TANGENTIAL AND OSCULATORY INTERPOLATION

BY

Cecil O. Schwalbe

Submitted in partial fulfillment

of the requirements for the

Degree of Master of Arts

Texas Technological College

May 18, 1936

TANGENTIAL AND OSCULATORY INTERPOLATION

THESIS

Approved:

Thesis Committee

Approved:

Head of the Department of Mathematics

Approved:

Dean of the Division of
Arts and Sciences

Approved:

Chairman of the Graduate Committee

TABLE OF CONTENTS

TANGENTIAL AND OSCULATORY INTERPOLATION

INTRODUCTION

Interpolation is the process of obtaining intermediate terms of a series of which particular terms are known, and comes from the Latin word *interpolare*: to alter or insert something fresh, connected with *polire* to polish. The term interpolation is usually limited to those cases in which there are two quantities x and U which are so related that when x has any arbitrary value, lying perhaps between certain limits, the value of U can be determined. Either of the two quantities x or U may be regarded as a function of the other, but it is generally convenient to regard x as the independent variable and U as the dependent variable, that is, U is expressed as a function of x. The independent variable x is called the argument, and U the dependent variable is called the entry.

John Wallis, an English mathematician, first used interpolation about 1650. He was studying the quadrature of the circle and attempted to determine the value of π by interpolation. He did not succeed in making the interpolation itself, because he did not employ literal or general exponents with the series which he used. This difficulty led Sir Isaac Newton to the discovery of the binomial theorem and the development of finite differencing, and his interpolation formulas.

Among the other mathematicians who have contributed to interpolation such men as Lagrange, Euler, Gauss, Bessel, Everett, Stirling, and Woolhouse have been outstanding.

In this discussion of Tangential and Osculatory interpolation only a central interval into which we wish to place a certain number of values will be considered. Tangential and Osculatory interpolation will both be defined as they are discussed.

It is not the purpose of this thesis to develop a series of tangential and osculatory formulas but to discuss the theory on which such a type of interpolation is based, and to show its application to Newton's fundamental forward difference formula.

CHAPTER I

STRAIGHT LINE INTERPOLATION

The form of interpolation most generally used in actual practice is known as straight line interpolation, which is very easy to apply. The principle involved can best be illustrated by an actual example.

Suppose we have $\log x_1 = n$, and $\log x_1 + h = m$ and we wish to find the $\log (x_1 + \frac{i}{t} h)$ where $\frac{i}{t}$ is some fraction, then to the value of $\log x_1$ is added $\frac{i}{t}(m - n)$ and the result is $n + \frac{i}{t}(m - n)$.

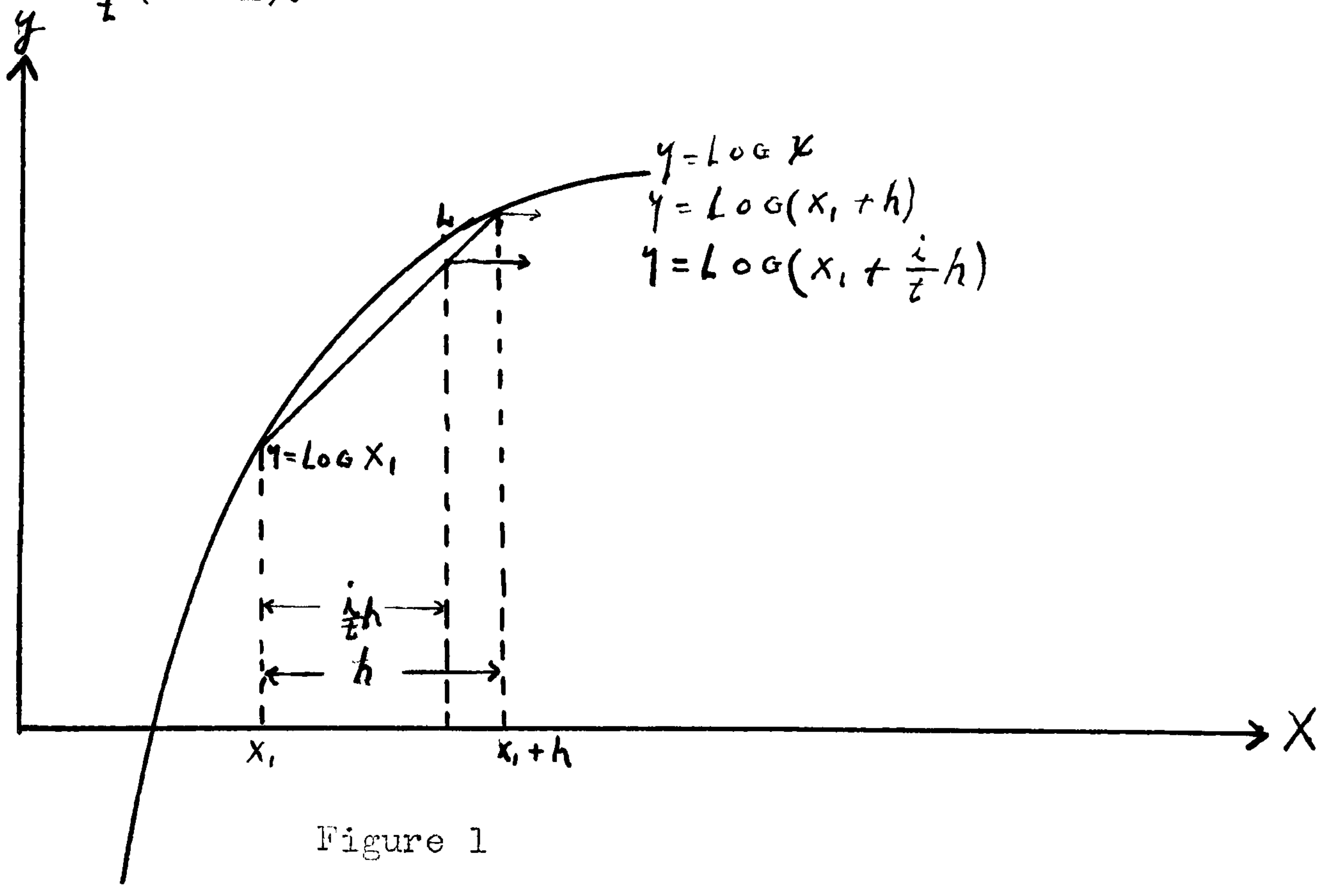

Figure 1

The graph above shows what is really done in applying straight line interpolation. The point L shows the point at which $y = \log (x_1 + \frac{i}{t} h)$ should fall and k is the computed point.

