\theta_1}) + \log(1 - re^{i\theta_2}) + \log(re^{i\theta_2} - re^{i\theta_1})]$$

since the first three terms of E are constants, and for any complex number a, we have $\log a = \log|a| + i \arg a$ where $\arg a$ is also a constant.

By taking partial derivatives, we find

$$\frac{\partial E^*}{\partial \theta_1} = -\frac{-rie^{i\theta_1}}{1 - re^{i\theta_1}} - \frac{-rie^{i\theta_1}}{re^{i\theta_2} - re^{i\theta_1}}$$
$$\frac{\partial E^*}{\partial \theta_2} = -\frac{-rie^{i\theta_2}}{1 - re^{i\theta_2}} - \frac{rie^{i\theta_2}}{re^{i\theta_2} - re^{i\theta_1}}$$

If we combine these, we have

$$\frac{\partial E^*}{\partial \theta_1} + \frac{\partial E^*}{\partial \theta_2} = \frac{rie^{i\theta_1}}{1 - re^{i\theta_1}} + \frac{rie^{i\theta_2}}{1 - re^{i\theta_2}} + \frac{ri(e^{i\theta_1} - e^{i\theta_2})}{re^{i\theta_2} - re^{i\theta_1}}$$
$$= \frac{rie^{i\theta_1}}{1 - re^{i\theta_1}} + \frac{rie^{i\theta_2}}{1 - re^{i\theta_2}} - i$$

To find the maximum with respect to θ_1 and θ_2, we can look at $\frac{\partial E^*}{\partial \theta_1} + \frac{\partial E^*}{\partial \theta_2} = 0$. Also, note that $\frac{rie^{i\theta_1}}{1 - re^{i\theta_1}}$ and $\frac{rie^{i\theta_2}}{1 - re^{i\theta_2}}$ have real and imaginary parts. If we only consider the real parts (since the imaginary parts are equated to 0 already), then we have

$$\Re\left(\frac{rie^{i\theta_1}}{1 - re^{i\theta_1}} + \frac{rie^{i\theta_2}}{1 - re^{i\theta_2}} + i\right) = 0$$
$$\Re\left(\frac{rie^{i\theta_1}}{1 - re^{i\theta_1}}\right) + \Re\left(\frac{rie^{i\theta_2}}{1 - re^{i\theta_2}}\right) = 0$$

Note that for the complex number $a + bi$, we have
$\Re i(a + bi) = \Re(ai - b) = -\Im(a + bi)$. After some algebra on

$$\Re\left(\frac{\partial L^*}{\partial \theta_1} + \frac{\partial L^*}{\partial \theta_2}\right) = \Im\left(\frac{e^{i\theta_1}}{1 - re^{i\theta_1}} + \frac{e^{i\theta_2}}{1 - re^{i\theta_2}}\right) = 0,$$

we can simplify the above equation as follows:

$$0 = \frac{e^{i\theta_1}}{1 - re^{i\theta_1}} + \frac{e^{i\theta_2}}{1 - re^{i\theta_2}} = \frac{1}{e^{-i\theta_1} - r} + \frac{1}{e^{-i\theta_2} - r}$$
$$= \frac{r - e^{i\theta_1}}{r^2 - re^{i\theta_1} - re^{-i\theta_1} + 1} + \frac{r - e^{i\theta_2}}{r^2 - re^{i\theta_2} - re^{-i\theta_2} + 1}$$
$$= \frac{r - \cos\theta_1 - i\sin\theta_1}{r^2 - 2r\cos\theta_1 + 1} + \frac{r - \cos\theta_2 - i\sin\theta_2}{r^2 - 2r\cos\theta_2 + 1}$$

We can thus look at

$$\frac{\sin\theta_1}{r^2-2r\cos\theta_1+1}+\frac{\sin\theta_2}{r^2-2r\cos\theta_2+1}=0.$$

Using this equation, we have the following:

$$\begin{aligned}
0 &= \sin\theta_1(r^2-2r\cos\theta_2+1)+\sin\theta_2(r^2-2r\cos\theta_1+1)\\
&= (\sin\theta_1+\sin\theta_2)(r^2+1)-2r[\sin(\theta_1+\theta_2)]\\
&= 2(r^2+1)\left(\sin\frac{\theta_1+\theta_2}{2}\cos\frac{\theta_1-\theta_2}{2}\right)-2r\sin\left(\frac{\theta_1+\theta_2}{2}+\frac{\theta_1+\theta_2}{2}\right)\\
&= \left(\sin\frac{\theta_1+\theta_2}{2}\right)\left[(r^2+1)\left(\cos\frac{\theta_1-\theta_2}{2}\right)-2r\left(\cos\frac{\theta_1+\theta_2}{2}\right)\right]
\end{aligned}$$

In general, $\sin\alpha=0$ for $\alpha=k\pi$ where $k\in\mathbb{N}$. This gives us $\frac{\theta_1+\theta_2}{2}=k\pi$, or $\theta_1=-\theta_2+2k\pi$. Since the sines of coterminal angles are always equal, we only need the result for $\theta_1=-\theta_2$ for $0\le\theta_1<2\pi$.

We also need to examine whether $(r^2+1)(\cos\frac{\theta_1-\theta_2}{2})-2r(\cos\frac{\theta_1+\theta_2}{2})=0$. By rearranging, we will examine

$$\frac{r^2+1}{2r}=\frac{\cos\frac{\theta_1+\theta_2}{2}}{\cos\frac{\theta_1-\theta_2}{2}}.$$

Since we assume $r>1$, then $\frac{r^2+1}{2r}>1$. Thus what we want to show is whether $(\cos\frac{\theta_1+\theta_2}{2})/(\cos\frac{\theta_1-\theta_2}{2})>1$ is possible. From Lemma 3.6, we know that $\frac{\pi}{2}<\theta_1<\pi$ and $\pi<\theta_2<\pi+\theta_1$. We have two possibilities to consider then:

- $\frac{\pi}{2}<\theta_1<\pi$ and $\pi<\theta_2<\frac{3\pi}{2}$. In this case, we have $|\frac{\theta_1-\theta_2}{2}|<\frac{\pi}{2}$, and this implies $\cos\frac{\theta_1-\theta_2}{2}>0$. Now we can look at $\frac{\cos(\theta_1+\theta_2)}{2}$. We consider $\theta_1=\pi-\theta_1'$ and $\theta_2=\pi+\theta_2'$, where θ_1' and θ_2' are the respective reference angles. Then $\theta_1+\theta_2=(\pi-\theta_1')+(\pi+\theta_2')=2\pi+(\theta_2'-\theta_1')$. Note that $(\theta_1+\theta_2)/2=\frac{1}{2}[2\pi+(\theta_2'-\theta_1')]=\pi+\frac{1}{2}(\theta_2'-\theta_1')$. Then $0<\theta_2'-\theta_1'<\pi \implies 0<\frac{1}{2}(\theta_2'-\theta_1')<\frac{\pi}{2} \implies \pi<\pi+\frac{1}{2}(\theta_2'-\theta_1')<\frac{3\pi}{2}$, i.e. we have $\pi<\frac{1}{2}(\theta_1+\theta_2)<\frac{3\pi}{2}$. Thus $\cos\frac{\theta_1+\theta_2}{2}<0$ always, and therefore this possibility does not yield a solution.

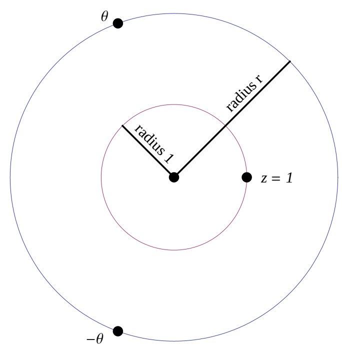

Figure 3.1. Three charges in a system (two charges on outer radius).

- $\frac{\pi}{2} < \theta_1 < \pi$ and $\frac{3\pi}{2} < \theta_2 < \pi + \theta_1$. Let $\theta_1 = \pi - \theta_1'$ and $\theta_2 = 2\pi - \theta_2'$, where θ_1' and θ_2' are the respective reference angles. Then $\theta_1 + \theta_2 = (\pi - \theta_1') + (2\pi - \theta_2') = 3\pi - (\theta_1' + \theta_2') \implies \frac{1}{2}(\theta_1 + \theta_2) = \frac{3\pi}{2} - \frac{1}{2}(\theta_1 + \theta_2)$. Note that $\max(\theta_1' + \theta_2') = \frac{\pi}{2} \implies \frac{1}{2}(\theta_1 + \theta_2) < \frac{3\pi}{2} - \frac{\pi/2}{2} = \frac{3\pi}{2} - \frac{\pi}{4} = \frac{5\pi}{4}$. Thus $\frac{5\pi}{4} < \frac{1}{2}(\theta_1 + \theta_2) < \frac{3\pi}{2}$. Thus $\cos\frac{\theta_1+\theta_2}{2}$ is negative. Note also that $\theta_1 - \theta_2 = (\pi - \theta_1') - (2\pi - \theta_2') = -\pi + (\theta_2' - \theta_1')$, which means $\frac{1}{2}(\theta_2' - \theta_1') = -\frac{\pi}{2} + \frac{1}{2}(\theta_2' - \theta_1')$. Since $\theta_2' > \theta_1'$, then we know $\frac{3\pi}{2} < -\frac{\pi}{2} + \frac{1}{2}(\theta_2' - \theta_1') < 2\pi \implies \cos\frac{\theta_1-\theta_2}{2} > 0$. Again, this does not yield a solution to the equation.

Therefore the energy field is maximized when $\theta_1 = -\theta_2$ for the two points $z_1 = re^{i\theta_1}$ and $z_2 = re^{i\theta_2}$ on a circle of radius r. This arrangement is illustrated by Figure 3.1. □

Now that we know when the logarithmic potential energy is maximized, and thus the potential energy is minimized, it is an exercise to ensure that the result found does indeed match known results, to verify our findings. In order to do this, we shall use the result of the Stieltjes problem on the unit circle. Before we show that the result of our work coincides, however, we shall find θ as a function of r in order to facilitate our understanding of what happens to the angle of separation of the points as the value of r increases. It will be beneficial to us to establish monotonicity for a function that we shall encounter.

Lemma 3.8. *The function*

$$f(r) = \frac{1 + r^2 - \sqrt{1 + 34r^2 + r^4}}{8r} \tag{3.6}$$

is a monotonic increasing function for $r > 1$*.*

Proof. By computation, we find that

$$f'(r) = \frac{(-1 + r^2)(-1 - r^2 + \sqrt{1 + 34r^2 + r^4})}{8r^2\sqrt{1 + 34r^2 + r^4}}$$

and now we need to evaluate what happens to $f'(r)$ for all $r > 1$.

For $r > 1$, we see that

- $(-1 + r^2) > 1$
- $8r^2 > 1$
- $\sqrt{1 + 34r^2 + r^4} > 1$
- We have $1 + r^2 = \sqrt{(1 + r^2)^2} = \sqrt{1 + 2r^2 + r^4} < \sqrt{1 + 34r^2 + r^4}$. This shows us that $-(1 + r^2) + \sqrt{1 + 34r^2 + r^4} > 0$

Thus every factor in the expression is positive, so equation (3.6) is always positive. The function (3.6) is therefore monotonically increasing. □

The information we have found provides us now with the ability to write the function

$$\begin{aligned} E_{\min}(1, re^{i\theta}, re^{-i\theta}) &= -(\log|re^{i\theta} - 1| + \log|re^{-i\theta} - 1| + \log|re^{i\theta} - re^{-i\theta}|) \\ &\quad + \frac{1}{2}(\log|re^{i\theta}| + \log|re^{-i\theta}| + \log|1|) \end{aligned}$$

where we specifically denote this equation because it provides the minimal potential energy in the system.

Theorem 3.9. *For the energy equation* $E_{\min}(1, re^{i\theta}, re^{-i\theta})$*, we have* $\theta = \frac{2\pi}{3}$ *when* $r = 1$ *and* $\theta \to \frac{\pi}{2}$ *when* $r \to \infty$*.*

Proof. For further examination, we will only need to concern ourselves with the equation that gives us the minimum potential energy; that is, we will only need to be concerned with the energy equation when we know $\theta_1 = -\theta_2$. With this condition, we establish the following rewriting of the minimum energy equation:

$$\begin{aligned}
E_{\min}(r,\theta) &= -(\log|re^{i\theta}-1| + \log|re^{-i\theta}-1| + \log|re^{i\theta}-re^{-i\theta}|) \\
&\quad + \frac{1}{2}(\log|re^{i\theta}| + \log|re^{-i\theta}| + \log|1|) \\
&= -2\log|re^{i\theta}-1| - \log|re^{i\theta}-re^{-i\theta}| + \frac{1}{2}(2\log r) \\
&= -2\log|r(\cos\theta + i\sin\theta)-1| - \log|r\cdot(2\sin\frac{-\theta-\theta}{2})| + \log r \\
&= -2\log[(r\cos\theta-1)^2 + (r\sin\theta)^2]^{1/2} - \log 2r - \log\sin\theta + \log r \\
&= -2\log[r^2\cos^2\theta - 2r\cos\theta + 1 + r^2\sin^2\theta]^{1/2}\log\sin\theta + \log\frac{r}{2r} \\
&= -2\log[r^2(\sin^2\theta+\cos^2\theta) - 2r\cos\theta + 1]^{1/2} - \log\sin\theta + \log\frac{1}{2} \\
&= -\log(r^2 - 2r\cos\theta + 1) - \log\sin\theta + \log\frac{1}{2}
\end{aligned}$$

To find the maximum energy, we can take the derivative of E with respect to θ. This gives us

$$\frac{\partial E_{\min}}{\partial\theta} = -\frac{-2r(-\sin\theta)}{r^2-2r\cos\theta+1} - \frac{\cos\theta}{\sin\theta}.$$

By setting $\frac{\partial E_{\min}}{\partial\theta} = 0$, we have

$$\begin{aligned}
0 &= -\frac{-2r(-\sin\theta)}{r^2-2r\cos\theta+1} - \frac{\cos\theta}{\sin\theta} \\
&= 2r\sin^2\theta + \cos\theta(r^2 - 2r\cos\theta + 1) \\
&= 2r - 2r\cos^2\theta + r^2\cos\theta - 2r\cos^2\theta + \cos\theta \\
&= (4r)\cos^2\theta - (r^2+1)\cos\theta - 2r
\end{aligned}$$

Now we can solve for $\cos\theta$:

$$\cos\theta = \frac{(1+r^2) \pm \sqrt{1+34r^2+r^4}}{8r}.$$

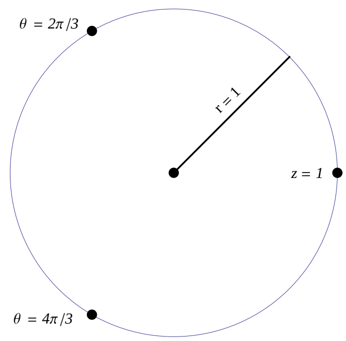

Figure 3.2. The Stieltjes problem with three points

Since we know $\frac{\pi}{2} < \theta < \pi$, then $\cos\theta < 0$, and this means, since $\sqrt{1+34r^2+r^4} > 1+r^2$, we can use the root $\cos\theta = \frac{(1+r^2)-\sqrt{1+34r^2+r^4}}{8r}$. Solving for θ, we have $\theta = \cos^{-1}\left(\frac{(1+r^2)-\sqrt{1+34r^2+r^4}}{8r}\right)$. Also, since $\frac{\pi}{2} < \theta < \pi$, we can work with the reference angle and thus use $\theta = \pi - \cos^{-1}\left(\frac{(1+r^2)-\sqrt{1+34r^2+r^4}}{8r}\right)$.

