LEGENDRE FUNCTIONS AS SOLUTIONS TO THE INHOMOGENEOUS HEAT EQUATION

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CONTENTS

ABSTRACT

This thesis is a study of the classical and weighted heat equations, with a detailed examination of solutions for a particular case of the weighted equation. This particular case involves the substitution of a weighting factor into the equation which reduces it to Legendre's differential equation. Several methods of finding solutions for this particular case are given, along with solutions computed using the given techniques.

CHAPTER I

INTRODUCTION

Consider the classical heat equation in the form

$$\frac{\partial}{\partial x}\left(\frac{\partial u(x,t)}{\partial x}\right)=\frac{\partial u}{\partial t} \quad , \quad x\in\Re \quad , \quad t>0 \quad , \tag{1.1}$$

and the weighted heat equation in the form

$$\frac{\partial}{\partial x}\left[w(x)\frac{\partial u}{\partial x}\right]=\frac{\partial u}{\partial t} \quad , \tag{1.2}$$

where the w in $w(x)$ stands for a weight. Although it might seem that (1.2) is quite similar to (1.1), "Little attention has been paid to diffusion in nonhomogeneous media in which the diffusion coefficients vary with distance measured in the direction of diffusion" [6].

The outer partial in (1.2) with respect to x is indicated as an iterated partial to emphasize that a major purpose of this paper is to insert a variable conductivity between the outer and inner partials. This paper treats conductivity variable in position in (1.2) rather than the nonlinear case, noted in Ford [10], as in

$$\frac{\partial}{\partial x}\left[w(u)\frac{\partial u}{\partial x}\right]=\frac{\partial u}{\partial t} \quad , \tag{1.3}$$

which has been studied by Aronson [2] who gives a particular closed form solution for the particular nonlinear version of the heat equation in

$$\frac{\partial}{\partial x}\left[mu^{m-1}\frac{\partial u}{\partial x}\right]=\frac{\partial^2}{\partial x^2}(u^m)=\frac{\partial u}{\partial t} \quad , \quad m>1 \quad . \tag{1.4}$$

Consider the heat equation itself in (1.1). So that the computations in later discussions may be fully understood, let us review the method of solution of this particular equation through separation of variables. To begin, assume the solution is of the form

$$u(x,t)=v(x)z(t) \quad . \tag{1.5}$$

Then

$$\frac{\partial u}{\partial x} = \frac{dv(x)}{dx} z(t) \quad , \tag{1.6}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2 v(x)}{dx^2} z(t) \quad , \tag{1.7}$$

and

$$\frac{\partial u}{\partial t} = v(x) \frac{dz(t)}{dt} \quad . \tag{1.8}$$

Upon substituting (1.7), and (1.8) into (1.1), we find

$$\frac{d^2 v(x)}{dx^2} z(t) = v(x) \frac{dz(t)}{dt} \quad , \tag{1.9}$$

and dividing by $v(x)z(t)$ gives

$$\frac{\frac{d^2 v(x)}{dx^2}}{v(x)} = \frac{\frac{dz(t)}{dt}}{z(t)} \quad . \tag{1.10}$$

As can be seen, the left side of (1.10) is dependent on x only and the right side is dependent on t only, hence, this equation must be equal to a constant.

Set

$$\frac{\frac{d^2 v(x)}{dx^2}}{v(x)} = \frac{\frac{dz(t)}{dt}}{z(t)} = -\lambda^2 \quad , \tag{1.11}$$

which implies

$$\frac{\frac{d^2 v(x)}{dx^2}}{v(x)} = -\lambda^2 \tag{1.12}$$

and

$$\frac{\frac{dz(t)}{dt}}{z(t)} = -\lambda^2 \quad . \tag{1.13}$$

Upon multiplying (1.12) by $v(x)$ and (1.13) by $z(t)$ and rearranging both equations, we have

$$\frac{d^2v(x)}{dx^2} + \lambda^2 v(x) = 0 \tag{1.14}$$

and

$$\frac{dz(t)}{dt} + \lambda^2 z(t) = 0 \quad . \tag{1.15}$$

By (1.14), we find the solution to $v(x)$ is

$$v(x) = A\sin\lambda x + B\cos\lambda x \quad , \tag{1.16}$$

where A and B are constants. Integrate (1.15) with respect to t to find

$$\ln z(t) = -\lambda^2 t + C$$

or

$$z(t) = De^{-\lambda^2 t} \quad , \tag{1.17}$$

where again, C and D are constants. Recalling $u(x,t) = v(x)z(t)$ and substituting the above results gives

$$u(x,t) = (A\sin\lambda x + B\cos\lambda x)De^{-\lambda^2 t} \quad . \tag{1.18}$$

This is a well-known solution to the classical heat equation [20].