Another graphical presentation of straight line interpolation can be given as in figure 2.

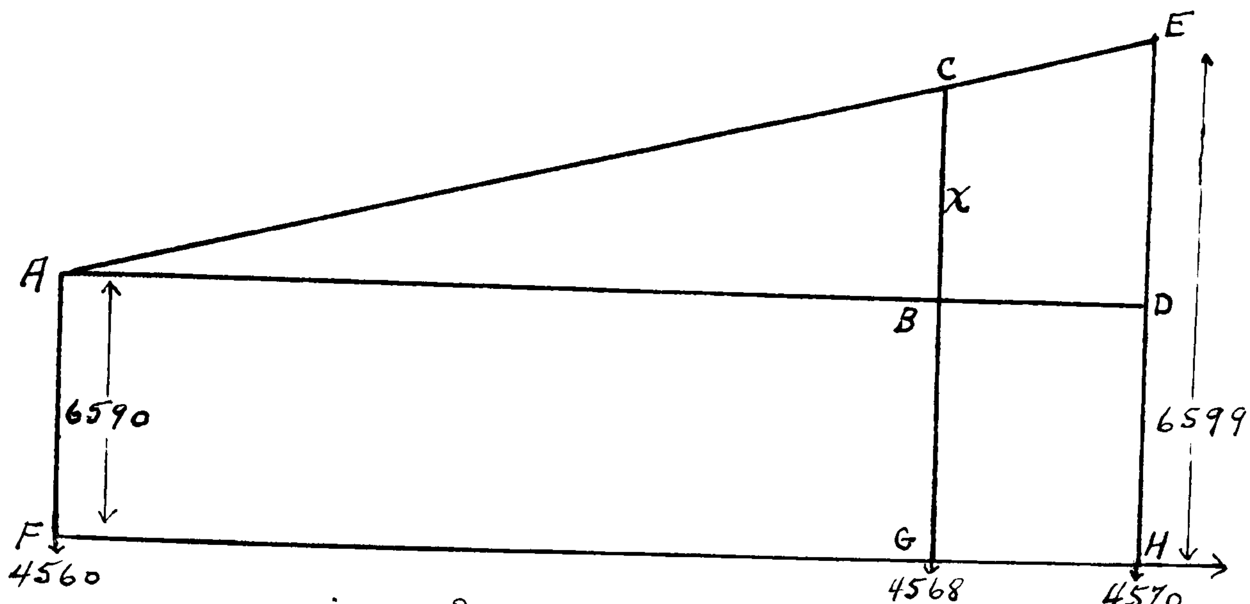

Figure 2

Suppose that we wish to find the mantissa for the number 4568. Laying off the number on the horizontal and the corresponding mantissas on the vertical. FA = 6590 is the mantissa for 4570, and HE = 6599 is the mantissa for 4570. GC, the mantissa for 4568 is to be found. To do this find x = BC, the amount to be added to GB = FA = 6590. DE = HE − FA = 6599 − 6590 = 9. From the similar triangles ADE and ABC: $\frac{x}{DE} = \frac{AB}{AD}$ or $x = \frac{AB}{AD} \times DE$.

$x = \frac{4568 - 4560}{4570 - 4560} \times 9. \quad x = \frac{8}{10} \times 9. \quad x = \frac{72}{10} = 7.2.$

Then GC = 6590 + 7.2 = 6597.2 which is the required mantissa.

Figure 1 shows a deviation in the computed value of log $(1 + \frac{i}{t})$ from the value that should be obtained. This deviation is due to the application of straight line interpolation to a non-linear function.

Most of the tables which are used in various branches of mathematics have been computed from a parabolic equation. When intermediate values are interpolated a slight deviation from the true value is always found. However fairly accurate values can be interpolated by the straight line method provided the interval in which the interpolation is to be made is extremely small. In figure 1 the nearer that x_1 approaches $x_1 + h$ the more accurate will be the interpolated value.

Despite the deviations that are always found in straight line interpolation it is used in practically all branches of science and mathematics, especially in trigonometry, business mathematics, life insurance, astronomy, and physics.

CHAPTER II

FINITE DIFFERENCING AND NEWTON'S FUNDAMENTAL FORMULA

In order to follow the application of Tangential and Osculatory interpolation an introduction to finite differencing and the development of Newton's Fundamental Formula is necessary. If we denote a function of x by U_x where U_x corresponds to $f(x)$ as used in ordinary mathematical analysis, the finite difference of U_x, denoted by the symbol ΔU_x may be defined by the general relation $\Delta U_x = U_{x+h} - U_x$ where h is any real constant. In the same manner the second difference of U_x is $\Delta^2 U_x$ and is equal to $\Delta U_{x+h} - \Delta U_x$. Likewise $\Delta^3 U_x = \Delta^2 U_{x+h} - \Delta^2 U_x$, etc. Ordinarily h is considered as unity and $\Delta U_x = U_{x+1} - U_x$. $\Delta^2 U_x = \Delta U_{x+1} - \Delta U_x$, etc.

It is convenient to arrange the tabular values and their differences for increasing values of the argument in what is called a difference table, as follows:

Argument	Entry			
x	U_x			
		ΔU_x		
$x + h$	U_{x+h}		$\Delta^2 U_x$	
		ΔU_{x+h}		$\Delta^3 U_x$
$x + 2h$	U_{x+2h}		$\Delta^2 U_{x+h}$	
		ΔU_{x+2h}		$\Delta^3 U_{x+h}$
$x + 3h$	U_{x+3h}		$\Delta^2 U_{x+2h}$	
⋮	⋮	ΔU_{x+3h}		$\Delta^3 U_{x+2h}$
⋮	⋮		$\Delta^2 U_{x+3h}$	
⋮	⋮	⋮		$\Delta^3 U_{x+3h}$
⋮	⋮	⋮	⋮	
⋮	⋮	⋮		$\Delta^3 U_{x+(n-2)h}$
⋮	⋮	⋮	$\Delta^2 U_{x+(n-1)h}$	
		ΔU_{x+nh}		
$x+(n+1)h$	$U_{x+(n+1)h}$			

And similarly for differences of order higher than the third. The first entry U_x is called the leading term, and the differences of U_x that is ΔU_x, $\Delta^2 U_x$, $\Delta^3 U_x$.... are called the leading differences. Evidently each difference in the table is the number (with its proper algebraic sign) obtained by subtracting the number immediately above and to the left from the number immediately below and to the left.