If we test the known value for the Stieltjes' Problem (i.e. where $r = 1$), we see that

$$\begin{aligned}\theta &= \pi - \cos^{-1}\left(\frac{\sqrt{1+34+1}-(1+1)}{8}\right)\\ &= \frac{2\pi}{3}\end{aligned}$$

This matches the result from Theorem 3.4, and this is illustrated by Figure 3.2.

We also want to know what happens to θ as $r \to \infty$. Thus, we need to look at the following limit:

$$\begin{aligned}&\lim_{r\to\infty} \frac{(1+r^2)-\sqrt{1+34r^2+r^4}}{8r}\\ &\quad= \lim_{r\to\infty} \frac{(1+r^2)^2-(1+34r^2+r^4)}{8r(1+r^2)+8r\sqrt{1+34r^2+r^4}}\\ &\quad= \lim_{r\to\infty} \frac{-32}{8r+\frac{8}{r}+8\sqrt{r^2+34+\frac{1}{r^2}}} \to 0\end{aligned}$$

Looking at our results, Lemma 3.8 demonstrates that for $r = 1$ then $\cos\theta = \frac{2\pi}{3}$.

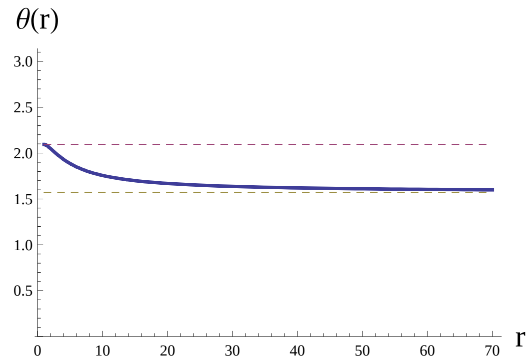

Figure 3.3. Behavior of θ as $r \to \infty$

Our expression for $\cos\theta$, in terms of r, is a monotonically inceasing function, and from our result above, we see that $\cos\theta$ is negative and monotonically increasing to 0 as $r \to \infty$. Thus $\cos\theta \to 0$, that is, $\theta \to \frac{\pi}{2}$. Therefore θ decreases from $\frac{2\pi}{3}$ to $\frac{\pi}{2}$ monotonically (see Figure 3.3). We then see that our two points on the circle of radius r approach $z_1 \to ir$ and $z_2 \to -ir$ monotonically, respectively. This is shown illustrated by Figure 3.4. □

Finding the Logarithmic Potential Energy

We have

$$E_{\min}(r,\theta) = -\log(r^2 - 2r\cos\theta + 1) - \log\sin\theta + \log\frac{1}{2} \tag{3.7}$$

from Theorem 3.9. This is the same as the initial equation we noted at the beginning of Theorem 3.9 except for the conversion of the equation so that it is written in terms of trigonometic functions as opposed to complex values.

Before we begin examining the behavior of the energy of the system, we should consider how this energy behaves; specifically we should examine whether the energy behaves monotonically.

Theorem 3.10. *Consider four electric charges with charge $1/n$: one located at the origin, one charge on a circle of radius a, and two charges located on a circle of radius b, where $b > a$. The minimum potential energy decreases monotonically as $r \to \infty$.*

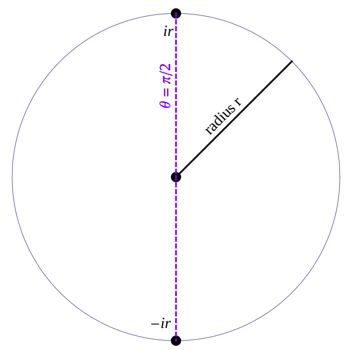

Figure 3.4. The case for $r \to \infty$

Proof. From Theorem 3.9, we have the minimal energy equation (3.7). In order to examine what happens to the energy when r increases to $+\infty$, we need to look at $E_{\min}$ strictly in terms of r. By using one of the results found in Theorem 3.9, we have

$$E_{\min}(r) = -\log\left[r^2 - 2r\left(\frac{(1+r^2)-\sqrt{1+34r^2+r^4}}{8r}\right)+1\right] \\ -\log\sqrt{1-\left(\frac{(1+r^2)-\sqrt{1+34r^2+r^4}}{8r}\right)^2}+\log\frac{1}{2}$$

By computation, we see that

$$\frac{dE_{\min}}{dr} = 2[1-3r^8-\sqrt{1+34r^2+r^4}+r^2(24-7\sqrt{1+34r^2+r^4}) \\ +r^6(-112+3\sqrt{1+34r^2+r^4})-r^4(342+67\sqrt{1+34r^2+r^4})]/ \\ [r\sqrt{1+34r^2+r^4}(3+3r^2+\sqrt{1+34r^2+r^4}) \\ (-1-r^4+\sqrt{1+34r^2+r^4}+r^2(14+\sqrt{1+34r^2+r^4}))]$$

and we want to examine the numerator and denominator of this function. In expanded form, we have

$$\frac{dE_{\min}}{dr} = 2[1 - 3r^8 - \sqrt{1+34r^2+r^4} + 24r^2 - 7r^2\sqrt{1+34r^2+r^4} \\ - 112r^6 + 3r^6\sqrt{1+34r^2+r^4} - 342r^4 - 67r^4\sqrt{1+34r^2+r^4}] \\ [r\sqrt{1+34r^2+r^4}(3+3r^2+\sqrt{1+34r^2+r^4}) \\ (-1 - r^4 + \sqrt{1+34r^2+r^4} + 14r^2 + r^2\sqrt{1+34r^2+r^4})].$$

For $r > 1$, we see that

- $112r^6 > 3r^6$
- $342r^4 > 24r^2$
- $3r^8 > 1$.

Since all other terms in the numerator are negative, this shows us that the numerator of $\frac{dE_{\min}}{dr}$ is always negative for $r > 1$. For the denominator of $\frac{dE_{\min}}{dr}$, we see that

- $r\sqrt{1+34r^2+r^4} > 0$
- $3 + 3r^2 + \sqrt{1+34r^2+r^4} > 0$
- $\sqrt{1+34r^2+r^4} > 1$
- $r^2\sqrt{1+34r^2+r^4} > r^4$, since $\sqrt{1+34r^2+r^4} > r^2$ for $r > 1$

and this shows that the denominator is always positive for $r > 1$.

We thus know that $\frac{dE_{\min}}{dr} < 0$ for $r > 1$, and this tells us that $E_{\min}$ is a monotonically decreasing function.

□

There are two cases to consider.

When $r = 1$, then we know $\theta = \frac{2\pi}{3}$. This is illustrated by Figure 3.2. Thus we have

$$\begin{aligned} E\left(1, \frac{2\pi}{3}\right) &= -\log\left(1^2 - 2(1)\cos\frac{2\pi}{3} + 1\right) - \log\sin\frac{2\pi}{3} + \log\frac{1}{2} \\ &= -\log 3 - \log\frac{\sqrt{3}}{2} + \log\frac{1}{2} \\ &= -\frac{3}{2}\log 3. \end{aligned}$$

From Theorem 3.4, we see that our result is correct. From Theorem 3.10, we see that the energy monotonically decreases from this value, and in fact it will decrease to $-\infty$ as $r \to \infty$.

One point on the outer circle.

In this case, we again look at the system having four electrical charges, each of charge $1/n$: one point that lies on our outer circle, two points that lie on the inner circle, and one point located at the origin. Both circles are concentric.

Lemma 3.11. *Consider four electric charges with charge $1/n$: one located at the origin, two charges on a circle of radius a, and one charge located on a circle of radius b, where $b > a$. A minimum potential energy among the four points is achieved when the points on the circle of radius a are separated by no more than π radians.*

Proof. By using rotations and stretching/shrinking, Without loss of generality we consider the specific case where the inner circle has radius 1 and the outer circle has radius r, where $r > 1$. Let the two points on $C(0,1)$ by denoted by $z_1 = e^{i\theta_1}$ and $z_2 = e^{i\theta_2}$. We want to maximize the logarithmic potential energy.

We start by considering z_1 and z_2 with $0 < \theta_1, \theta_2 < \pi/2$, respectively. Reflect z_1 over the imaginary axis. Label this reflected point $z_1^* = e^{i(\pi - \theta_1)}$. Then $|r - z_1| \leq |r - z_1^*|$ and $|z_2 - z_1| < |z_2 - z_1^*|$. This maximizes the logarithmic potential energy, and thus we will use $\pi/2 < \theta_1^* < \pi$ for z_1^*.

We now reflect z_2 over the real axis. Label this point z_2^* where $3\pi/2 < \theta_2^* < 2\pi$. We find that $|z_2 - z_1^*| < |z_2^* - z_1^*|$ and $|z_2^* - r| = |z_2 - r|$. Since this inreases the logarithmic potential energy, we want $3\pi/2 < \theta_2^* < 2\pi$ for z_2^*.

By our established conditions, note that $\max|z_2^* - z_1^*| = 2$. Consider where a diameter passes through z_1^* and 0. Let z_2^* vary from the diameter by angle ε, and consider the two perturbations of z_2^* given by $e^{i(\theta_2^*+\varepsilon)}$ and $e^{i(\theta_2^*-\varepsilon)}$. Not that both of these points are the same distance from z_1^*, but $|r - e^{i(\theta_2^*+\varepsilon)} < |r - e^{i(\theta_2^*-\varepsilon)}|$.

Thus the maximum logarithmic potential energy occurs when $\pi/2 < \theta_1^* < \pi$ and $\theta_1^* < \theta_2^* < \pi + \theta_1^*$. □

Theorem 3.12. *Consider four electric charges with charge $1/n$: one located at the origin, two charges $z_1 = ae^{i(\theta_1+\beta)}$ and $z_2 = ae^{i(\theta_2+\beta)}$ on a circle of radius a, and one charge $z = be^{i\beta}$ located on a circle of radius b, where $b > a$. Then the minimum potential energy among the four points is achieved when $\theta_1 = -\theta_2$.*

Proof. We can represent the total energy of the field by the equation

$$E(r, z_1, z_2) = -\frac{1}{2}\left(\log\frac{1}{|r|} + \log\frac{1}{|z_1|} + \log\frac{1}{|z_2|}\right) \\ - (\log|r - z_1| + \log|r - z_2| + \log|z_2 - z_1|)$$

We use the rotation and stretching/shrinking argument used in the Lemma 3.6. Since z_1 and z_2 both lie on a circle of radius 1, we can consider $z_1 = e^{i\theta_1}$ and $z_2 = e^{i\theta_2}$. To minimize the energy field, we want to look at $\frac{\partial E}{\partial\theta_1}$ and $\frac{\partial E}{\partial\theta_2}$.

Note that by eliminating some terms and reducing some notation, we can work with

$$E^*(r, z_1, z_2) = \frac{1}{2}\log\frac{1}{r} - \log(r - e^{i\theta_1}) - \log(r - e^{i\theta_2}) - \log(e^{i\theta_2} - e^{i\theta_1})$$

Taking both partial deriatives with respect to θ_1 and θ_2, we have

$$\frac{\partial E^*}{\partial\theta_1} = \frac{ie^{i\theta_1}}{r - e^{i\theta_1}} + \frac{ie^{i\theta_1}}{e^{i\theta_2} - e^{i\theta_1}}$$
$$\frac{\partial E^*}{\partial\theta_2} = \frac{ie^{i\theta_2}}{r - e^{i\theta_2}} - \frac{ie^{i\theta_2}}{e^{i\theta_2} - e^{i\theta_1}}$$

By adding the derivatives together, we have

$$\begin{aligned}\frac{\partial E^*}{\partial \theta_1}+\frac{\partial E^*}{\partial \theta_2} &= \frac{ie^{i\theta_1}}{r-e^{i\theta_1}}+\frac{ie^{i\theta_1}}{e^{i\theta_2}-e^{i\theta_1}}+\frac{ie^{i\theta_2}}{r-e^{i\theta_2}}-\frac{ie^{i\theta_2}}{e^{i\theta_2}-e^{i\theta_1}}\\ &= \frac{ie^{i\theta_1}}{r-e^{i\theta_1}}+\frac{ie^{i\theta_2}}{r-e^{i\theta_2}}-i\end{aligned}$$

To find the minimum values for θ_1 and θ_2, we need to look at $\frac{\partial E^*}{\partial \theta_1}+\frac{\partial E^*}{\partial \theta_2}=0$. Also, we want to note that in general the terms $\frac{ie^{i\theta_1}}{r-e^{i\theta_1}}$ and $\frac{ie^{i\theta_2}}{r-e^{i\theta_2}}$ have both real and imaginary parts. If we consider only the real-number parts, we have

$$\begin{aligned}\Re\left(\frac{ie^{i\theta_1}}{r-e^{i\theta_1}}+\frac{ie^{i\theta_2}}{r-e^{i\theta_2}}-i\right)&=0\\ \Re\left(\frac{ie^{i\theta_1}}{r-e^{i\theta_1}}+\Re\frac{ie^{i\theta_2}}{r-e^{i\theta_2}}-i\right)&=0\end{aligned}$$

Since $\Re i(a+bi)=\Re(ai-b)=-\Im(a+bi)$, then we have

$$\begin{aligned}-\Im\left(\frac{e^{i\theta_1}}{r-e^{i\theta_1}}-\Im\frac{e^{i\theta_2}}{r-e^{i\theta_2}}-i\right)&=0\\ \Im\left(\frac{e^{i\theta_1}}{r-e^{i\theta_1}}+\frac{e^{i\theta_2}}{r-e^{i\theta_2}}\right)&=0\end{aligned}$$

Now we can find our optimal values for θ_1 and θ_2, looking at only the imaginary parts of the following:

$$\begin{aligned}0 &= \frac{e^{i\theta_1}}{r-e^{i\theta_1}}\cdot\frac{e^{-i\theta_1}}{e^{-i\theta_1}}+\frac{e^{i\theta_2}}{r-e^{i\theta_2}}\cdot\frac{e^{-i\theta_2}}{e^{-i\theta_2}}\\ &= \frac{1}{re^{-i\theta_1}-1}+\frac{1}{re^{-i\theta_2}-1}\\ &= \frac{re^{i\theta_1}-1}{r^2-re^{i\theta_1}-re^{-i\theta_1}+1}+\frac{re^{i\theta_2}-1}{r^2-re^{i\theta_2}-re^{-i\theta_2}+1}\\ &= \frac{(r\cos\theta_1-1)+i(r\sin\theta_1)}{r^2-2r\cos\theta_1+1}+\frac{(r\cos\theta_2-1)+i(r\sin\theta_2)}{r^2-2r\cos\theta_2+1}\end{aligned}$$

By looking only at the imaginary parts, we look at

$$\frac{r\sin\theta_1}{r^2-2r\cos\theta_1+1}+\frac{r\sin\theta_2}{r^2-2r\cos\theta_2+1}=0.$$

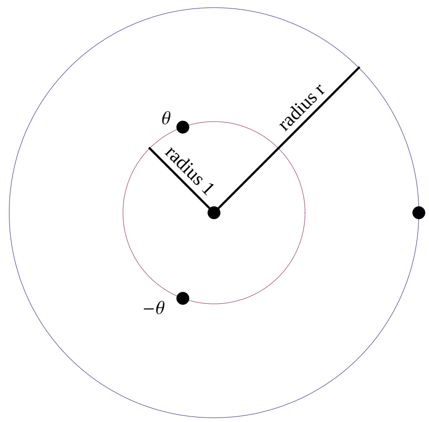

Figure 3.5. Three charges in a system (one charges on outer radius).