CHAPTER II
THE WEIGHTED HEAT EQUATION

Let us now return to the weighted heat equation

$$\frac{\partial}{\partial x}\left[w(x)\frac{\partial u}{\partial x}\right]=\frac{\partial u}{\partial t} \tag{2.1}$$

and apply methods similar to those used to find a solution to the unweighted heat equation to attempt to find a solution to (2.1). If we again assume the solution $u(x,t)$ can be expressed in the form

$$u(x,t)=v(x)z(t) \tag{2.2}$$

then, as before,

$$\frac{\partial u}{\partial x}=\frac{dv(x)}{dx}z(t) \quad , \tag{2.3}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{d^2v(x)}{dx^2}z(t) \quad , \tag{2.4}$$

and

$$\frac{\partial u}{\partial t}=v(x)\frac{dz(t)}{dt} \quad . \tag{2.5}$$

Upon substituting (2.3) and (2.5) into (2.1), we find

$$\frac{\partial}{\partial x}\left[w(x)\frac{dv(x)}{dx}z(t)\right]=v(x)\frac{dz(t)}{dt} \tag{2.6}$$

or, equivalently

$$\frac{dw(x)}{dx}\frac{dv(x)}{dx}z(t)+w(x)\frac{d^2v(x)}{dx^2}z(t)=v(x)\frac{dz(t)}{dt} \quad . \tag{2.7}$$

As before, divide by $v(x)z(t)$ to get

$$\frac{\frac{dw(x)}{dx}\frac{dv(x)}{dx}}{v(x)}+\frac{w(x)\frac{d^2v(x)}{dx^2}}{v(x)}=\frac{\frac{dz(t)}{dt}}{z(t)} \tag{2.8}$$

which again can be seen to be equal to a constant, due to the fact that the left side is dependent on x only and the right side is dependent on t only.
Hence, set

$$\frac{\frac{dw(x)}{dx}\frac{dv(x)}{dx}}{v(x)}+\frac{w(x)\frac{d^2v(x)}{dx^2}}{v(x)}=\frac{\frac{dz(t)}{dt}}{z(t)}=-\lambda^2 \tag{2.9}$$

which implies

$$\frac{\frac{dw(x)}{dx}\frac{dv(x)}{dx}}{v(x)}+\frac{w(x)\frac{d^2v(x)}{dx^2}}{v(x)}=-\lambda^2 \tag{2.10}$$

and

$$\frac{\frac{dz(t)}{dt}}{z(t)}=-\lambda^2 \quad . \tag{2.11}$$

Upon multiplying (2.10) by $v(x)$ and (2.11) by $z(t)$ and rearranging both equations, we have

$$\frac{dw(x)}{dx}\frac{dv(x)}{dx}+w(x)\frac{d^2v(x)}{dx^2}+\lambda^2v(x)=0 \tag{2.12}$$

and

$$\frac{dz(t)}{dt}+\lambda^2z(t)=0 \quad . \tag{2.13}$$

By (2.13), we arrive at the same solution for $z(t)$ as before, namely

$$z(t)=Ae^{-\lambda^2 t} \quad . \tag{2.14}$$

As it is most likely that "The term $w(x)$ in (2.7) makes the medium inhomogeneous, and the solution in closed form then ranges from difficult (for special cases) to impossible" [21], the rest of this paper will be limited to considering one special case of the weighted heat equation, namely when $w(x)=1-x^2$. Therefore, consider (2.12) again, but this time make the substitution $w(x)=1-x^2$. Then we have

$$\frac{d}{dx}\left(1-x^2\right)\frac{dv(x)}{dx}+\left(1-x^2\right)\frac{d^2v(x)}{dx^2}+\lambda^2v(x)=0 \tag{2.15}$$

or, carrying through the derivation and rearranging gives

$$\left(1-x^2\right)\frac{d^2 v(x)}{dx^2}-2x\frac{dv(x)}{dx}+\lambda^2 v(x)=0 \quad . \tag{2.16}$$

Compare (2.16) with Legendre's differential equation

$$\left(1-x^2\right)\frac{d^2 v(x)}{dx^2}-2x\frac{dv(x)}{dx}+\nu(\nu+1)v(x)=0 \quad . \tag{2.17}$$