It might be stated here that the sum of the entries in any column of differences is equal to the difference between the first and last entries of the preceding column. This affords a numerical check on the accuracy of the table. Thus in the above table we have $\Delta^2 U_{x+3h} = \Delta^2 U_x + \Delta^3 U_x + \Delta^3 U_{x+h} + \Delta^3 U_{x+2h}$.

As an example of a difference table we have:

x	U_x	ΔU_x	$\Delta^2 U_x$	$\Delta^3 U_x$
0	$U_0 = 2$			
		1		
1	$U_1 = 3$		6	
		7		6
2	$U_2 = 10$		12	
		19		6
3	$U_3 = 29$		18	
		37		6
4	$U_4 = 66$		24	
		61		6
5	$U_5 = 127$		30	
		91		
6	$U_6 = 218$			

The function from which the above table was taken is $U_x = x^3 + 2$. It is found that the third difference gives a constant value and the fourth difference is zero.

We must at this time establish a general property on which most of the interpolation formulas are based. Consider the general case where $U_x = Ax^n + Bx^{n-1} + Cx^{n-2} + \ldots + Lx + M$.

$$\Delta U_x = A\left\{(x+h)^n - x^n\right\} + B\left\{(x+h)^{n-1} - x^{n-1}\right\} + \ldots + Lh, \text{ but}$$

$$(x+h)^n = x^n + nhx^{n-1} + \frac{n(n-1)}{2!} h^2 x^{n-2} + \ldots + h^n \text{ so that}$$

$$\Delta U_x = A\left\{nhx^{n-1} + \frac{n(n-1)}{2!} h^2 x^{n-2} + \ldots + h\right\}$$

$$+ B\left\{(n-1)hx^{n-2} + \frac{(n-1)(n-2)}{2!} h^2 x^{n-3} + \ldots + h\right\} + \ldots + Lh$$

which is a polynomial of degree (n-1) in x and therefore the first differences of a polynomial represent another polynomial of degree less by one unit. By repeated applicationof this result we see that the second differences represent a polynomial of degree (n-2), etc. Then the nth differences represent a polynomial of degree of (n-n) or 0; that is the nth differences are constant. It follows that the (n+1)th differences of a polynomial of the nth degree are all zero which is the general property that forms the foundation for several interpolation formulas.

Among the polynomials of degree n there is one of special importance and interest in the theory of interpolation. If we denote the product $x(x-1)(x-2)(x-3)\ldots\ldots(x-n\ \ 1)$ by $x^{(n)}$ we have what is called a factorial. Likewise

$$(x+1)x(x-1)(x-2)\ \ldots.(x-n+2) = (x+1)^{(n)}.$$

$$(x+2)(x+1)(x)(x-1)(x-2)\ \ldots.\ (x-n+3) = (x+2)^{(n)}, \text{ etc.}$$

Then $\Delta x^{(n)} = (x+1)^{(n)} - x^{(n)}$ or

$$\Delta^n x^{(n)} = (x+1)(x)(x-1)(x-2)\ \ldots..(x-n+2) - x(x-1)(x-2)\ldots(x-n+1).$$

$\Delta x^{(n)} = (x+1-x+n-1)\left[(x)(x-1)(x-2) \;\ldots\ldots\; (x-n+\;)\right]$

$\Delta x^{(n)} = nx^{(n-1)}.$

By the proper procedure the process of differencing can be carried out very easily. All polynomials can be expressed as a series of factorials. Let $\phi_k(x)$ denote a polynomial in x of degree k. $\phi_k(x) = g + (x-n+k)\phi_{k-1}(x)$ where g is the remainder and $\phi_{k-1}(x)$ is the quotient when $\phi_k(x)$ is divided by $(x-n+k)$ so $\phi_{k-1}(x)$ is of degree (k-1). By repeating this process we obtain an expression for a polynomial of the nth degree in terms of factorials. $\phi_n(x) = a + bx + cx^{(2)} + dx^{(3)} + \ldots zx^{(n)}$ ◯ where a, b, c, ... Z are constants.

Suppose that we wished to express the function $U_x = x^5 - 6x^4 + 9x^3 - 2x^2 + 4x - 6$ and its successive differences in the factorial notation. Using detached coefficients when dividing by x, x-1, x-2 ...

1	1	- 6	+ 9	- 2	+ 4	- 6
	0	+ 1	- 5	+ 4	+ 2	
2	1	- 5	+ 4	+ 2	+ 6	
	0	+ 2	- 6	- 4		
3	1	- 3	- 2	- 2		
	0	+ 3	+ 0			
4	1	+ 0	- 2			
	0	4				
	1	4				
	1					

We obtain the value of U_x in the form

$U_x = x^{(5)} + 4x^{(4)} - 2x^{(3)} - 2x^{(2)} + 6x - 6.$

$\Delta U_x = 5x^{(4)} + 16x^{(3)} - 6x^{(2)} - 4x + 6.$

$\Delta^2 U_x = 20x^{(3)} + 48x^{(2)} - 12x - 4.$

$\Delta^3 U_x = 60x^{(2)} + 96x - 12.$

$\Delta^4 U_x = 120x + 96.$

$\Delta^5 U_x = 120$

$\Delta^6 U_x = 0$

Let x_1 be one of the tabulated values of the argument of a polynomial of degree n, and let h be the interval between successive values of the argument. Writing $f(x_1 + xh)$ for $\phi_n(x)$ in (I) and applying $\Delta x^{(n)} = nx^{(n-1)}$ to both sides of (I) we find that $f(x_1 + xh) = a + bx + cx^{(2)} + dx^{(2)} + ex^{(3)} \dots + zx^{(n)}$ (I)

$\Delta f(x_1 + xh) = b + 2cx^{(1)} + 3\ ex^{(2)} + \dots + nzx^{(n-1)}$ (II)