We see here that by Theorem 3.7, we have a similar result, and thus the results here will mirror our earlier findings. Thus we have $\theta_1 = -\theta_2$, by Theorem 3.7. This case is illustrated by Figure 3.5. □

For this situation, we find that we have a similar situation as our previously-studied case. So once again we want to examine what happens when $r = 1$ and as $r \to \infty$. We will show this this result still coincides with the Stieltjes problem on the unit circle.

The expression for $\cos\theta$ is the same for this case as for our previous case, and thus Lemma 3.8 applies directly in this situation.

The information we have found allows us to write the function

$$E_{\min}(e^{i\theta}, e^{-i\theta}, r) = -(\log|r - e^{\theta}| + \log|r - e^{-i\theta}| + \log|e^{i\theta} - e^{-i\theta}|) + \frac{1}{2}(\log|r| + \log|e^{i\theta}| + \log|e^{i\theta}|)$$

where we specifically denote this equation because it provides the minimal potential energy in the system.

Theorem 3.13. *For the energy equation $E_{\min}$, we have $\theta = \frac{2\pi}{3}$ when $r = 1$ and $\theta \to \pi/2$ when $r \to \infty$.*

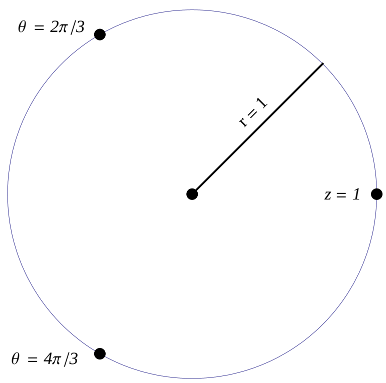

Figure 3.6. The Stieltjes problem with three points

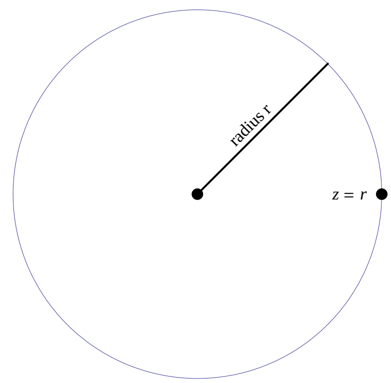

Figure 3.7. The case for $r \to \infty$

Proof. We have the equation, with appropriate values,

$$\begin{aligned} E_{\min}(r,\theta) &= -\log|r-e^{\theta}| - \log|r-e^{-i\theta}| - \log|e^{i\theta}-e^{-i\theta}| \\ &\quad + \frac{1}{2}(\log|r| + \log|e^{i\theta}| + \log|e^{i\theta}|) \\ &= -2\log|r-(\cos\theta + i\sin\theta)| - \log|2\sin(\frac{-\theta-\theta}{2})| + \frac{1}{2}\log r \\ &= -\log(r^2 - 2r\cos\theta + 1) - \log\sin\theta + \log\frac{1}{2} + \frac{1}{2}\log r \end{aligned}$$

To find the maximum logarithmic potential energy, we want to take the derivative of L with respect to θ. This gives us

$$\frac{\partial E}{\partial \theta} = -\frac{-2r(-\sin\theta)}{r^2 - 2r\cos\theta + 1} - \frac{\cos\theta}{\sin\theta}$$

By the work done in Theorem 3.9, we see that $\theta = \frac{2\pi}{3}$. Since the work for this theorem mirrors Theorem 3.9, we see that as $r \to +\infty$, then $z_1 \to i$ and $z_2 \to -i$ as well. □

Figure 3.6 and Figure 3.5 depict the cases described by Theorem 3.13.

Finding the Logarithmic Potential Energy

From Theorem 3.13, we have the equation

$$E_{\min}(r,\theta) = -\log(r^2 - 2r\cos\theta + 1) - \log\sin\theta + \log\frac{1}{2} + \frac{1}{2}\log r \tag{3.8}$$

written so that we can examine the equation in terms of r and θ instead of the complex positions of the charges.

As in the previous case, we want to ensure that the energy behaves monotonically.

Theorem 3.14. *Consider four electric charges with charge $1/n$: one located at the origin, two charges on a circle of radius a, and one charge located on a circle of radius b, where $b > a$. The minimum potential energy decreases monotonically as $r \to \infty$.*

Proof. From Theorem 3.13, we have the minimal energy equation (3.8). In order to examine what happens to the energy when r increases to $+\infty$, we need to look at equation (3.8) strictly in terms of r. By using one of the results found in Theorem 3.9, where we wrote $\cos\theta$ in terms of r, we have

$$\begin{aligned} E_{\min}(r) = &-\log\left[1 + r^2 + \frac{1}{4}(-1 - r^2 + \sqrt{1+34r^2+r^4})\right] \\ &- \log\sqrt{1 - \left(\frac{(1+r^2) - \sqrt{1+34r^2+r^4}}{8r}\right)^2} - \log 2 + \frac{1}{2}\log r \end{aligned}$$

By computation, we see that

$$\begin{aligned} \frac{dE_{\min}}{dr} = &[3 - 5r^8 - 3\sqrt{1+34r^2+r^4} + 5r^6(-36 + \sqrt{1+34r^2+r^4}) \\ &+ 23r^2(4 + \sqrt{1+34r^2+r^4}) - r^4(342 + 97\sqrt{1+34r^2+r^4})]/ \\ &[r\sqrt{1+34r^2+r^4}(3 + 3r^2 + \sqrt{1+34r^2+r^4}) \\ &(-1 - r^4 + \sqrt{1+34r^2+r^4} + r^2(14 + \sqrt{1+34r^2+r^4}))] \end{aligned}$$

and in expanded form we have

$$\begin{aligned}\frac{dE_{\min}}{dr} = {} & [3 - 5r^8 - 3\sqrt{1+34r^2+r^4} - 180r^6 + 5r^6\sqrt{1+34r^2+r^4} \\ & + 92r^2 + 23r^2\sqrt{1+34r^2+r^4} - 342r^4 - 97r^4\sqrt{1+34r^2+r^4}] \\ & [r\sqrt{1+34r^2+r^4}(3+3r^2+\sqrt{1+34r^2+r^4}) \\ & (-1 - r^4 + \sqrt{1+34r^2+r^4} + 14r^2 + r^2\sqrt{1+34r^2+r^4})].\end{aligned}$$

Looking at the denominator, we see that, for $r > 1$, we have

- $180r^6 > 5r^6$
- $342r^4 > 92r^2$
- $97r^4\sqrt{1+34r^2+r^4} > 23r^2\sqrt{1+34r^2+r^4}$
- $3\sqrt{1+34r^2+r^4} > 3$

and thus see that the numerator of $\frac{dE_{\min}}{dr}$ is negative for $r > 1$. The denominator of $\frac{dE_{\min}}{dr}$ is the same as the denominator of the function in equation (3.10), and thus we know the denominator is always positive for $r > 1$. This tells us that equation (3.8) is a monotonically decreasing function. □

We have two cases to consider: when $r = 1$ and what happens when $r \to \infty$. When $r = 1$, we know that $\theta = \frac{2\pi}{3}$. By computation, we find that

$$E_{\min}\left(1, \frac{2\pi}{3}\right) = -\log 2 - \log 3 - \log\frac{\sqrt{3}}{2} = \log\frac{1}{3\sqrt{3}} = -\frac{3}{2}\log 3.$$

From the above theorem, we see that the energy decreases monotonically from this point, and thus the energy decreases to $-\infty$ as $r \to \infty$.

Four Charges on Two Concentric Circles

This section expands our focus to consider the scenario where there are four points located on the two concentric circles and one point located at the origin. We shall consider the system where the fixed negative charge located at the origin has force of absolute value $n/2 - 1$ and there are four free unit charges located at $z_1, \ldots, z_4$. The logarithmic potential energy for this system is given by

$$E(z_1, \ldots, z_4) = \frac{3}{2}\sum_{k=1}^{4} \log|z_k| - \sum_{1 \le j < k \le 4} \log|z_k - z_j|. \tag{3.9}$$

We shall examine two possible cases for this system:

- The Stieltjes Problem (all four points $z_1, \ldots, z_4$ are located on one circle),
- two free unit charges are located on a circle of radius a and two free unit charges are located on a circle of radius b, where $b > a$,

Note that we need to consider only these cases, as opposed to cases where the free unit charges may reside in the annulus between the circles of radius a and b. This is due to the Maximum Principle [12].

3.4 The Stieltjes Problem

It is worth examining the Stieltjes Problem for this scenario, as this will give us the minimum energy possible for having four free unit charges on a circular conductor and one fixed charge located at the origin of the circle. In order to standardize our work and simplify the work necessary, we shall consider the four free unit charges residing on a unit circle and the one fixed negative charge be located at the origin.

From our general results in the previous section, we know that the free unit charges will be equally spaced and will be the roots of the equation $x^4 - 1 = 0$, up to rotation. The energy for the system is thus computed, using the points $z_1, \ldots, z_4$

where $z_k = re^{(\pi k/2)i}$.

$$\begin{aligned}
E(z_1,\dots,z_4) &= \frac{3}{2}\sum_{k=1}^{4}\log|re^{(\pi k/2)i}| - \sum_{1\le j<k\le 4}\log|z_j - z_k| \\
&= \frac{3}{2}(\log|r| + \log|ir| + \log|-r| + \log|-ir|) \\
&\quad - (\log|r-ir| + \log|r-(-r)| + \log|r-(-ir)| \\
&\quad + \log|ir-(-r)| + \log|ir-(-ir)| + \log|-r-(-ir)|) \\
&= \frac{3}{2}(4\log r) - (4\log r\sqrt{2} + 2\log 2r) \\
&= 6\log r - (4\log r\sqrt{2} + 2\log 2r) \\
&= \log r^6 - \log 16r^6 \\
&= \log\frac{1}{16}
\end{aligned}$$

3.5 Two Points on each circle

Geometric Restrictions

We shall consider the case where there are two points on the smaller circle and two points on the larger circle. We need to consider the geometric restrictions that will allow us to find the minimal logarithmic potential energy. We can also look at maximizing the product of the distances between pairs of charges.

To begin, we can transform the inner circle (by stretching or shrinking) so that this inner circle has a radius of length 1. Having done this, we can rotate the concentric circles such that the two points on the inner circle are located at $w_1 = e^{i\alpha}$ and $w_2 = e^{-i\alpha}$. Now consider two points $z_1 = re^{i\theta_1}$ and $z_2 = re^{i\theta_2}$, where $r > 1$ and $\pi/2 > \theta_1 > \theta_2 > 0$. By reflecting z_1 across the imaginary axis (calling this new point $\widehat{z_1}$), we see the following:

- $|\widehat{z_1} - w_1| > |z_1 - w_1|$,
- $|\widehat{z_1} - w_2| > |z_1 - w_2|$, and
- $|\widehat{z_1} - z_2| > |z_1 - z_2|$.

This reflection of z_1 decreases the logarithmic potential energy.

If we reflect z_2 across the real axis (call this reflected point $\widehat{z_2}$), we see the following:

- $|\widehat{z_2} - w_2| = |z_2 - w_1|$,
- $|z_2 - w_2| = |\widehat{z_2} - w_1|$, and
- $|\widehat{z_1} - z_2| < |\widehat{z_1} - \widehat{z_2}|$.

This reflection of z_2 decreases the logarithmic potential energy, as desired.

Minimal Energy

Points $re^{i\theta}$ and $re^{-i\theta}$ on the outer circle

Using equation (3.9), we will determine the location of the charges that will give a minimal logarithmic potential energy when we have $z_1 = e^{i\alpha}$, $z_2 = e^{-i\alpha}$, $z_3 = re^{i\theta}$, and $z_4 = re^{-i\theta}$, where $r > 1$. We also need to specify that $0 < \alpha < \pi/2$ and $\pi/2 < \theta < \pi$.

We start with

$$E(z_1, \ldots, z_{n-1}) = \frac{3}{2}\sum_{k=1}^{4} \log|z_k| - \sum_{1\le j<k\le 4} \log|z_j - z_k|,$$

and using our values for the points z_j, we have

$$\begin{aligned} E(z_1, \ldots, z_4) = \frac{3}{2}&[\log|e^{i\alpha}| + \log|e^{-i\alpha}| + \log|re^{i\theta}| + \log|re^{-i\theta}|] \\ &- [\log|e^{i\alpha} - e^{-i\alpha}| + \log|e^{i\alpha} - re^{i\theta}| + \log|e^{i\alpha} - re^{-i\theta}| \\ &+ \log|e^{-i\alpha} - re^{i\theta}| + \log|e^{-i\alpha} - re^{-i\theta}| + \log|re^{i\theta} - re^{-i\theta}|]. \end{aligned}$$

If we look at the real part of every term, we have the following:

$$\begin{aligned} E(z_1, \ldots, z_4) = 3\log r - [&\log(e^{i\alpha} - e^{-i\alpha}) + \log(e^{i\alpha} - re^{i\theta}) + \log(e^{i\alpha} - re^{-i\theta}) \\ &+ \log(e^{-i\alpha} - re^{i\theta}) + \log(e^{-i\alpha} - re^{-i\theta}) + \log(re^{i\theta} - re^{-i\theta}). \end{aligned}$$

Now we have the following partial derivatives:

$$\frac{\partial E}{\partial \alpha} = -\Re\left[\frac{ie^{i\alpha}+ie^{-i\alpha}}{e^{i\alpha}-e^{-i\alpha}} + \frac{ie^{i\alpha}}{e^{i\alpha}-re^{i\theta}} + \frac{ie^{i\alpha}}{e^{i\alpha}-re^{-i\theta}} + \frac{-ie^{-i\alpha}}{e^{-i\alpha}-re^{i\theta}} + \frac{-ie^{-i\alpha}}{e^{-i\alpha}-re^{-i\theta}}\right]$$

$$\frac{\partial E}{\partial \theta} = -\Re\left[\frac{-rie^{i\theta}}{e^{i\alpha}-re^{i\theta}} + \frac{rie^{-i\theta}}{e^{i\alpha}-re^{-i\theta}} + \frac{-rie^{i\theta}}{e^{-i\alpha}-re^{i\theta}} + \frac{rie^{-i\theta}}{e^{-i\alpha}-re^{-i\theta}} + \frac{rie^{i\theta}+rie^{-i\theta}}{re^{i\theta}-re^{-i\theta}}\right] \tag{3.10}$$

After some simplifying, and noting that we are looking for the minimum, we have the following sum:

$$\begin{aligned}
0 &= \Re\left(\frac{\partial E}{\partial \alpha} + \frac{\partial E}{\partial \theta}\right) \\
&= -\Re\left[\frac{i(e^{i\alpha}+e^{-i\alpha})}{e^{i\alpha}-e^{-i\alpha}} + i + \frac{i(e^{i\alpha}+re^{-i\theta})}{e^{i\alpha}-re^{-i\theta}} + \frac{-i(e^{-i\alpha}+re^{i\theta})}{e^{-i\alpha}-re^{i\theta}} + (-i) + \frac{i(e^{i\theta}+e^{-i\theta})}{e^{i\theta}-e^{-i\theta}}\right] \\
&= \frac{\cos\alpha}{\sin\alpha} + \frac{\cos\theta}{\sin\theta} + \frac{2r\sin(\alpha+\theta)}{1+r^2-r\cos(\alpha+\theta)} \\
&= \frac{\sin(\alpha+\theta)}{\sin\alpha\sin\theta} + \frac{2r\sin(\alpha+\theta)}{1+r^2-r\cos(\alpha+\theta)} \\
&= \sin(\alpha+\theta)[1+r^2-r\cos(\alpha+\theta)+2r\sin\alpha\sin\theta] \\
&= \sin(\alpha+\theta)[1+r^2-2r\cos(\alpha+\theta)+r\cos(\alpha-\theta)]
\end{aligned}$$

This presents us two scenarios. For $\sin(\alpha+\theta)=0$, we see that we have two different possibilites:

- $\alpha+\theta=0$. This leads us to $\theta=-\alpha$.
- $\alpha+\theta=\pi$. This leads us to $\theta=\pi-\alpha$.

We also have $1-2r\cos(\alpha+\theta)+r\cos(\alpha-\theta)+r^2=0$. Since we defined $0<\alpha<\pi/2$ and $\pi/a<\theta<\pi$, then $\cos(\alpha-\theta)>0$ because $\alpha-\theta<0$. We also know $\pi/2<\alpha+\theta<3\pi/2$, thus $\cos(\alpha+\theta)<0$, which gives us $-2\cos(\alpha+\theta)>0$. Specifically, we know $0<-2\cos(\alpha+\theta)<2$. All together, this means that $1-2r\cos(\alpha+\theta)+r\cos(\alpha-\theta)+r^2=0$ has no real solutions.

At this point, we will examine $\frac{\partial E}{\partial \alpha}$ and, letting $\theta=\pi-\alpha$, we want to show that there is one solution and that $\frac{\partial E}{\partial \alpha}$ is monotonic.

Using equation (3.10), we have the following:

$$\begin{aligned}
\frac{\partial E}{\partial \alpha}(\alpha, \pi-\alpha, r) &= -\Re\left[\frac{i(e^{i\alpha}+e^{-i\alpha})}{e^{i\alpha}-e^{-i\alpha}} + \frac{ie^{i\alpha}}{e^{i\alpha}-re^{i(\pi-\alpha)}} + \frac{ie^{i\alpha}}{e^{i\alpha}-re^{-i(\pi-\alpha)}}\right. \\
&\qquad \left. + \frac{-ie^{-i\alpha}}{e^{-i\alpha}-re^{i(\pi-\alpha)}} + \frac{-ie^{-i\alpha}}{e^{-i\alpha}-re^{-i(\pi-\alpha)}}\right] \\
&= -\Re\left[\frac{i(e^{i\alpha}+e^{-i\alpha})}{e^{i\alpha}-e^{-i\alpha}} + \frac{i}{1+re^{-2i\alpha}} + \frac{i}{1+r} + \frac{-i}{1+r} + \frac{-i}{1+re^{2i\alpha}}\right] \\
&= -\Re\left[\frac{i(e^{i\alpha}+e^{-i\alpha})}{e^{i\alpha}-e^{-i\alpha}} + \frac{i[1+re^{2i\alpha}-1-re^{-2i\alpha}]}{1+re^{2i\alpha}+re^{-2i\alpha}+r^2}\right] \\
&= -\Re\left[\frac{\cos\alpha}{\sin\alpha} - \frac{2r\sin 2\alpha}{1+r^2+2r\cos 2\alpha}\right] \\
&= \frac{-\cos\alpha(1+r^2+2r\cos 2\alpha)+2r\sin 2\alpha\sin\alpha}{\sin\alpha(1+r^2+2r\cos 2\alpha)} \\
&= \frac{\cos\alpha[-(1+r^2+2r\cos 2\alpha)+4r\sin^2\alpha]}{\sin\alpha(1+r^2+2r\cos 2\alpha)} \\
&= \frac{\cos\alpha(8r\sin^2\alpha - r^2-2r-1)}{\sin\alpha(1+r^2+2r\cos 2\alpha)}
\end{aligned}$$

Since we want to examine this equation when $\frac{\partial E}{\partial \alpha}(\alpha, \pi-\alpha, r) = 0$, we can clearly see that we have a solution for $\cos\alpha = 0$, i.e. when $\alpha = \pi/2$. We also have to examine the equation $8r\sin^2\alpha - r^2 - 2r - 1 = 0$.

For this second equation, let us consider where the radius r is a fixed value. That is, we want to examine what happens to the free unit charges on the circles as α is allowed to vary. We should notice that, for $8r\sin^2\alpha - r^2 - 2r - 1 = 0$, if we solve for $\sin\alpha$ we arrive at $\sin\alpha = \frac{r+1}{2\sqrt{2r}}$. Since the maximum value of $\sin\alpha$ is 1, this means that we should note the value $r = 3+2\sqrt{2}$, which is one of the roots of the equation $\frac{r+1}{2\sqrt{2r}} = 1$.

The reason for paying attention to $r = 3+2\sqrt{2}$ is because at this value the graph of $(\cos\alpha)(8r\sin^2\alpha - r^2 - 2r - 1) = 0$ changes from have one root strictly between between 0 and $\pi/2$ (for $r < 3+2\sqrt{2}$) to where $(\cos\alpha)(8r\sin^2\alpha - r^2 - 2r - 1) = 0$ has no roots strictly between 0 and $\pi/2$ (for $r \geq 3+2\sqrt{2}$).

It is worth noting here that when we let $r = 1$, we find that the solutions to

$$(\cos\alpha)(8r\sin^2\alpha - r^2 - 2r - 1) = 0$$

are $\alpha_1 = \pi/4$ and $\alpha_2 = \pi/2$. As the value of r increases, the value of α_1 increases as well, up to where $\alpha_1 = \alpha_2 = \pi/2$ when $r \geq 3 + 2\sqrt{2}$.

Points r and $-r$ on the outer circle

We would like to exmaine the system when we have antipodal symmetry between the points on the outer boundary of the annulus. In order to ease the computations, we will consider the charges z_3 and z_4 to be antipodal, and moreover we will consider them located at the points $z_3 = r$ and $z_4 = -r$, without loss of generality. In doing so, we can consider the points z_1 and z_2 to be located anywhere on the inner boundary of the annulus. Using equation (3.9), in this case we will determine the location of charges z_1 and z_2 that give a minimal logarithmic potential energy when we have the points $z_1 = e^{i\alpha}$, $z_2 = e^{i\theta}$, $z_3 = r$, and $z_4 = -r$, where $0 \leq \alpha, \theta \leq 2\pi$ and $r > 1$.

Using equation (3.9) and our values for the points z_j, we have

$$\begin{aligned} E(z_1, \ldots, z_4) = &\frac{3}{2}[\log|e^{i\alpha}| + \log|e^{i\theta}| + \log|r| + \log|-r|] \\ &- [\log|e^{i\alpha} - e^{i\theta}| + \log|e^{i\alpha} - r| + \log|e^{i\alpha} + r| \\ &+ \log|e^{i\theta} - r| + \log|e^{i\theta} + r| + \log|r + r|] \end{aligned} \tag{3.11}$$

Since we want to consider only the real parts of equation (3.11), in finding the derivative $\partial E/\partial\alpha$, we can look at

$$\begin{aligned} \frac{\partial E}{\partial \alpha} &= -\Re\left[\frac{ie^{i\alpha}}{e^{i\alpha} - r} + \frac{ie^{i\alpha}}{e^{i\alpha} + r} + \frac{ie^{i\alpha}}{e^{i\alpha} - e^{i\theta}}\right] \\ &= -\frac{r\sin\alpha}{1 - 2r\cos\alpha + r^2} + \frac{r\sin\alpha}{1 + 2r\cos\alpha + r^2} - \frac{\sin(\alpha - \theta)}{2 - 2\cos(\alpha - \theta)} \end{aligned}$$

and in the same way the derivative $\partial E/\partial\alpha$ is given by

$$\begin{aligned} \frac{\partial E}{\partial \theta} &= -\Re\left[\frac{-ie^{i\theta}}{e^{i\alpha} - e^{i\theta}} + \frac{ie^{i\theta}}{e^{i\theta} + r} + \frac{ie^{i\theta}}{e^{i\theta} - r}\right] \\ &= -\frac{\sin(\theta - \alpha)}{2 - 2\cos(\theta - \alpha)} + \frac{r\sin\theta}{1 + 2r\cos\theta + r^2} - \frac{r\sin\theta}{1 - 2r\cos\theta + r^2} \end{aligned}$$

when fully simplified. The sum of both derivatives is given by

$$\frac{\partial L}{\partial \alpha}+\frac{\partial L}{\partial \theta}= -\frac{r\sin\alpha}{1-2r\cos\alpha+r^2}+\frac{r\sin\alpha}{1+2r\cos\alpha+r^2}-\frac{r\sin\theta}{1-2r\cos\theta+r^2}+\frac{r\sin\theta}{1+2r\cos\theta+r^2}$$

and if we set $\frac{\partial L}{\partial \alpha}+\frac{\partial L}{\partial \theta}=0$ and clear out the denominators, we have

$$\begin{aligned}0 &= r^2\cos\alpha\sin\alpha+2r^4\cos\alpha\sin\alpha+r^6\cos\alpha\sin\alpha-4r^4\cos\alpha\cos^2\theta\sin\alpha \\ &\quad + r^2\cos\theta\sin\theta+2r^4\cos\theta\sin\theta+r^6\cos\theta\sin\theta-4r^4\cos^2\alpha\cos\theta\sin\theta\end{aligned}$$

which simplifies down to

$$0=r^2[(1+r^4)\cos(\alpha-\theta)-2r^2\cos(\alpha+\theta)]\sin(\alpha+\theta). \tag{3.12}$$

The equation $(1+r^4)\cos(\alpha-\theta)-2r^2\cos(\alpha+\theta)=0$ has no real solutions. If we solve this equation for r, we have

$$r^2=\frac{2\cos(\alpha+\theta)\pm\sqrt{-16\sin\alpha\cos\alpha\sin\theta\cos\theta}}{2\cos(\alpha-\theta)}. \tag{3.13}$$

By definition of α and θ, we see that the radicand of equation (3.13) is negative, meaning that r^2 has complex roots. Thus equation (3.12) has no real roots.

If we look at $\sin(\alpha+\theta)=0$, we see that $\alpha+\theta=0$ or $\alpha+\theta=\pi$, which gives us $\theta=-\alpha$ or $\theta=\pi-\alpha$. Since we found, through our geometric argument, that $0\le\alpha\le\pi/2$ and that $\pi\le\theta\le 3\pi/2$ and we are not considering coterminal angles, then $\alpha+\theta=\pi$ is impossible.

Now we need to find the critical point, so we will look at $\frac{\partial E}{\partial \alpha}=0$ and let $\theta=-\alpha$. We have the equation

$$\frac{\partial E}{\partial \alpha}(\alpha,-\alpha,r)=-\Re\left[\frac{ie^{i\alpha}}{e^{i\alpha}-r}+\frac{ie^{i\alpha}}{e^{i\alpha}+r}+\frac{ie^{i\alpha}}{e^{i\alpha}-e^{-i\alpha}}\right] \tag{3.14}$$

which simplifies down to

$$\frac{\partial E}{\partial \alpha}(\alpha, -\alpha, r) = -\frac{r \sin \alpha}{1 - 2r \cos \alpha + r^2} + \frac{r \sin \alpha}{1 + 2r \cos \alpha + r^2} - \frac{\cos \alpha}{2 \sin \alpha}.$$

If we set $\frac{\partial E}{\partial \alpha}(\alpha, -\alpha, r) = 0$ and clear out denominators, we have

$$-\cot \alpha - 2r^2 \cot \alpha - r^4 \cot \alpha + 4r^2 \cos^2 \alpha \cot \alpha - 4r^2 \cos \alpha \sin \alpha = 0$$

and this equation simplifies to

$$-[(1 + r^2)^2 - 4r^2 \cos 2\alpha] \cot \alpha = 0.$$

This gives us two different possible solutions. We have

$$\begin{aligned} \cot \alpha &= 0 \\ \alpha &= \frac{\pi}{2}, \frac{3\pi}{2} \end{aligned}$$

but we see that that these are the only solutions. This is because

$$\begin{aligned} (1 + r^2)^2 - 4r^2 \cos 2\alpha &= 0 \\ \cos 2\alpha &= \frac{(1 + r^2)^2}{4r^2} \end{aligned}$$

and since we specified that $r > 1$, this always gives $\frac{(1+r^2)^2}{4r^2} > 1$. Since $\cos 2\alpha > 1$ is impossible, no solution is possible from this case.

3.6 Charges in a Square

In this section, we deal with three and four charges on a square. The problem on distributions of more than four charges on a square which minimize the potential energy remains open.

Three Charges on a Square

We begin by looking at three charges being placed on a square. The inspiration comes from the Stieltjes problem, in that we will consider three points on the boundary of a square and determine the placement of the charges so that the system has minimal total potential energy. (Note that the charges will be located on the boundary of the square, due to the Maximum Principle.) As with our earlier cases on the circle, this will involve working within geometric restrictions and determining the ideal positions using the found restrictions.

To begin, let $Q = \{z = x + iy : |x| = 1, |y| = 1\}$. Consider Problem 1 for three unit charges $\{\nu_1, \nu_2, \nu_3\}$ distributed on the boundary of Q, denoted by ∂Q. Consider the position of the charges z_1, z_2, and z_3, respectively, for the charges on ∂Q. We will also begin by assuming that the three points on the square have the following coordinates:

- $z_1 = 1 + iy_1$ where $0 < y_1 < 1$,
- $z_2 = 1 + iy_2$ where $0 < y_2 < 1$, and
- $z_3 = x + i$ where $0 < x < 1$.