$\Delta^2 f(x_1 + xh) = 2c + 6ex^{(1)} + \dots + n(n-1)zx^{(n-2)}$ (III)

$\Delta^3 f(x_1 + xh) = 6e + \dots + n(n-1)(n-2)zx^{(n-3)}$ (IV), etc.

for higher differences of $f(x_1 + xh)$. If we substitute $x = 0$ in the equation $\phi_n(x) = a + bx^{(1)}\ cx^{(2)} + dx^{(3)} + \dots + zx^{(n)}$ we find $\phi_n(0) = a$, and by substituting $x = 0$ in

$\Delta f(x_1 + xh) = b + 2cx^{(1)} + 3dx^{(2)} + \dots + nzx^{(n-1)}$ we find

$\Delta f(x_1) = b$. Likewise letting $x = 0$ in

$\Delta^2 f(x_1 + xh) = 2c + 6dx^{(1)} + 12ex^{(2)} + \dots + n(n-1)zx^{(n-2)}$.

$\Delta^2 f(x_1) = 2c$; $c = \frac{1}{2}\Delta^2 f(x_1)$. In the same manner we find:

$\Delta^3 f(x_1) = 6d$ $\qquad d = 1/6\ \Delta^3 f(x_1)$

$\Delta^4 f(x_1) = 24e$ $\qquad e = 1/24 \Delta^4 f(x_1)$

etc.

Then we can write equation (I) in the following form:

$$U_{(x_1 + xh)} = f(x_1 + xh) = f(x_1) + x\,\Delta f(x_1) + \frac{x(x-1)}{2!}\Delta^2 f(x_1) + \frac{x(x-1)(x-2)}{3!}\Delta^3 f(x_1) + \dots + \frac{x(x-1)(x-2)\dots(x-n+1)}{n!}\Delta^n f(x_1)$$

or replacing $f(x_1)$ by U_{x_1} we have

$$U_{(x_1 + xh)} = U_{x_1} + x\,\Delta U_{x_1} + \frac{x(x-1)}{2!}\Delta^2 U_{x_1} + \frac{x(x-1)(x-2)}{3!}\Delta^3 U_1 + \dots + \frac{x(x-1)(x-2)(x-3)\dots(x-n\ 1)}{n!}\Delta^n U_1$$

Letting $x_1 = 0$ we have:

$$U_{(0+xh)} = U_0 + \Delta U_0 \; \frac{x(x-1)}{2!}\Delta^2{}_0 + \frac{x(x-1)(x-2)}{3!}\Delta^3 U_0$$

$$+ \cdot \cdot \cdot \cdot + \frac{x(x-1)(x-2)(x-3) \; \cdots \; (x-n+1)}{N!}\Delta^n U_0$$

which is Newton's Interpolation Formula.

This formula proves well suited for interpolation from the point of view of simplicity and represents a curve which passes through the points $(0,U_0)$, $(1, U_1)$, $(2, U_2)$, etc., whose equation is rational integral and in general of the (n-1)th degree if there are n points.

If we restrict the use of Newton's Formula to first differences there is obviously straight line interpolation and it amounts to fitting a straight line to two of the points. $U_x = U_0 + x\Delta U_0$. Then the interpolated value would be a point falling on the straight line $U_x = U_0 + x\Delta U_0$, shown in figure 3.

Restricting the use to second differences means fitting a parabola, $U_x = U_0 + x\Delta U_0 + \frac{x(x-1)}{2!}\Delta^2 U_0$, to three of the points, shown in figure 4.

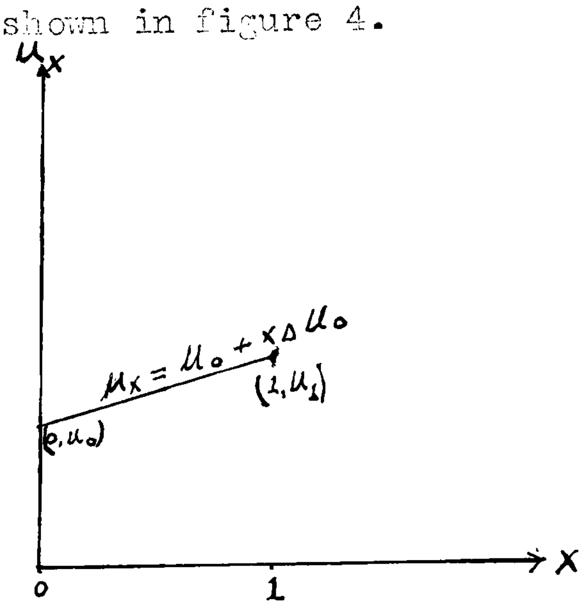

Figure 3

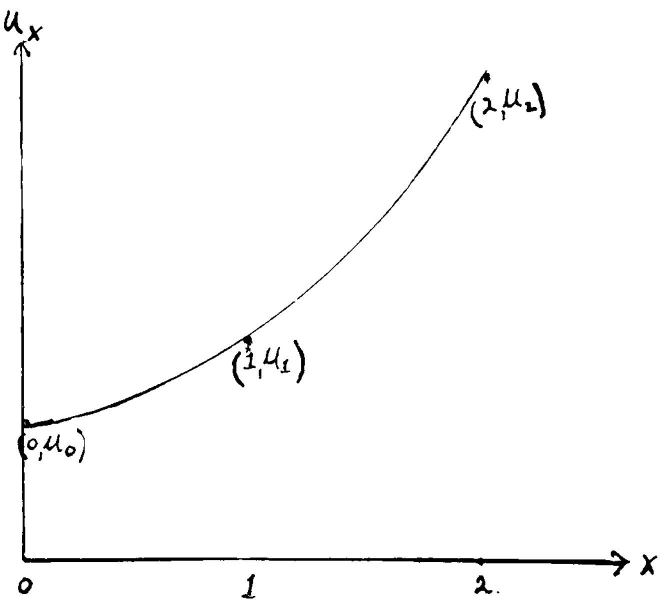

Figure 4

Then the interpolated values fall on the parabolic curve $U_x = U_0 + x\Delta U_0 + \frac{x(x-1)}{2!}\Delta^2 U_0$ where x takes the necessary values from x = o to x = 2. Likewise as the use of Newton's Formula is restricted to third, fourth, fifth ... nth differences we find the fitted curves to be of the third, fourth, fifth .... nth degrees.

The problem of finding a smooth curve to pass through the given values of $U_0, U_1, U_2, \ldots U_n$ has no single unique solution; in fact an infinite number of curves satisfying these conditions can be found, but for practical purposes the simplest solution or equation should be used. Newton's Formula, as derived, has been the foundation for various other formulas. It was first given in 1687 in Newton's "Philosophia Naturalis Principia Mathematica".

Since the other formulas used in interpolation are based on Newton's Formula we shall restrict the Tangential and Osculatory interpolation to this fundamental formula.

CHAPTER III

TANGENTIAL INTERPOLATION

If any formula based upon finite differences is to be employed to interpolate several values in the same interval, the work can be simplified by formulas for the leading differences of these interpolated values.

Taking Newton's formula we can show how it can be employed to interpolate t - 1 values in a central interval.

Newton's Formula:

$$u_x = U_0 + x\Delta U_0 + \frac{x(x-1)}{2!}\Delta^2 U_0 + \frac{x(x-1)(x-2)}{3!}\Delta^3 U_0 + \cdots$$

Let $x = \frac{t}{t}$

$$u_{\frac{t}{t}} = u_0 + \frac{t}{t}\Delta u_0 + \frac{\frac{t}{t}\left(\frac{t}{t}-1\right)}{2!}\Delta^2 u_0 + \frac{\frac{t}{t}\left(\frac{t}{t}-1\right)\left(\frac{t}{t}-2\right)}{3!}\Delta^3 u_0$$

$x = \frac{t+1}{t}$

$$u_{\frac{t+1}{t}} = u_0 + \frac{t+1}{t}\Delta u_0 + \frac{\left(\frac{t+1}{t}\right)\left(\frac{t+1}{t}-1\right)}{2!}\Delta^2 u_0 + \frac{\left(\frac{t+1}{t}\right)\left(\frac{t+1}{t}-1\right)\left(\frac{t+1}{t}-2\right)}{3!}\Delta^3 u_0$$

$x = \frac{t+2}{t}$

$$u_{\frac{t+2}{t}} = u_0 + \frac{t+2}{t}\Delta u_0 + \frac{\left(\frac{t+2}{t}\right)\left(\frac{t+2}{t}-1\right)}{2!}\Delta^2 u_0 + \frac{\left(\frac{t+2}{t}\right)\left(\frac{t+2}{t}-1\right)\left(\frac{t+2}{t}-2\right)}{3!}\Delta^3 u_0$$

$x = \frac{t+3}{t}$

$$u_{\frac{t+3}{t}} = u_0 + \frac{t+3}{t}\Delta u_0 + \frac{\left(\frac{t+3}{t}\right)\left(\frac{t+3}{t}-1\right)}{2!}\Delta^2 u_0 + \frac{\frac{t+3}{t}\left(\frac{t+3}{t}-1\right)\left(\frac{t+3}{t}-2\right)}{3!}\Delta^3 u_0$$

Differencing the above equations to third differences we find the leading differences to be:

(1) $$\frac{\Delta u_0}{t} + \frac{t+1}{2}\cdot\frac{\Delta^2 u_0}{t^2} - \frac{t^2-1}{6}\cdot\frac{\Delta^3 u_0}{t^3}$$

(2) $$\frac{\Delta^2 U_0}{t^2} + \frac{\Delta^3 U_0}{t^3}$$

(3) $$\frac{\Delta^3 U_0}{t^3}$$

Then t can take on different values according to the number of values that we wish to interpolate in the central interval.

If the method of interpolating several values in an interval as explained above is applied to several succeeding intervals the interpolation curve passing through the values interpolated between U_1 and U_2 will not in general be continuous with the curve passing through the values interpolated between U_2 and U_3, etc. Then in the final series of interpolated values there will be discontinuities at the points U_2, U_3, U_4, U_5, etc.

It is possible to adjust whatever interpolation formula is employed so that any two interpolation curves will have the same slope at the point of their intersection; that is, at the point which constitutes the "end" point of one interval and the "beginning" point of the next interval. Such a scheme applied to interpolation is known as tangential interpolation.

We shall assume the tangential formula to be of the form:

$$U_x = a + bx + cx^2 + dx^3 \qquad \text{(A)}$$

where no more than third differences are applied in the interpolation formula.

If the corresponding curve passes through U_1 and U_2 we have:

$U_1 = a + b + c + d = U_0 + \Delta U_0$ (B)

$U_2 = a + 2b + 4c + 8d = U_0 + 2\Delta U_0 + \Delta^2 U_0$ (C)

Since the next requirement is that the slope of curve (A) at U_1 shall be the same as that of the parabola through U_0, U_1, and U_2 or

$U_x = U_0 + x\Delta U_0 + \frac{x(x-1)}{2} \Delta^2 U_0$ at the same point. Then

since $U_x = a + bx + cx^2 + dx^3$

$\frac{d}{dx}(U_x) = b + 2cx + 3dx^2$

when $x = 1$

$\frac{d}{dx}(U_x) = b + 2c + 3d$

and $U_x = U_0 + x\Delta U_0 + \frac{x(x-1)}{2!} \Delta^2 U_0$

$\frac{d}{dx}(U_x) = \Delta U_0 + \frac{1}{2}\Delta^2 U_0$

Therefore the following relation must be true:

$b + 2c + 3d = \Delta U_0 + \frac{1}{2}\Delta^2 U_0$ (D)

The next requirement is that the slope of (A) at U_2 shall be the same as the slope of the parabola through U_1 U_2 and U_3.

$U_x = a + bx + cx^2 + dx^3$

$\frac{d}{dx}(U_x) = b + 2cx + 3dx^2$

when $x = 2$.