In order to maximize the distance, we begin by reflecting z_3 across the imaginary axis, giving us the point $z_3^* = -x + i$. This gives us $|z_3^* - z_1| > |z_3 - z_1|$ since

$$\begin{aligned}|z_3^* - z_1| &= |(-x + i) - (1 + iy_1)| = |-(x + 1) + i(1 - y_1)| \\ &= \sqrt{(x + 1)^2 + (1 - y_1)^2} = \sqrt{x^2 + y_1^2 + 2x - 2y_1 + 2}\end{aligned}$$

$$\begin{aligned}|z_3 - z_1| &= |(x + i) - (1 + iy_1)| = |(x - 1) + i(1 - y_1)| \\ &= \sqrt{(x - 1)^2 + (1 - y_1)^2} = \sqrt{x^2 + y^2 - 2x - 2y_1 + 2}\end{aligned}$$

and we see that $\sqrt{x^2 + y_1^2 + 2x - 2y_1 + 2} > \sqrt{x^2 + y^2 - 2x - 2y_1 + 2}$. We also have $|z_3^* - z_2| > |z_3 - z_2|$ since

$$\begin{aligned}|z_3^* - z_2| &= |(-x + i) - (1 + iy_2)| = \sqrt{(x + 1)^2 + (1 - y_2)^2} \\ &= \sqrt{x^2 + y^2 + 2x - 2y^2 + 2}\end{aligned}$$

$$\begin{aligned}|z_3 - z_2| &= |(x + i) - (1 + iy_2)| = \sqrt{(x - 1)^2 + (1 - y_2)^2} \\ &= \sqrt{x^2 + y_2^2 - 2x - 2y_2 + 2}\end{aligned}$$

and we see that $\sqrt{x^2+y^2+2x-2y^2+2} > \sqrt{x^2+y_2^2-2x-2y_2+2}$.

We can further maximize the sum of the distances by moving z_3^* to the point $\widehat{z_3} = -1+i$. Note that $|\widehat{z_3} - z_1| > |z_3^* - z_1|$ since

$$\begin{aligned} |\widehat{z_3} - z_1| &= |(-1+i) - (1+iy_1)| = \sqrt{(-2)^2 + (1-y_1)^2} \\ &= \sqrt{y_1^2 - 2y_1 + 5} \end{aligned}$$

$$|z_3^* - z_1| = \sqrt{x^2 + y_1^2 + 2x - 2y_1 + 2}$$

and thus $\sqrt{y_1^2 - 2y_1 + 5} > \sqrt{x^2 + y_1^2 + 2x - 2y_1 + 2}$ since $0 < x < 1$ by assumption.

In a similar fashion, we can reflect z_2 across the real axis (obtaining the point $\overline{z_2}$) to increase the sum of the distances between the charges. In fact, we can move $\overline{z_2}$ to the point $\widehat{z_2} = 1 - i$ to produce the maximum distance. We see this by

$$|\widehat{z_3} - \widehat{z_2}| = |(-1+i) - (1-i)| = 2\sqrt{2}$$

$$|\widehat{z_3} - \overline{z^2}| = |(-1+i) - (1-iy_2)| = \sqrt{y_2^2 + 2y_2 + 5}$$

and noting that $0 < y_2 < 1$ implies that $|\widehat{z_3} - \overline{z^2}| < 2\sqrt{2}$. Thus for this system, we have the distances $\sqrt{4+(1-y_1)^2}$, $y_1 + 1$, and $2\sqrt{2}$.

Now we wish to consider moving the point z_1 to the point $\widehat{z_1} = 1 + i$. In doing so, this system has distances 2, 2, and $2\sqrt{2}$. We want to show that the system of charges $\{\widehat{z_1}, \widehat{z_2}, \widehat{z_3}\}$ has greater energy than the system $\{z_1, \widehat{z_2}, \widehat{z_3}\}$, and thus we can look at the product of the distances for each configuration. Indeed, we have the product

$$f(z_1, \widehat{z_2}, \widehat{z_3}) = |z_1 - \widehat{z_2}||z_1 - \widehat{z_3}||\widehat{z_2} - \widehat{z_3}| = (y_1+1)(\sqrt{y_1^2 - 2y_1 + 5})(2\sqrt{2})$$

which is a strictly increasing function ranging from $f(0) = 2\sqrt{10}$ to $f(1) = 8\sqrt{2}$. This means the system of charges $\{\widehat{z_1}, \widehat{z_2}, \widehat{z_3}\}$ provides a smaller energy than $\{z_1, \widehat{z_2}, \widehat{z_3}\}$.

It is also worth investitaging the result of varying the real coordinate of the extreme charge $\widehat{z_2}$. Let us consider the system with charges $\widehat{z_1}$, $\widetilde{z_2} = x - i$ for

$0 < x < 1$, and $\widehat{z_3}$. (Note that symmetry eliminates the need for us to consider $-1 < x < 1$.) The product of distance between these three charges is given by

$$f(\widehat{z_1}, \widetilde{z_2}, \widehat{z_3}) = |\widehat{z_1} - \widetilde{z_2}||\widehat{z_1} - \widehat{z_3}||\widetilde{z_2} - \widehat{z_3}| = 2\sqrt{25 + 6x^2 + x^4}$$

which is a strictly increasing function ranging from $f(0) = 10$ to $f(1) = 8\sqrt{2}$. Thus as f is increasing, the maximum product is attained at $x = 1$, the largest allowable value of x in our system.

Four Charges on a Square

This is a variation on the previous result. This time we will consider a system containing found unit charges contained in a square conductor. According to the Maximum Principle, these charges will reside on the boundary of the square. Let us define the square $Q = \{z = x + iy : |x| = 1, |y| = 1\}$. Consider Problem 1 for four unit charges $\{\nu_1, \nu_2, \nu_3\, \nu_4\}$ distributed on the boundary of Q, denoted by ∂Q, and consider the position of the charges z_1, z_2, z_3, and z_4, respectively, for the charges on ∂Q. Without loss of generality let us assign the four charges on the square the following coordinates:

- $z_1 = 1 + iy_1$,
- $z_2 = 1 + iy_2$,
- $z_3 = 1 + iy_3$, and
- $z_4 = 1 + iy_4$

where $0 < y_4 < y_3 < y_2 < y_1 < 1$. We want to find the location of the charges that will product the maximum value of the products of the distances between the points, as this will produce the minimum logarithmic potential energy in the system. Through a geometric argument, we will show that the desired configuration is to position one charge at each of the corners of the square.

We can move the charge ν_1 to the point $\widetilde{z_1} = 1 + i$. Then we can reflect the charge ν_2 across the polar line $\theta = \pi/4$ to the point $\widehat{z_2} = y_2 + i$. This second geometric argument increases the distance between z_2 and the two points z_3 and z_4 while maintaining the same distance from z_1.

By reflecting the point z_3 across the imaginary axis to the point $\widehat{z_3} = -1 + iy_3$, the distance between $\widehat{z_3}$ and the points $\widetilde{z_1}$, $\widehat{z_2}$, and z_4 all increase. Further, we can reflect z_4 across the real axis to the point $\overline{z_4} = 1 - iy_4$ to increase the distances between $\overline{z_4}$ and $\widetilde{z_1}$, $\widehat{z_2}$, and $\widehat{z_3}$. Continuing in this vein, move the point $\overline{z_4}$ to the point $\widetilde{z_4} = 1 - i$.

This establishes the following respective coordinates for the charges $\nu_1, \dots, \nu_4$ (allowing for relabeling of the charge coordinates):

- $z_1 = 1 + i$,
- $z_2 = y_2 + i$,
- $z_3 = -1 + iy_3$, and
- $z_4 = 1 - i$

where $0 < y_3 < y_2 < 1$. In continuing the process, we can reflect z_2 across the real axis to the point $\overline{z_2} = y_2 - i$. While the distances between z_1 and z_4 are preserved (by symmetry), the distance between $\overline{z_2}$ and z_3 is increased.

Now we can consider moving z_3 to the point $\widetilde{z_3} = -1 + i$. In doing so, note that the distances $|z_1 - \widetilde{z_3}|$, $|z_2 - \widetilde{z_3}|$, and $|\widetilde{z_3} - z_4|$ are affected. (The other three distances $|z_1 - z_2|$, $|z_1 - z_4|$, and $|z_2 - z_4|$ are remain contant.) Let us take note of the two difference product of distances:

$$
\begin{aligned}
D &= |z_1 - z_3| \cdot |z_2 - z_3| \cdot |z_3 - z_4| \\
&= |(1 + i) - (-1 + iy_3)| \cdot |(y_2 - i) - (-1 + iy_3)| \cdot |(-1 + iy_3) - (1 - i)| \\
&= \sqrt{5 - 2y_3 + y_3^2}\sqrt{2 + 2y_2 + 2y_3 + y_2^2 + y_3^2}\sqrt{5 + 2y^3 + y_3^2}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{D} &= |z_1 - \widetilde{z_3}| \cdot |z_2 - \widetilde{z_3}| \cdot |\widetilde{z_3} - z_4| \\
&= |(1 + i) - (-1 + i)| \cdot |(y_2 - i) - (-1 + i)| \cdot |(-1 + i) - (1 - i)| \\
&= 4\sqrt{2}\sqrt{5 + y_2 + y_2^2}.
\end{aligned}
$$

In considering D, if we look at $\frac{\partial D}{\partial y_3}$ and leave y_2 fixed, we have

$$
\frac{\partial D}{\partial y_3} = \frac{25 + (37 + 12y_2 + 6y_2^2)y_3 + 18y_3^2 + 2(8 + 2y_2 + y_2^2)y_3^3 + 5y_3^4 + 3y_3^5}{\sqrt{5 - 2y_3 + y_3^2}\sqrt{5 + 2y_3 + y_3^2}\sqrt{2 + 2y_2 + y_2^2 + 2y_3 + y_3^2}}.
$$

This shows that D is a monotonically increasing function, and thus the product of distances D is reach a maximum value when z_3 is located at the point $-1+i$, since $-1+i$ provides the largest possible value of y_3. Let us relabel $z_3 = -1+i$ for ease of notation.

Now we have to examine the optimal position of the point $z_2 = y_2 - i$. We can reflect this point across the imaginary axis to get the point $\widehat{z_2} = -y_2 - i$, which gives us three new distances between points in the system: $|z_1 - \widehat{z_2}|$, $|z_3 - \widehat{z_2}|$, and $|z_4 - \widehat{z_2}|$. The other three distances in the system remain constant. This gives us a product D of distances:

$$\begin{aligned} D &= |z_1 - \widehat{z_2}| \cdot |z_3 - \widehat{z_2}| \cdot |z_4 - \widehat{z_2}| \\ &= |(1+i) - (-y_2 - i)| \cdot |(-1+i) - (-y_2 - i)| \cdot |(1-i) - (-y_2 - i)| \\ &= \sqrt{y_2^2 + 2y_2 + 5}\sqrt{y_2^2 - 2y_2 + 5}\sqrt{y_2^2 + 2y_2 + 1}. \end{aligned}$$

If we look at $\frac{dD}{dy_2}$, then we have

$$\frac{dD}{dy_2} = \frac{25 + 6y_2 + 12y_2^2 + 2y_2^3 + 3y_2^4}{(1+y_2)\sqrt{5 - 2y_2 + y_2^2}} \sqrt{\frac{(1+y_2)^2}{5 + 2y_2 + y_2^2}}. \tag{3.15}$$

Since $y_2 \in [0,1]$, then we see that equation (3.15) is always positive, and this means that D is an increasing function. Thus the maximum value of D occurs when $\widehat{z_2}$ is located at the point $-1-i$.

This means our maximum product of distances $\prod_{1 \le j < k \le 4} |z_j - z_k|$ occurs when each of the the four charges are located in the corners of the square.

CHAPTER 4
SOME RESULTS ON THE DISTRIBUTION OF CHARGES IN VARYING DOMAINS

In this chapter, we discuss some questions on the behavior of energy minimizing configurations of charges in varying domains. As an example demonstrating the so-called *break of symmetry effect*, we will consider three charges situated on a family of rectangles continuously depending on a parameter which transforms a square into a segment.

4.1 Configurations of Charges in Varying Domains

First we introduce necessary notation and explain some terminology. For this chapter, we will denote the energy described by Problem 1 by L.

Definition 4.1. *Consider the complex plane* $\mathbb{C}$ *with the usual Euclidean metric, and for a subset* $E \subset \mathbb{C}$ *and* $\varepsilon > 0$*, let* E^ε *be the* ε*-neighborhood of* E*; that is,* $E^\varepsilon = \{z \in \mathbb{C} : |z - \zeta| < \varepsilon \text{ for some } \zeta \in E\}$*. Let* $\mathcal{H}$ *be the collection of all (nonempty) closed, bounded subsets of* $\mathbb{C}$*. If* $E,\ F \in \mathcal{H}$*, define* $\rho(E, F) = \inf\{\varepsilon : E \subset F^\varepsilon \text{ and } F \subset E^\varepsilon\}$*. The metric* ρ *is called the* **Hausdorff metric.**

Definition 4.2. *A sequence* $E_1, \ldots, E_j, \ldots$ *of nonempty compact subsets in* $\mathbb{C}$ *is said to* **converge in the Hausdorff metric** *to the nonempty compact set* E *in* $\mathbb{C}$ *if* $\rho(E_j, E) \to 0$ *as* $j \to \infty$*.*

Theorem 4.3. *Let* E_j*, where* $j = 1, 2, \ldots$*, be a sequence of compact sets on* $\mathbb{C}$ *such that* E_j *converges to a compact set* E *which contains at least* n *distinct points in the Hausdorff metric. Let* $\{z_1^{(j)}, \ldots, z_n^{(j)}\}$ *be a set of* n *extremal points for Problem 1 for the set* E_j*. Suppose further that for each* k *we also have* $z_k^{(j)} \to z_k^* \in E$ *as* $j \to \infty$*. Then the limit set* $\{z_1^*, \ldots, z_n^*\}$ *is extremal for Problem 1 for the set* E*.*

Proof. For each compact set E_j, we have the logarithmic potential energy

$$L_n(E_j) = -\sum_{1 \le m < \ell \le n} \log |z_m^{(j)} - z_\ell^{(j)}|. \tag{4.1}$$

First we need to show that all points $\{z_1^*, \ldots, z_n^*\}$ are distinct. This can be shown by a contradiction argument. For the set of points $\{z_1^*, \ldots, z_n^*\}$, assume $z_m^* = z_\ell^*$ where $1 \le m < \ell \le n$. Then $|z_m^* - z_\ell^*| = 0$ which gives us $\log|z_m^* - z_\ell^*| = -\infty$. Since E has a finite diameter the latter means means that for E we have $L_n(E) = +\infty$ contradicting the minimality property. Thus all points $\{z_1^*, \ldots, z_n^*\}$ must be distinct.

It is worth mentioning that $\{z_1^*, \ldots, z_n^*\} \subset E$. Indeed, if some $z_m^* \notin E$, then $d(z_m^*, E) > 0$, where d is the usual Euclidean metric. This means we have a sequence of points $z_m^{(j)} \to z_m^*$, where $z_m^{(j)} \in E_j$ for all $j = 1, 2, \ldots$. Choose $\delta > 0$ such that

$$d(z_m^*, E) = d(E_j, E) + \delta.$$

Let $\mathcal{H} = \{E \cup \{E_j\}_{j=1}^\infty\}$ in the Hausdorff metric. Since $\mathbb{C}$ with the usual Euclidean metric is complete, then $\mathcal{H}$ in the Hausdorff metric is complete as well. This means $\delta \to 0$, since $\{z_k^{(j)}\}_{j=1}^\infty$ is a convergent sequence in E_j as $j \to \infty$, the set E_j is closed, and thus the limit point of $\{z_k^{(j)}\}_{j=1}^\infty$ must lie in E_j. Since we assume $z_m^* \notin E$, then this implies there exists a value $\gamma \in \mathbb{N}$ such that $z_m^* \notin E_\gamma$ for all $\gamma \ge J$. This is a contradiction, since the limit point of the sequence does not lie in the set containing the sequence, and thus is impossible. Therefore z_m^* must lie in E.