$\frac{d}{dx}(U_x) = b + 4c + 12d$

$U_2 = U_1 + x\Delta U_1 + \frac{x(x-1)}{2!} \Delta^2 U_1$

$\frac{d}{dx}(U_x) = \Delta U_1 + \frac{x^2}{2!} \Delta^2 U_1 - \frac{x}{2!} \Delta^2 U_1$

and

$b + 4c + 12d = \Delta U_0 + 3/2\, \Delta^2 U_0 + \frac{1}{2} \Delta^3 U_0$ (E)

Solving the following equations simultaneously

$a + b + c + d = U_0 + \Delta U_0$ (B)

$a + 2b + 4c + 8d = U_0 + 2\Delta U + \Delta^2 U_0$ (C)

$$b + 2c + 3d = \Delta U_0 + \frac{1}{2}\Delta^2 U_0 \qquad (D)$$

$$b + 4c + 12d = \Delta U_0 + 3/2\ \Delta^2 U_0 + \frac{1}{2}\Delta^3 U_0 \qquad (E)$$

we find

$$d = \frac{1}{2}\Delta^3 U$$

$$c = \frac{1}{2}\Delta^2 U_0 - 2\Delta^3 U_0$$

$$b = 5/2\ \Delta^3 U_0 + \Delta U_0 - \frac{1}{2}\Delta^2 U_0$$

$$a = U_0 - \Delta^3 U_0.$$

Then

$$U_x = U_0 - \Delta^3 U_0 + x\left[5/2\ \Delta^3 U_0 - \frac{1}{2}\Delta^2 U_0 + \Delta U_0\right] + x^2\left[\frac{1}{2}\Delta^2 U_0 - 2\Delta^3 U_0\right] + x^3\left[\frac{1}{2}\Delta^3 U_0\right].$$

$$U_x = U_0 + x\Delta U_0 + \frac{x(x-1)}{2!}\Delta^2 U_0 + \frac{(x-1)^2(x-2)}{2!}\Delta^3 U_0$$

which is the tangential interpolation formula worked out from Newton's Formula.

Let us consider the values U_0, U_1, U_2 as the "first set" and U_1, U_2, U_3 as the "second set" for the interval from U_1 to U_2. Then the slope of the interpolation curve at U_1 is determined by the first set of values and the slope at U_2 is determined by the second set of values. Likewise the slope at U_3 is determined by the first set in the interval from U_2 to U_3 which is the second set in the interval from U_1 to U_2. Hence the interpolation curves for the two intervals, U_1 to U_2 and U_2 to U_3, must have the same slope at their point of intersection U_2 and so on for all succeeding intervals.

Substituting the values $x = \frac{t}{t}$, $x = \frac{t+1}{t}$, $x = \frac{t+2}{t}$, and $x = \frac{t+3}{t}$ in the tangential interpolation formula we have

$$U_{\frac{t}{t}} = U_0 + \frac{t}{t}\Delta U_0$$

$$U_{\frac{t+1}{t}} = U_0 + \frac{t+1}{t} \Delta U_0 + \frac{t+1}{2t^2} \Delta^2 U_0 + \frac{1-t}{2t^3} \Delta^3 U_0$$

$$U_{\frac{t+2}{t}} = U_0 + \frac{t+2}{t} \Delta U_0 + \frac{t+2}{t^2} \Delta^2 U_0 + \frac{4-2t}{t^3} \Delta^3 U_0$$

$$U_{\frac{t+3}{t}} = U_0 + \frac{t+3}{t} \Delta U_0 + \frac{3(t+3)}{2t^2} \Delta^2 U_0 + \frac{9(3-t)}{2t^3} \Delta^3 U_0$$

we find the leading differences of the four equations to be:

1st. $\frac{\Delta U_0}{t} + \frac{(t+1)\Delta^2 U_0}{2t^2} - \frac{t-1}{2t^3} \Delta^3 U_0$

2nd. $\frac{\Delta^2 U_0}{t^2} - (t-3) \frac{\Delta^3 U_0}{t^3}$

3rd. $\frac{3\Delta^3 U_0}{t^3}$

When $t = 5$ we have:

1st $.2\Delta U_0 + .12\Delta^2 U_0 - .016\Delta^3 U_0$

2nd $.04\Delta^2 U_0 - .016\Delta^3 U_0$

3rd $.024\Delta^3 U_0$

As a very simple example, which will show the application of tangential interpolation, suppose that we wish to insert four values between the log of 2.7183 and the log of 2.7184 having the following logs given in a set of tables.

U_0 = log 2.7182 = 0.4342814081

U_1 = log 2.7183 = 0.4342973851

U_2 = log 2.7184 = 0.4343133615

U_3 = log 2.7185 = 0.4343293373

U_4 = log 2.7186 = 0.4343453126

Differencing the above values we find the leading differences to be:

ΔU_0 = .0000159771

$\Delta^2 U_0$ = -.0000000007

$\Delta^3 U_0$ = .0000000001

Applying the leading differences found in the tangential formula for $t = 5$ we have:

1st $.2(.0000159771) + .12(-.0000000007) - .012(.0000000001)$

1st $.0000031953344$

2nd $.04(-.0000000007) - .016(0000000001)$

2nd $- .0000000000296$

3rd $.024(.0000000001)$

3rd $.0000000000024$

Building a new difference table using the leading differences found for $t = 5$ in the tangential formula we find the values that we wish to insert between 0.4342973851 and 0.4343133615 to be 0.4343005804, 0.4343037757, o.4343069710, and 0.4343101663 correct to ten decimal places.

A graphic interpretation is given in figure 5. If we plot the equation corresponding to Newton's Formula with x as the abscissa and U_x as the ordinate we obtain the graph as shown by (I). Using the same set of axes for plotting the tangential equation (II) corresponding to the first set (U_0, U_1, U_2) for the interval U_1 to U_2 there exists a correspondence of all points between U_0 and U_2. Between the points U_2 and U_3 the graph of equation (II) deviates from that of equation (I).

Using O' as the origin and a new set of axes to plot equation (III), corresponding to the second set $(U_1\ U_2\ U_3)$ for the interval U_1 to U_2, all points between U_1 and U_3 as found by plotting equation (III) correspond to all points in the same interval as found by plotting equation (II). The graph of equation (III) deviates from that of equation (I) between the points U_3 and U_4, etc.

In general the graph of equation (I) deviates from the graph of each successive tangential equation in the interval U_i to U_{i+1} where $i =$ [illegible]$,2,3 \ldots N$.

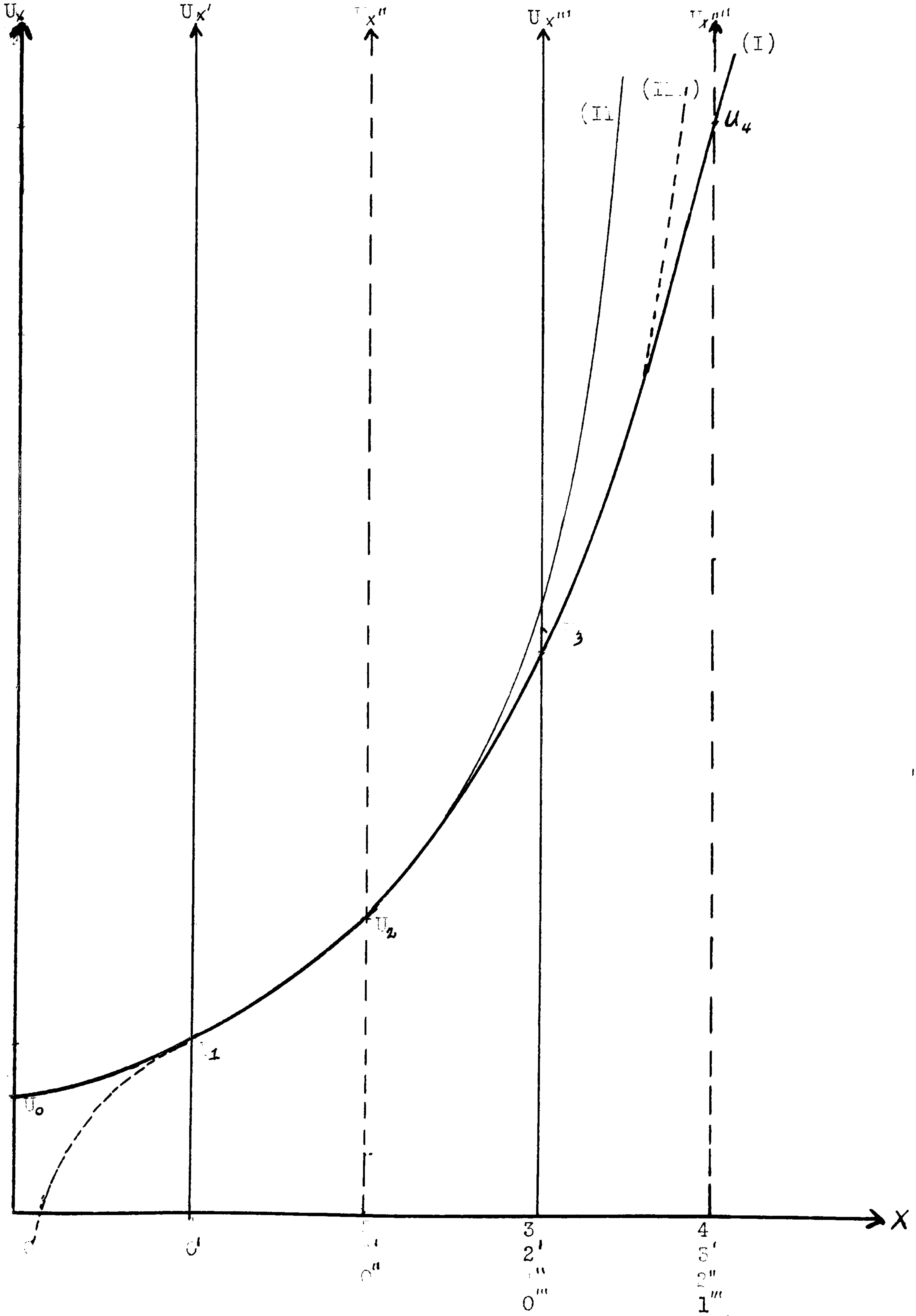

Figure 5

CHAPTER IV

OSCULATORY INTERPOLATION

If two interpolated curves are required to have not only the same slopes but also the same curvatures at their points of intersection the interpolation is called osculatory. Osculatory interpolation under such circumstances calls for fifth differences which in ordinary interpolation is rather cumbersome, but despite this disadvantage we shall develop an osculatory interpolation formula based on Newton's Fundamental Interpolation Formula.

Let us assume the osculatory interpolation formula to be of the form: $y_x = a + bx + cx^2 + dx^3 + ex^4 + fx^5$.(K)

If the corresponding curve passes through y_1 and y_2 we have

$$y_1 = a + b + c + d + e + f = U_0 + \Delta U_0 \quad \text{(L)}$$

$$y_2 = a + 2b + 4c + 8d + 16e + 32f = U_0 + 2\Delta U_0 + \Delta^2 U_0 . \quad \text{(M)}$$

Next we require that the slope of curve (K) at y_2 shall be the same as that of the curve through y_0, y_1, and y_2 or

$$U_x = U_0 + x\Delta U_0 + \frac{x(x-1)}{2!}\Delta^2 U_0 + \frac{x(x-1)(x-2)}{3!}\Delta^3 U_0 + \frac{x(x-1)(x-2)(x-3)}{4!}\Delta^4 U_0$$

at the same point.

This requires that

$$\frac{d}{dx}(a + bx + cx^2 + dx^3 + ex^4 + fx^5) = \frac{d}{dx}\Big[U_0 + x\Delta U_0 + \frac{x(x-1)}{2!}\Delta^2 U_0 + \frac{x(x-1)(x-2)}{3!}\Delta^3 U_0 + \frac{x(x-1)(x-2)(x-3)}{4!}\Delta^4 U_0\Big]$$

$$\frac{d}{dx}(a + bx + cx^2 + dx^3 + ex^4 + fx^5) = b + 2cx + 3dx^2 + 4ex^3 + 5fx^4$$

The slope at the point y_1 is

$$b + 2c + 3d + 4e + 5f$$

$$\frac{d}{dx}\left[U_0 + x\Delta U_0 + \frac{x(x-1)}{2!}\Delta^2 U_0 + \frac{x(x-1)(x-2)}{3!}\Delta^3 U_0 + \frac{x(x-1)(x-2)(x-3)}{4!}\Delta^4 U_0\right]$$

$$= \Delta U_0 + (x-\tfrac{1}{2})\Delta^2 U_0 + (\tfrac{1}{2}x^2 - x + 1/3)\Delta^3 U_0 + \left[\frac{x^3}{6} - \frac{3x^2}{4} + \frac{11x}{12} - 1/4\right]\Delta^4 U_0$$

When $x = 1$ the slope is: $\Delta U_0 + \tfrac{1}{2}\Delta^2 U_0 - 1/6\,\Delta^3 U_0 + 1/12\,\Delta^4 U_0$

Then:

$$b + 2c + 3d + 4e + 5f = \Delta U_0 + \tfrac{1}{2}\Delta^2 U_0 - 1/6\,\Delta^3 U_0 + 1/12\,\Delta^4 U_0 \qquad (N)$$

Likewise when $x = 2$:

$$b + 4c + 12d + 32e + 80f = \Delta U_0 + 3/2\,\Delta^2 U_0 + 1/3\,\Delta^3 U_0 - 1/[illegible]\,\Delta^4 U_0 + 1/12\,\Delta^5 U_0 \qquad (O)$$

Imposing the last condition which required that the two curves must have the same curvature at their points of intersection.