Now we need to show that the logarithmic potential energy associated with $\{z_1^*, \ldots, z_n^*\}$ is extremal and satisfies Problem 1. Notice that the right-hand side of equation (4.1) is continuous. Thus we see that, as $n \to \infty$, then $L_n(E_j)$ approaches the limit

$$L_n(E) = -\sum_{1 \le m < \ell \le n} \log|z_m^* - z_\ell^*|. \tag{4.2}$$

Suppose that $L_n(E) \le -\sum_{1 \le m < \ell \le n} \log|z_m^* - z_\ell^*|$. Assume there are points $\{\widehat{z_1}, \ldots, \widehat{z_n}\} \subset E$, where at least one $\widehat{z_m} \notin \{z_1^*, \ldots, z_n^*\}$, such that the minimum logarithmic potential energy is given by

$$L_n(E) = -\sum_{1 \le m < \ell \le n} \log|\widehat{z_m} - \widehat{z_\ell}|.$$

There must also be points $\{\widehat{z_1}^{(j)}, \ldots, \widehat{z_n}^{(j)}\}$ in E_j such that $\widehat{z_k}^{(j)} \to \widehat{z_k}$, since each sequence $\{z_k^{(j)}\}_{j=1}^n$ already converges to z_k^* as $j \to \infty$. We can also write

$$-\sum_{1\le m<\ell\le n} \log|\widehat{z_m} - \widehat{z_\ell}| = -\sum_{1\le m<\ell\le n} \log|z_m^* - z_\ell^*| - \varepsilon_0$$

where $\varepsilon_0 > 0$. Since $\{z_1^{(j)}, \ldots, z_n^{(j)}\}$ are the extremal points for E_j, then we know

$$-\sum_{1\le m<\ell\le n} \log|z_m^{(j)} - z_\ell^{(j)}| \le -\sum_{1\le m<\ell\le n} \log|\widehat{z_m}^{(j)} - \widehat{z_\ell}^{(j)}|.$$

This gives us the following chain of inequalities:

$$\begin{aligned} L_n(E_j) &= -\sum_{1\le m<\ell\le n} \log|z_m^{(j)} - z_\ell^{(j)}| \\ &\le -\sum_{1\le m<\ell\le n} \log|\widehat{z_m}^{(j)} - \widehat{z_\ell}^{(j)}| \\ &< -\sum_{1\le m<\ell\le n} \log|\widehat{z_m} - \widehat{z_\ell}| \\ &= -\sum_{1\le m<\ell\le n} \log|z_m^* - z_\ell^*| - \varepsilon_0 \\ &= \left(-\sum_{1\le m<\ell\le n} \log|z_m^{(j)} - z_\ell^{(j)}| + \alpha_j\right) - \varepsilon_0 \end{aligned}$$

for some value $\alpha_j > 0$ where

$$\alpha_j = -\sum_{1\le m<\ell\le n} \log|z_m^* - z_\ell^*| + \sum_{1\le m<\ell\le n} \log|z_m^{(j)} - z_\ell^{(j)}| \tag{4.3}$$

depends on E_j. Notice that our resulting inequality is

$$-\sum_{1\le m<\ell\le n} \log|z_m^{(j)} - z_\ell^{(j)}| < -\sum_{1\le m<\ell\le n} \log|z_m^{(j)} - z_\ell^{(j)}| + \alpha_j - \varepsilon_0$$

which simplifies down to $0 < \alpha_j - \varepsilon_0$.

From equation (4.3), we see that α_j is continuous. Thus as $j \to \infty$, then we have $a_j \to 0$ since $E_j \to E$. Since ε_0 represents the amount of different between $-\sum_{1\le m<\ell\le n} \log|\widehat{z_m} - \widehat{z_\ell}|$ and $-\sum_{1\le m<\ell\le n} \log|z_m^* - z_\ell^*|$, this amount is constant.

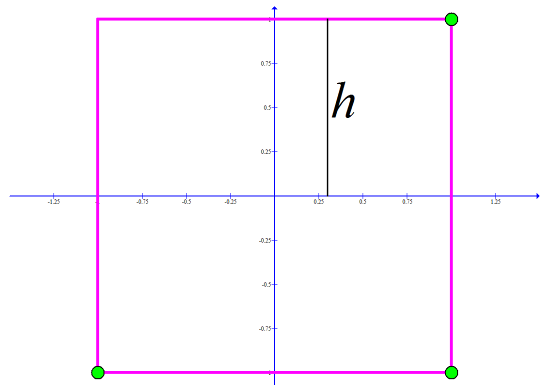

Figure 4.1. Three equilibrium points on a square.

Thus as $j \to \infty$, we have $0 < -\varepsilon_0$, which is impossible. Thus we must have equation (4.2) as true. $\square$

Corollary 4.4. *Let a compact set E, a sequence of compact sets $\{E_j\}_{j=1}^{\infty}$, and a sequence of sets of extrema points $\left\{\{z_k^{(j)}\}_{k=1}^{n}\right\}_{j=1}^{\infty}$ be defined as in Theorem 4.3. In addition, assume that E has a unique set of points $\{z_1^*, \ldots, z_n^*\}$ extremal for Problem 1. If the set of extremal points $\{z_1^*, \ldots, z_n^*\}$ is unique for E, then every sequence of extremal configurations $\{z_1^{(j)}, \ldots, z_n^{(j)}\} \subset E_j$ will converge in the Hausdorff metric to the extremal configuration $\{z_1^*, \ldots, z_n^*\}$ of E.*

Proof. Assume the conclusion is false. Then we assume there is a sequence $\{\widetilde{z}_1{}^{(j)}, \ldots, \widetilde{z}_n{}^{(j)}\}$ of extremal points in E_j that does not converge to $\{z_1^*, \ldots, z_n^*\}$. By Theorem 4.3, we know that the sequence must converge to a limit set of points $\{\widetilde{z}_1, \ldots, \widetilde{z}_n\} \subset E$ which is extremal. Thus we have a set of extremal points $\{\widetilde{z}_1, \ldots, \widetilde{z}_n\} \neq \{z_1^*, \ldots, z_n^*\}$ in E. This contradicts there existing only one unique set of extremal points in E. Therefore every sequence $\{z_1^{(j)}, \ldots, z_n^{(j)}\}$ must converge to $\{z_1^*, \ldots, z_n^*\}$. $\square$

4.2 Deforming a Square with Three Charges into a Segment

We want to examine what happens when we transform the square into a rectangle. Specifically, we will start with a square and reduce the height h of the rectangle. Figure 4.1 represents the system where $h = 1$.

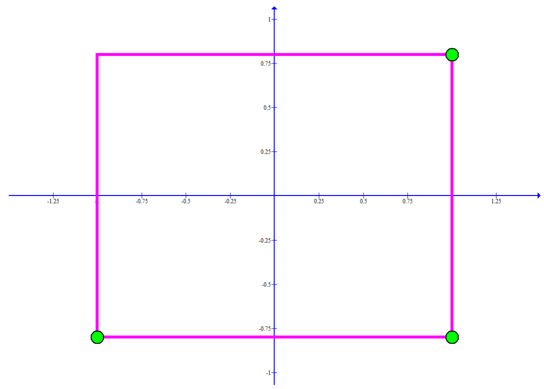

Figure 4.2. Three points on a rectangle.

We will construct the rectangles by reducing the height h from 1 to 0, in essence taking the square into rectangles and then finally into a straight line. In order to do this, we will consider the movement of the point at $z = 1 + i$. As illustated in Figure 4.2, we will consider the point at $z = 1 + i$ to be free-moving, and thus we have the following distances to consider:

- $|(-1 - ih) - (x + ih)| = \sqrt{x^2 + 2x + 1 + 4h^2}$
- $|(1 - ih) - (x + ih)| = \sqrt{x^2 - 2x + 1 + 4h^2}$

Look at the energy

$$L(x, h) = -\log\sqrt{x^2 + 2x + 1 + 4h^2} - \log\sqrt{x^2 - 2x + 1 + 4h^2}$$

and find the minimum value of L. We can look at

$$\frac{\partial L}{\partial x} = -\left(\frac{x+1}{x^2 + 2x + 1 + 4h^2} + \frac{x-1}{x^2 - 2x + 1 + 4h^2}\right)$$

and by setting $\frac{\partial L}{\partial x} = 0$, we want to find the zeros of

$$0 = \frac{2x(x^2 - 1 + 4h^2)}{(x^2 + 2x + 1 + 4h^2)(x^2 - 2x + 1 + 4h^2)}.$$

This reduces to looking only at the numerator of the above fraction. Thus we have the zeros $x = 0, \pm\sqrt{1-4h^2}$. By symmetry, we only need to look at the critical points at $x = 0$ and $x = \sqrt{1-4h^2}$.

Notice that $x = \sqrt{1-4h^2}$ is defined within our domain only for $h \in [-\frac{1}{2}, \frac{1}{2}]$, but for practical purposes we will use $h \in [0, \frac{1}{2}]$ with respect to the height of the rectangles.

When $h \geq \frac{1}{2}$, then there is only one critical point, namely where $x = 0$. For the derivative function $\frac{\partial L}{\partial x}$, note that the following is true when $h > \frac{1}{2}$:

- if $x < 0$, then $\frac{\partial L}{\partial x} > 0$, and
- if $x > 0$, then $\frac{\partial L}{\partial x} < 0$.

Thus when $h > \frac{1}{2}$, then $x = 0$ is a maximum point of L. For $x \in [0, 1]$ then L is always decreasing, and this meand the minimum value for L is achieved when $x = 1$, the largest value of x allowable.

When $h < \frac{1}{2}$, then there are three critical points, namely $x = 0$, $x = \sqrt{1-4h^2}$, and $x = -\sqrt{1-4h^2}$. By symmetry, we will only look at $x = 0$ and $x = \sqrt{1-4h^2}$. For the function $L(x, h)$, we have

$$\left.\frac{\partial^2 L}{\partial x^2}\right|_{x=\sqrt{1-4h^2}} = 1 - \frac{1}{4h^2}$$

and thus when $h < 1/2$ we have $\dfrac{\partial^2 L}{\partial x^2} < 0$. This implies the local minimum values must occur at either $x = 0$ or $x = 1$.

We want to find the value of h where the minimum of L changes from $x = 1$ to $x = 0$. That is, for

$$L(x, h) = -\log\sqrt{x^2 + 2x + 1 + 4h^2} - \log\sqrt{x^2 - 2x + 1 + 4h^2}$$

we want to find when $L(0, h) < L(1, h)$.

Look at $L(0, h) = -\log(1 + 4h^2)$ and $L(1, h) = -\frac{1}{2}\log(4 + 4h^2) - \frac{1}{2}\log(4h^2)$. Then we have

$$-\log(1 + 4h^2) < -\frac{1}{2}\log(4 + 4h^2) - \frac{1}{2}\log(4h^2).$$

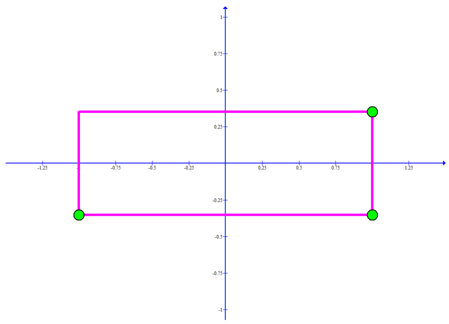

Figure 4.3. Extremal energy with $h = \frac{\sqrt{2}}{4}$ and charge at $1 + \frac{\sqrt{2}}{4}i$

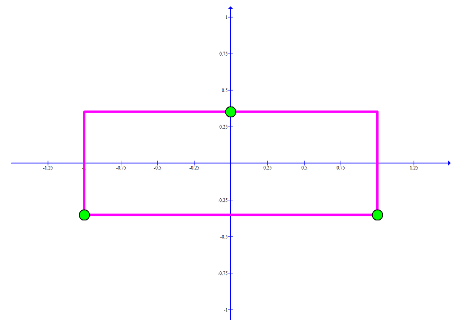

Figure 4.4. Extremal energy with $h = \frac{\sqrt{2}}{4}$ and charge at $\frac{\sqrt{2}}{4}i$

The latter equation is equivalent to the inequality

$$h < \frac{\sqrt{2}}{4},$$

which gives the value of h where the extremal minimum of L changes from $x = 1$ to $x = 0$.

When $h > \sqrt{2}/4$, then L has its extremal energy at $x = 1$. When $h < \sqrt{2}/4$, then L has its extremal energy at $x = 0$.

When $h = \sqrt{2}/4$, we see the system has the energy

$$L(0, \sqrt{2}) = L(1, \sqrt{2}) = -\log \frac{3}{2}.$$

This shows the charge configurations

$$\left\{-1 - \frac{\sqrt{2}}{4}i, 1 - \frac{\sqrt{2}}{4}i, 1 + \frac{\sqrt{2}}{4}i\right\} \quad \text{and} \quad \left\{-1 - \frac{\sqrt{2}}{4}i, 1 - \frac{\sqrt{2}}{4}i, \frac{\sqrt{2}}{4}i\right\}$$

both produce the extremal energy for the system of charges.

In order to have a smooth change in the location of the third charge, there would have to be a minimum energy that did not reside at either 0 or 1. However, such a local minimum does not exist, and so what occus is that the minimum energy occurs at the extremes of the domain values for x, i.e. 0 and 1.

CHAPTER 5
AFFINE TRANSFORMATIONS AND AFFINE ENERGY

In this chapter, we will discuss how affine transformations change the shape of geometric figures and the effect of affine transformations on some conformal characteristics of planar sets. The last section we will discuss some properties of the affine potential energy of charges.

5.1 Definitions and Basic Properties

An *affine transformation* is a function of $z \in \mathbb{C}$ such that

$$f(z) = az + b\bar{z} + c \tag{5.1}$$

with $a, b, c \in \mathbb{C}$ that has a non-zero determinant, i.e.

$$|a|^2 - |b|^2 \neq 0.$$

Consider the affine transformation $F(z) = az + b\bar{z}$ where $a = a_1 + ia_2$, $b = b_1 + ib_2$, and $z = x + iy$. Then we can look at $f(z) = f\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = M\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ where

$$M = \begin{pmatrix} a_1 + b_1 & -a_2 + b_2 \\ a_2 + b_2 & a_1 - b_1 \end{pmatrix}$$

and thus we find that the determinant of matrix M is given by

$$\begin{aligned} \det M &= (a_1 + b_1)(a_1 - b_1) - (a_2 + b_2)(-a_2 + b_2) \\ &= (a_1^2 + a_2^2) - (b_1^2 + b_2^2) \\ &= |a|^2 - |b|^2. \end{aligned}$$

The affine transformation $f(z)$ is $F(z)$ with a translation applied. This establishes that if $|a|^2 - |b|^2 \neq 0$, then the affine transformation f is nonsingular; this implies that f has an inverse affine transformation.