Since curvature $= \dfrac{\frac{d^2}{dx^2}}{\left[1 + \left(\frac{d}{dx}\right)^2\right]^{3/2}}$

The curvature of (K) at U_1 is:

$$\frac{2c + 6d + 12e + 20f}{\left[1 + (b + 2c + 3d + 4e + 5f)^2\right]^{3/2}}$$

In like manner the curvature of U_x at U_1 is:

$$\frac{\Delta^2 U_0 + 7/6\,\Delta^4 U_0}{\left[1 + (\Delta U_0 + \tfrac{1}{2}\Delta^2 U_0 - 1/6\,\Delta^3 U_0 + 1/12\,\Delta^4 U_0)^2\right]^{3/2}}$$

which leads to the relationship

$$2c + 6d + 12e + 20f = \Delta^2 U_0 + 7/6\,\Delta^4 U_0 \qquad (P)$$

Since the curvature at U_2 is determined by the same equation as the one that determines the curvature at U_1 by replacing U_0 by U_1, ΔU_0 by ΔU_1, . . . $\Delta^4 U_0$ by $\Delta^4 U_1$ we have the relationship

$$2c + 12d + 48e + 160f = \Delta^2 U_0 + 2\Delta^3 U_0 + 11/12\,\Delta^4 U_0 - 1[illegible]\,\Delta^5 U_0 \qquad (Q)$$

Solving the equations (L), (M), (N), (O), (P), and (Q) simultaneously we find:

$$a = U_0 - 2\Delta^3 U_0 + [illegible]\,\Delta^4 U_0 + 1/6\,\Delta^5 U_0$$

$$b = \Delta U_0 - \tfrac{1}{2}\Delta^2 U_0 + 25/3\,\Delta^3 U_0 - 117/12\,\Delta^4 U_0 - 2/3\,\Delta^5 U_0$$

$$c = \frac{1}{2}\Delta^2 U_o - 13\Delta^3 U_o + \frac{231}{24}\Delta^4 U_o + \frac{25}{24}\Delta^5 U_o$$

$$d = \frac{29}{3}\Delta^3 U_o - \frac{51}{8}\Delta^4 U_o - \frac{19}{24}\Delta^5 U_o$$

$$e = -\frac{7}{2}\Delta^3 U_o + \frac{37}{24}\Delta^4 U_o + \frac{7}{24}\Delta^5 U_o$$

$$f = \frac{1}{6}\Delta^3 U_o - \frac{1}{8}\Delta^4 U_o - \frac{1}{24}\Delta^5 U_o.$$

Substituting the values of a, b, c, d, e, and f in (K) we have the osculatory interpolation curve.

$$U_x = U_o + x\Delta U_o + \frac{x(x-1)}{2!}\Delta^2 U_o + \frac{3(x-1)^3(x-2)^2 + x(x-1)(x-2)}{3!}\Delta^3 U_o$$

$$+ \frac{3(x-1)^2(x-2)^2(6-x) + x(x-1)(x-2)(x-3)}{4!}\Delta^4 U_o - \frac{(x-1)^3(x-2)^2}{4!}\Delta^5 U_o$$

If we substitute the values $x = \frac{t}{t}$, $x = \frac{t+1}{t}$, $= \frac{t+2}{t}$,

$x = \frac{t+3}{t}$, $x = \frac{t+4}{t}$, and $x = \frac{t+5}{t}$ and difference these six equations we find the leading differences to be:

(1) $\frac{\Delta U}{t} + \frac{t+1}{2t^2}\Delta^2 U_o + \frac{3-6t+[illegible]^2 - t^4}{[illegible]^5}\Delta^3 U_o + \frac{2t^4+14t^3-35t^2+37t-3}{24t}\Delta^4 U$

$- \frac{(t-1)^2}{[illegible]^5}\Delta^5 U_o$

(2) $\frac{\Delta^2 U}{t^2} + \frac{45-42t+t^2}{[illegible]^5}\Delta^3 U_o + \frac{14t^3-105t^2+139t-45}{12t}\Delta^4 U_o$

$- \frac{(t-3)(3t-5)}{24[illegible]^5}\Delta^5 U_o$

(3) $\frac{75-36t+15t^2}{[illegible]^5}\Delta^3 U_o - \frac{5(21t^3-84t-45)}{12[illegible]^5}\Delta^4 U_o$

$- \frac{6t^2-52t+150}{2[illegible]^5}\Delta^5 U_o$

(4) $\frac{2369t - 280}{12t^5} \Delta^4 U_o \quad - \frac{5(2-t)}{t^5} \Delta^5 U_o$

(5) $\qquad - \frac{5(2-t)}{t^5} \Delta^5 U_o$

The application of this formula is the same as the tangential formula with the exception that osculatory interpolation is carried to fifth differences, where as the tangential formula is based on third differences. A graphic interpretation is the same as that for the tangential formula.

Tangential and Osculatory interpolation formulas are applicable to any finite difference formula whether it be based upon advancing, central, or backward differences.

BIBLIOGRAPHY

Forsyth, C. H., Mathematical Analysis of Statistics. John Wiley & Sons, New York, 1924

Ball, W. W. Rouse, A Short Account of the History of Mathematics, McMillan Co., New York, 1924

Cajori, Florian, History of Mathematics, Macmillan Co., 1926.

Boole, George, Finite Differences. Strechert Co., 1926.

Whittaker and Robinson, Interpolation. Blackie and Son, 1924.

Pearson, Karl, Tracts for Computers. Cambridge University Press, London, 1920.