First, we list some well-known properties of affine transformations of the form given in equation (5.1).

Property 5.1. *The inverse of $w = az + b\bar{z} + c$ is the affine transformation given by*

$$z = \frac{\bar{a}}{|a|^2 - |b|^2}w + \frac{b}{|b|^2 - |a|^2}\bar{w} + \frac{b\bar{c} - c\bar{a}}{|a|^2 - |b|^2}.$$

Property 5.2. *If we write $z = x + iy$, the transformation $f(z)$ has the form*

$$f(z) = g(x + iy) = Ax + By + C$$

where $A, B, C \in \mathbb{C}$. The transformation $g(x + iy)$ is affine if $\Im(A\bar{B}) \neq 0$

Property 5.3. *The composition of two affine transformations is an affine transformation.*

Proof. Let $w = az + b\bar{z} + c$ and let $\zeta = Az + B\bar{z} + C$. Then we have

$$w \circ \zeta(z) = (aA + b\bar{B})z + (aB + \bar{A}b)\bar{w} + (aC + b\bar{C} + c)$$

which is an affine transformation. □

Property 5.4. *Affine transformations map lines onto lines. In particular, parallel lines are invariant under affine transformations.*

Proof. Let $z = te^{i\theta} + z_1$ and $z = te^{i\theta} + z_2$ be two lines in $\mathbb{C}$, where $\theta \in [0, \pi)$ fixed, $t \in \mathbb{R}$, and $z_1 \neq z_2$. Applying affine transformation $f(z)$ to each line, we have

$$f(te^{i\theta} + z_1) = (az_1 + b\overline{z_1} + c) + (ae^{i\theta} + be^{-i\theta})t$$

and

$$f(te^{i\theta} + z_2) = (az_2 + b\overline{z_2} + c) + (ae^{i\theta} + be^{-i\theta})t.$$

Both lines after the affine transformation have idential argument parts, specifically $ae^{i\theta} + be^{-i\theta}$. Thus the lines are parallel under affine transformation. □

Property 5.5. *Affine transformations map ellipses onto ellipses. In particular, affine transformations map circles onto ellipses.*

Proof. Without loss of generality, use the unit circle, which is parametrized by $e^{i\theta}$. Applying an affine transformation to the unit circle, we have

$$\begin{aligned} f(e^{i\theta}) &= ae^{i\theta} + be^{-i\theta} + c \\ &= (a+b)\cos\theta + i(a-b)\sin\theta \end{aligned}$$

This shows that the unit circle is mapped to an ellipse. By composition with an appropriate inverse, any ellipse can then mapped to the unit circle and then mapped to another ellipse by approproate affine transformation. □

Property 5.6. *Affine transformations of the form $f(z)$ are the only harmonic polynomials which are one-to-one in $\mathbb{C}$.*

The properties above can be found in [24, p. 73]. The next five definitions and properties below can be found in [15].

Definition 5.7. *An affine transformation $f(z)$ is unimodular if*

$$|\,|a|^2 - |b|^2\,| = 1.$$

Unimodular affine transformations are those that preserve the area of planar figures under transformation.

Definition 5.8. *An affine transformation $f(z) = az + b\bar{z} + c$ is orthogonal if we have*

$$a = s(\sqrt{1-\alpha^2} + i\alpha) \ \ \textit{and} \ \ b = 0$$

or if we have

$$b = s(\beta + i\sqrt{1-\beta^2}) \ \ \textit{and} \ \ a = 0$$

where $\alpha, \beta \in \mathbb{R}$ and $s \in \mathbb{R}$ and c is arbitrary.

For a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to be orthogonal, we know that the conditions

$$\begin{aligned} a^2 + c^2 &= 1 \\ b^2 + d^2 &= 1 \\ ab + cd &= 0 \end{aligned}$$

must be met. Using $F(z) = az + b\bar{z}$, if we set $a = \sqrt{1-\alpha^2} + i\alpha$ and $b = 0$, then we can write the matrix for $F(z)$ as

$$M = \begin{pmatrix} \sqrt{1-\alpha^2} & -\alpha \\ \alpha & \sqrt{1-\alpha^2} \end{pmatrix}$$

which satisfies the orthogonality conditions. Similar work shows that
$F(z) = az + b\bar{z}$ is orthogonal when $b = \beta + i\sqrt{1-\beta^2}$ and $a = 0$.

Definition 5.9. *An affine transformation $f(z)$ is orthonormal if it is orthogonal and $|s| = 1$.*

Property 5.10. *Orthogonal affine transformations preserve angles.*

Property 5.11. *Orthonormal affine transformations preserve angles and distances.*

Of particular interest is the interaction that occurs with three noncollinear points in $\mathbb{C}$ when a mapped through an affine transformation.

Property 5.12. *For any triples $\{a_1, a_2, a_3\}$ and $\{A_1, A_2, A_3\}$ which are noncollinear, there is a unique affine transformation $f(z)$ such that $f(a_k) = A_k$ for $k = 1, 2, 3$.*

Proof. Let $g(z)$ be the affine transformation mapping

$$g : \{a_1, a_2, a_3\} \to \{1, e^{2\pi i/3}, e^{-2\pi i/3}\}$$

and let $f(z)$ be the affine transformation that maps

$$f : \{1, e^{2\pi i/3}, e^{-2\pi i/3}\} \to \{A_1, A_2, A_3\}.$$

It is enough to prove this property for the tranformation f, as the composition of affine transformations $f \circ g$ is itself an affine transformation by Property 5.3.

Using the affine transformation $f(z) = Az + B\bar{z} + C$, we will use the regular triangle with vertices located at $z_1 = 1$, $z_2 = e^{2\pi i/3}$, and $z_3 = e^{-2\pi i/3}$ and find the affine transformation that maps $z_1 \to A_1$, $z_2 \to A_2$, and $z_3 \to A_3$ in $\mathbb{C}$.

The coefficients A, B, and C need to be found, and since we are looking for a transformation that will affect z_1, z_2, and z_3, the the function f will be dependent upon the locations of the points A_1, A_2, and A_3. Given this, we have the system of equations

$$\begin{aligned} A(1) + B(1) + C &= A_1 \\ A\left(e^{2\pi i/3}\right) + B\left(e^{-2\pi i/3}\right) + C &= A_2 \\ A\left(e^{-2\pi i/3}\right) + B\left(e^{-4\pi i/3}\right) + C &= A_2 \end{aligned}$$

and using Mathematica, we find the coefficients are

$$\begin{aligned} A &= \frac{A_1}{3} + \left(\frac{-1 - i\sqrt{3}}{6}\right) A_2 + \left(\frac{-1 + i\sqrt{3}}{6}\right) A_3 \\ B &= \frac{A_1}{3} + \left(\frac{-1 + i\sqrt{3}}{6}\right) A_2 + \left(\frac{-1 - i\sqrt{3}}{6}\right) A_3 \\ C &= \frac{A_1 + A_2 + A_3}{3} \end{aligned}$$

for the affine transformation f. □

5.2 Affine Transformations and Polygons

In this section we discuss some well-known results about affine properties of Euclidean polygons. First we give necessary definitions.

Definition 5.13. *For any two points $a, b \in \mathbb{C}$, let $[a, b]$ denote the straight line segment from a to b; that is,*

$$[a, b] = \{z = a + (b - a)t : 0 \leq t \leq 1\}.$$

In the following definition and below, we will use the following cyclic convention. For the set of n vertices $\{v_0, \ldots, v_{n-1}\} \in \mathbb{C}$, we define

$$v_n = v_0 \text{ and } v_{n+1} = v_1$$

We can now define a polygon as follows.

Definition 5.14. *A polygon $\mathscr{P}$ with vertices $v_0, \ldots, v_{n-1} \in \mathbb{C}$ is a closed curve, which is the union of points of the line segments $[v_k, v_{k+1}]$ for $k = 0, \ldots, n-1$.*

A polygon $\mathscr{P}$ with n vertices is thus composed of n line segments, and these line segments are called the **sides** of $\mathscr{P}$. Every polygon will be considered as a closed curve oriented in the lexigraphical order of the vertices; this lexigraphical order will determine the uniqueness of the polygon as opposed to the location of the vertices in $\mathbb{C}$. Further, a polygon that is composed of n sides (and thus has n vertices) will be called an **n-gon**.

If $\mathscr{P}$ is a polygon without self-intersection, then it bounds a Jordan domain. This Jordan domain we will call an **inpolygon** and denote it by $\mathscr{P}^{\diamondsuit}$. The closure of $\mathscr{P}^{\diamondsuit}$ will be called the **filled polygon**, denoted by $\mathscr{P}^{\blacklozenge}$.

Definition 5.15. *Let α_k denote the inner angle of a polygon $\mathscr{P}$ at its vertex v_k. A polygon $\mathscr{P}$ is said to be **regular** if $\alpha_0 = \alpha_k$ for $k = 1, \ldots n-1$ and $[v_0, v_1] = [v_k, v_{k+1}]$ for $k = 1, \ldots n-1$. In addition to having angles of equal measure, a regular polygon also has sides of equal length.*

When referring to a **standard n-gon**, we mean the regular polygon with n vertices located in the roots of unity, i.e.

$$v_k = e^{2\pi i(k-1)/n} \qquad \text{for } k = 1, 2, \ldots, n.$$

For each n-gon, there will be an associated polynomial

$$p(z) = \prod_{k=1}^{n} (z - v_k)$$

where $v_1, \ldots, v_k$ are the vertices of the n-gon. Notice that an associated polynomial defines the set of vertices of $\mathscr{P}$ but not the sides of $\mathscr{P}$; in particular, $p(z)$ does not define the lexigraphical order of the vertices of $\mathscr{P}$ (i.e. the orientation of $\mathscr{P}$ is not defined by $p(z)$).

The standard n-gon will be associated with the polynomial

$$p_n(z) = z^n - 1.$$

A number of results have been discovered connecting the roots of a monic polynomial $p(z) = z^n + a_{(n-1)}z^{n-1} + \cdots + a_1 z + a_0$ and the roots of the derivative $p'(z) = nz^{n-1} + (n-1)a_{n-1}z^{n-2} + \cdots + a_1$. One of the best-known results is the Gauss-Lucas theorem (see [24]).

Theorem 5.16 (Gauss-Lucas)**.** *Let $p(z)$ be a polynomial of degree n with zeros $z_1, \ldots, z_n$. The critical points of $p(z)$ lie in the convex hull of the set $\{z_1, \ldots, z_n\}$. In addition, if an open segment (z_j, z_k) of the boundary of the convex hull contains a critical point of the polynomial p, then all zeros of p lie on the line passing through z_j and z_k.*

One connection between the roots of a polynomial and the roots of the derivative can be introduced through a result by Steiner. If we consider a triangle with vertices $z_1, z_2, , z_3$ (referring to said triangle as $\triangle z_1 z_2 z_3$) with distinct and noncollinear vertices, then to the triangle $\triangle z_1 z_2 z_3$ we can associate a polynomial

$$p_3(z) = (z - z_1)(z - z_2)(z - z_3).$$

For any such triangle, Steiner showed that there is always an ellipse that can be inscribed in the triangle that is tangent at the midpoints of the triangle.

Theorem 5.17 (Steiner)**.** *Given any triangle there is a unique ellipse inscribed in the triangle that passes through the midpoints of the sides of the triangle and is tangent to the sides of the triangle at these midpoints.*

The ellipse as described in Theorem 5.16 is called the **Steiner inellipse**. The center of the Steiner inellipse is located at the centroid of $\triangle z_1 z_2 z_3$. The Steiner inellipse degenerates to a circle if and only if $\triangle z_1 z_2 z_3$ is an equilateral triangle.

We observe that we can find a Steiner inellipse for a particular $\triangle z_1 z_2 z_3$ based on the correct affine transformation taking the vertices of an equilateral triangle to the points z_1, z_2, and z_3, respectively. This affine transformation will also map the inscreibed circle to the corresponding Steiner inellipse.

The concept above can also be applied to regular polygons. One important property is the fact that affine transformations preserve the midpoint of line segments. Any circle inscribed in a regular polygon is tangent to the polygon at the midpoint of each side, and thus each apothem of the polygon is a radius of the circle.

Of particular interest are those polygons that are affine images of regular polygons. We say that a convex n-gon is **affinely regular** if it is the image of a regular convex polygon with n vertices located at the roots of unity under an affine transformation. According to the Bôcher-Grace theorem stated below, all triangles are affinely regular, but in general this is not true for n-gons where $n \geq 4$.

The above paragraph is a summary of a generalization of the Bôcher-Grace Theorem. As stated in [6]:

Theorem 5.18 (Bôcher-Grace)**.** *Let p be a third-degree complex polynomial. Then there is a unique inscribed ellipse interpolating the midpoints of the triangle formed from the roots of p, and the foci of the ellipse are the critical points of p.*

Clifford and Lachance provide the following variation of this theorem for polygons.

Theorem 5.19 (Bôcher-Grace Theorem for Polygons)**.** *Let p be an n-th degree complex polynomial and let its critical points take the form*

$$\alpha + \beta \cos k\pi/n, \qquad k = 1, \ldots, n-1, \;\; \beta \neq 0.$$

There is an inscribed ellipse interpolating the midpoints of the convex polygon formed by the roots of p, and the foci of this ellipse are the two most extreme critical points of p: $\alpha \pm \beta \cos \pi/n$.

5.3 Affine Transformations and Conformal Invariants

According to a classical result, essentially due to L. Ahlfors, a one-to-one mapping f is conformal in a domain D if and only if f preserves moduli of all rectangles. In this section, we will discuss the following question:

Question 5.20. *How many moduli of "essentially different" rectangles does an affine mapping have to preserve to be conformal?*

The main goal is to prove the following:

Theorem 5.21. *Suppose that $f(z) = az + b\overline{z} + c$ is an affine mapping which preserves moduli of rectangles $R(m_1)$ and $R(m_2)$, with $0 < m_1 < m_2$, where $R(m) = \{z = x + iy : 0 < x < 1, 0 < y < m\}$. Then $b = 0$ and therefore $f(z) = az + c$ in a linear conformal mapping.*

Proof. If f preserves the moduli of $R(m_1)$ and $R(m_2)$, then the mapping $az + b\overline{z}$ preserves these moduli as well. Hence, we may assume that $c = 0$. Thus, $f(z) = az + b\overline{z}$. Next $f(z)$ preserves the moduli if and only if for every $c \neq 0$, the mapping $c \cdot f(z)$ preserves these moduli. Take $c = \frac{1}{f(1)}$, and consider the function $\widetilde{f(z)} = \frac{f(z)}{f(1)}$. Then we have $\widetilde{f(z)} = 1$, which implies that $b = 1 - a$. Therefore, in proving the theorem, we may restrict ourselves to the mapping

$$f(z) = az + (1-a)\overline{z}. \tag{5.2}$$

Let

$$A(m) = f(im) = aim - (1-a)im = \rho e^{i\pi\alpha}. \tag{5.3}$$

Here for a given value a, $\rho = \rho(m)$ and $\alpha = \alpha(m)$ are functions of the parameter $m > 0$. The function (5.2) maps $R(m)$ onto the parallelogram $Pl(\rho, \alpha)$ having its vertices at $w = 0$, $w = 1$, $w = A(m) + 1$, and $w = A(m)$. To find the modulus of $Pl(\rho, \alpha)$, we consider the following Schwarz-Christoffel mapping

$$f(z, \alpha, \tau) = B \int_0^z t^{\alpha-1}(t-1)^{(1-\alpha)-1}(t-\tau)^{\alpha-1}\, dt + C$$

where $\alpha = \alpha(m)$ is defined by equation (5.3) and the parameter τ will be determined later.

Since $f(0, \alpha, \tau) = 0$, we have $C = 0$. Thus we have the mapping

$$f(z, \alpha, \tau) = B \int_0^z t^{\alpha-1}(t-1)^{-\alpha}(t-\tau)^{\alpha-1}\, dt.$$

Since $f(1, \alpha, \tau) = 1$, we must have

$$B = \frac{1}{\int_0^1 t^{\alpha-1}(t-1)^{-\alpha}(t-\tau)^{\alpha-1}\, dt}.$$

Since $A = \rho e^{i\pi\alpha}$, to find A we need to find $(1 + \rho e^{i\pi\alpha}) - 1 = f_1(\tau) - 1$ and so we have

$$\begin{aligned}
A &= f_1(\tau) - 1 \\
&= B \int_0^\tau t^{\alpha-1}(t-1)^{-\alpha}(t-\tau)^{\alpha-1}\, dt - 1 \\
&= B \left(\int_0^1 t^{\alpha-1}(t-1)^{-\alpha}(t-\tau)^{\alpha-1}\, dt + \int_1^\tau t^{\alpha-1}(t-1)^{-\alpha}(t-\tau)^{\alpha-1}\, dt \right) - 1 \\
&= B \left(\frac{1}{B} + \int_1^\tau t^{\alpha-1}(t-1)^{-\alpha}(t-\tau)^{\alpha-1}\, dt \right) - 1 \\
&= B \int_1^\tau t^{\alpha-1}(t-1)^{-\alpha}(t-\tau)^{\alpha-1}\, dt
\end{aligned}$$

and this gives us

$$A = \rho e^{i\pi\alpha} = \frac{\int_1^\tau t^{\alpha-1}(t-1)^{-\alpha}(t-\tau)^{\alpha-1}\, dt}{\int_0^1 t^{\alpha-1}(t-1)^{-\alpha}(t-\tau)^{\alpha-1}\, dt} = e^{i\pi\alpha} \frac{\int_1^\tau t^{\alpha-1}(t-1)^{-\alpha}(\tau-t)^{\alpha-1}\, dt}{\int_0^1 t^{\alpha-1}(1-t)^{-\alpha}(\tau-t)^{\alpha-1}\, dt}$$

Hence

$$\rho = \frac{\int_1^\tau t^{\alpha-1}(t-1)^{-\alpha}(\tau-t)^{\alpha-1}\, dt}{\int_0^1 t^{\alpha-1}(1-t)^{-\alpha}(\tau-t)^{\alpha-1}\, dt}. \tag{5.4}$$

Using the tables of integrals (see [11]), or alternatively by using Mathematica or Maple, we can express ρ in terms of the hypergeometric functional as

$$\rho(\tau, \alpha) = \tau^\alpha \frac{F(\alpha, \alpha; 1; 1-\tau)}{F(1-\alpha, \alpha; 1; \tau^{-1})} \tag{5.5}$$

where F is the ordinary hypergeometric function. Changing variables in the integral in the numerator of equation (5.4) via

$$x = \frac{\tau - t}{t(\tau - 1)}, \qquad 1 \le t \le \tau,$$

we can rewrite equation (5.4) as

$$\rho = \rho(\tau, \alpha) = \tau^\alpha \frac{\int_0^1 x^{\alpha-1}(1-x)^{-\alpha}(1+(\tau-1)x)^{-\alpha}\, dx}{\int_0^1 x^{\alpha-1}(1-x)^{-\alpha}(1-\tau^{-1}x)^{\alpha-1}\, dx}.$$

Now we fix α and then choose $\tau = \tau(m)$ such that the modulus of the parallelogram $Pl(\rho, \alpha)$ equals to the modulus of the rectangle $R(m)$; i.e.

$$\text{Mod}(Pl(\rho, \alpha)) = m.$$

Since $R(m) = Pl(m, 1/2)$, from equation (5.5) we have

$$m = \tau^{1/2} \frac{F(1/2, 1/2; 1; 1-\tau)}{F(1/2, 1/2; 1; \tau^{-1})}.$$

Solving equation (5.3) for a, we find

$$a = \frac{1}{2} - \frac{i}{2} \frac{\rho(\tau, \alpha)}{m(\tau)} e^{i\pi\alpha}. \tag{5.6}$$

Therefore, if the moduli of two rectangles $R(m_1)$ and $R(m_2)$, with $0 < m_1 < m_2$, are preserved under the mapping (5.2) the corresponding quotients in (5.6) must be equal, meaning we have the equality

$$\frac{\rho(\tau(m_1), \alpha)}{m_1} = \frac{\rho(\tau(m_2), \alpha)}{m_2}, \qquad 0 < m_1 < m_2.$$

This contradicts the monotonicity property of Lemma 5.22 stated below unless $\alpha = 1/2$. In the latter case, using equation (5.6), we find that $a = 1$ and therefore $f(z) = z$. Hence, in general $f(z) = az + c$ is a linear mapping. Thus the proof is complete. □

Our proof of Theorem 5.21 is based on Lemma 5.22 below. For $0 < \alpha \leq 1/2$ and $0 < x \leq 1$, define the function

$$g(x, \alpha) = x^{1/2-\alpha} \frac{F\left(\alpha, \alpha; 1; \frac{x-1}{x}\right)}{F\left(1/2, 1/2; 1; \frac{x-1}{x}\right)} \cdot \frac{F\left(1/2, 1/2; 1; x\right)}{F\left(1-\alpha, \alpha; 1; x\right)}.$$

Lemma 5.22. *For each fixed $0 < \alpha < 1/2$, the function $g(x, \alpha)$ strictly increases as x increases from 0 to 1.*

Let

$$g_1(x, \alpha) = \frac{F\left(1/2, 1/2; 1; x\right)}{F\left(1-\alpha, \alpha; 1; x\right)}$$

and

$$g_2(x,\alpha) = x^{1/2-\alpha} \frac{F\left(\alpha,\alpha;1;\frac{x-1}{x}\right)}{F\left(1/2,1/2;1;\frac{x-1}{x}\right)}.$$

Let $z = (x-1)/x$. Using formulas (15.3.4) in [1], we obtain

$$g_2(x,\alpha) = \frac{F(1-\alpha,\alpha;1;1-x)}{F(1/2,1/2;1;1-x)}.$$

Hence,

$$g_2(x,\alpha) = \frac{1}{g_1(1-x,\alpha)}.$$

Since $g(x,\alpha) = g_1(x,\alpha)/g_1(1-x,\alpha)$ the required monotonicity property of $g(x,\alpha)$ will follow from the following.

Lemma 5.23. *For each fixed* $0 < \alpha < \pi/2$*, the function* $g_1(x,\alpha)$ *strictly increases on the interval* $0 < x < 1$*.*

Proof. Consider the Taylor expansions

$$F(1/2,1/2;1;x) = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n}{(1)_n} \frac{x^n}{n!}$$

and

$$F(1-\alpha,\alpha;1;x) = \sum_{n=0}^{\infty} \frac{(1-\alpha)_n\,(\alpha)_n}{(1)_n} \frac{x^n}{n!},$$

each of which converges for $|x| < 1$. The corresponding coefficients

$$a_n = \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n}{(1)_n} \frac{1}{n!}$$

and

$$b_n = \frac{(1-\alpha)_n\,(\alpha)_n}{(1)_n} \frac{1}{n!}$$

are positive for $0 < \alpha < 1/2$.

Consider the quotient

$$\frac{a_{n+1}}{b_{n+1}} = \frac{\left(\frac{1}{2}\right)_{n+1}\left(\frac{1}{2}\right)_{n+1}}{(1-\alpha)_{n+1}(\alpha)_{n+1}} = \frac{a_n}{b_n}\,\frac{(1/2+n)^2}{(1-\alpha+n)(\alpha+n)}.$$

The quadratic function

$$(1-\alpha+n)(\alpha+n) = -\alpha^2+\alpha+n$$

has its maximal value at $\alpha = 1/2$. Hence,

$$(1-\alpha+n)(\alpha+n) < (1/2+n)^2 \quad \text{for } 0 \le \alpha < 1/2.$$

This implies that

$$\frac{a_{n+1}}{b_{n+1}} > \frac{a_n}{b_n}.$$

Thus, the sequence a_n/b_n strictly increases. Therefore, by the result established by M. Vuorinen (see [3]) the quotient

$$g_1(x,\alpha) = \frac{F(1/2,1/2;1;x)}{F(1-\alpha,\alpha;1;x)}$$

strictly increases for $0 < x < 1$. □

5.4 The Affine Energy of Charges

We will begin by defining the affine energy in a manner similar to that used in Definition 3.1.

Definition 5.24. *Let $X = \{\nu_1,\ldots,\nu_n\}$ be a system of n charges in the complex plane. Let m_k be the mass of charge ν_k, and let $M = \{m_1,\ldots,m_n\}$ be the set of masses associated with their respective charges in X. We assume that the charge ν_k is situated at the point $z_k \in \mathbb{C}$. For an affine transformation $\phi : \mathbb{C} \to \mathbb{C}$ with ϕ onto, the affine potential energy $E_{@}(X,M)$ of this system is defined by*

$$E_{@}(X,M) = \inf_{\phi} E(\phi(X),M) = \inf_{\phi}\left(-\sum_{1\le j<k\le n} m_j m_k\,\log|\phi(z_k)-\phi(z_j)|\right) \quad (5.7)$$

where the infimum is taken over all affine maps ϕ such that

$$\mathrm{diam}(\phi(X)) \leq \mathrm{diam}(X). \tag{5.8}$$

It is immediate from Equations 5.7 and 5.8 that

$$E_{@}(X, M) \leq E(X, M).$$

If we look at the simplest case, where $X = \{z_1, z_2\}$, then we obviously have $E_{@}(X, M) = E(X, M)$. For a system consisting of three unit charges in $\mathbb{C}$ the affine energy can be calculated as in the following proposition.

Proposition 5.25. *For a system of unit charges denoted by $X = \{z_1, z_2, z_3\}$, we have*

$$E_{@}(X) = -3\log(\mathrm{diam}(X)).$$

The formula in the right hand side of this equation gives the usual energy of charges situated at the vertices of an equilateral triangle.

Proof. Assume we have a system of three unit charges with locations denoted by $X = \{z_1, z_2, z_3\}$. In this system, there are three distance vectors formed. Since labelling of the points is arbitrary, let us say

$$|z_3 - z_2| \geq |z_3 - z_1| \geq |z_2 - z_1|.$$

Note that $\mathrm{diam}(X) = |z_3 - z_2|$. Also, we have

$$\log|z_3 - z_2| \geq \log|z_3 - z_1| \geq \log|z_2 - z_1|$$

which leads us to

$$-\log|z_3 - z_2| \leq -\log|z_3 - z_1| \leq -\log|z_2 - z_1|.$$

The smallest possible logarithmic potential energy for a system of three charges is

$$-3\log(\mathrm{diam}(X))$$

where all three distance vectors are as large as possible.

For the affine energy $E_{@}(X)$, recall that for our affine maps ϕ, defined in Equation 5.1, we have the condition $\text{diam}(\phi(X)) \leq \text{diam}(X)$. Thus, in looking at the affine maps ϕ applied to the X, and specifically applied to the three distance vectors formed by the charge locations in X, then

$$|\phi(z_k) - \phi(z_j)| \leq \text{diam}(X) \qquad \text{for } 1 \leq j < k \leq 3$$

which implies that

$$-\log|\phi(z_k) - \phi(z_j)| \geq -\log(\text{diam}(X)) \qquad \text{for } 1 \leq j < k \leq 3.$$

Thus we have

$$-\sum_{1 \leq j < k \leq 3} \log|\phi(z_k) - \phi(z_j)| \geq -3\log(\text{diam}(X))$$

and this imples $E_{@}(X) = -3\log(\text{diam}(X))$. □

We have proved that for two or three charges the affine energy coincides with the usual potential energy of these charges. It seems possible that in most cases the affine energy of several charges is strictly smaller than the potential energy of these charges. To state a precise conjecture, we will need the following definition.

Definition 5.26. *A system of points is called* ***generic*** *if no three points in the system belong to the same line segment and all three pairwise distances are distinct.*

Conjecture 5.27. *If the charges $z_1, \ldots, z_n$ are in a generic position, then*

$$E_{@}(z_1, \ldots, z_n) < E(z_1, \ldots, z_n).$$

Now we turn to collinear charges. It occurs that in this case the affine energy always coincides with the usual energy of these charges.

Lemma 5.28. *If the charges $z_1, \ldots, z_n$ are collinear, then*

$$E_{@}(z_1, \ldots, z_n) = E(z_1, \ldots, z_n).$$

Proof. If we consider the case where all of the points are collinear, we can use the affine map ϕ that satisfies $\phi(0) = 0$ and $\phi(1) = 1$. Thus we have

$$\phi(0) = a(0) + b(0) = 0, \quad \phi(1) = a + b = 1$$

which gives us the constraint $b = 1 - a$. This defines our affine map

$$\phi(z) = az + (1 - a)\bar{z}.$$

Note that if all of the charges considered $X = \{z_1, \ldots, z_n\}$ lie on a line, we can use necessary transformations, a rotation, and stretching/shrinking to reposition the system of charges X such that we have the system $X^* = \{z_1^*, \ldots, z_n^*\}$ where $z_1^* = 0$ and $z_n^* = 1$. By having all of the charges located on the real axis, we thus have the affine map

$$\phi(x) = ax + (1 - a)x = x$$

and thus the affine map is the identity map for the system of charges X^*. This tells us that the affine energy of the the system X^* is the same as the potential energy. □

Working on extremal problems related to the affine energy of charges, we need a criterion whether or not a given system of charges provides a local minimum under certain geometrical conditions. In this direction we have a simple result saying that a collinear system of charges never provides a local minimum for the affine energy.

Lemma 5.29. *Collinear systems of $n \geq 3$ charges cannot produce the local minimal affine potential energy.*

Proof. We can consider, through necessary translation, rotation, and stretching or shrinking, the system of charges $X = \{z_1, \ldots, z_n\}$ to be located on the standard real line segment where $0 = z_1 < \cdots < z_k < \cdots z_n = 1$. Then we can also consider a perturbed system of charges $X^* = \{z_1, \ldots, z_k^*, \ldots, z_n\}$ where $a_k^* = a_k + i\varepsilon$ for some small $\varepsilon > 0$. We can see from this perturbed system X^* that the distance between a_k^* and any other point in X^* increases and thus the perturbed system has a smaller potential energy and therefore a smaller affine potential energy. □

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