It is obvious these are the same equation with the identification

$$\nu(\nu+1)=\lambda^2 \quad . \tag{2.18}$$

Carrying through the multiplication and rearranging we have

$$\nu^2+\nu-\lambda^2=0 \tag{2.19}$$

and upon solving for ν using the quadratic formula, we find

$$\nu=\frac{-1\pm\sqrt{1+4\lambda^2}}{2} \quad . \tag{2.20}$$

Make the following designations for clarity in future calculations:

$$\nu_1=\frac{-1+\sqrt{1+4\lambda^2}}{2} \tag{2.21}$$

and

$$\nu_2=\frac{-1-\sqrt{1+4\lambda^2}}{2} \quad . \tag{2.22}$$

Solutions to Legendre's equation (2.17) are

$$S=a_0\left[1-\frac{\nu(\nu+1)}{2!}x^2+\frac{\nu(\nu+1)(\nu-2)(\nu+3)}{4!}x^4 - \frac{\nu(\nu+1)(\nu-2)(\nu+3)(\nu-4)(\nu+5)}{6!}x^6+\ldots\right] \tag{2.23}$$

and

$$T = a_1\left[x - \frac{(v-1)(v+2)}{3!}x^3 + \frac{(v-1)(v+2)(v-3)(v+4)}{5!}x^5 - \frac{(v-1)(v+2)(v-3)(v+4)(v-5)(v+6)}{7!}x^7 + \ldots\right] \tag{2.24}$$

where a_0 and a_1 are arbitrary constants [16]. For comparison with later calculations, call the terms inside the brackets above S_λ and T_λ, respectively. As S_λ and T_λ are linearly independent, and each series is a solution of Legendre's equation, any linear combination of S_λ and T_λ is a solution. Hence, the general solution is given by

$$v(x) = a_0 S_\lambda + a_1 T_\lambda \quad . \tag{2.25}$$

Given the computation of v_1 and v_2 in (2.21) and (2.22), we are now in a position to substitute into (2.23) and (2.24) to determine two linearly independent solutions to (2.16). Upon substituting v_1 into (2.23), we have

$$\begin{aligned} S_\lambda(x) = 1 &- \frac{1}{2!}\left[\left(-\frac{1}{2}+\frac{1}{2}\sqrt{1+4\lambda^2}\right)\left(\frac{1}{2}+\frac{1}{2}\sqrt{1+4\lambda^2}\right)\right]x^2 \\ &+ \frac{1}{4!}\left[\left(-\frac{1}{2}+\frac{1}{2}\sqrt{1+4\lambda^2}\right)\left(\frac{1}{2}+\frac{1}{2}\sqrt{1+4\lambda^2}\right)\left(-\frac{5}{2}+\frac{1}{2}\sqrt{1+4\lambda^2}\right)\left(\frac{5}{2}+\frac{1}{2}\sqrt{1+4\lambda^2}\right)\right]x^4 \\ &- \frac{1}{6!}\left[\left(-\frac{1}{2}+\frac{1}{2}\sqrt{1+4\lambda^2}\right)\left(\frac{1}{2}+\frac{1}{2}\sqrt{1+4\lambda^2}\right)\left(-\frac{5}{2}+\frac{1}{2}\sqrt{1+4\lambda^2}\right)\left(\frac{5}{2}+\frac{1}{2}\sqrt{1+4\lambda^2}\right)\right. \\ &\qquad \left.\left(-\frac{9}{2}+\frac{1}{2}\sqrt{1+4\lambda^2}\right)\left(\frac{9}{2}+\frac{1}{2}\sqrt{1+4\lambda^2}\right)\right]x^6 + \ldots \end{aligned}$$

$$S_\lambda(x) = 1 - \frac{1}{2!}\left[\lambda^2\right]x^2 + \frac{1}{4!}\left[(\lambda^2)(\lambda^2-6)\right]x^4 - \frac{1}{6!}\left[(\lambda^2)(\lambda^2-6)(\lambda^2-20)\right]x^6 + \ldots$$

$$\therefore S_\lambda(x) = 1 - \frac{\lambda^2}{2!}x^2 - \frac{6\lambda^2-\lambda^4}{4!}x^4 - \frac{120\lambda^2-26\lambda^4+\lambda^6}{6!}x^6 + \ldots \tag{2.26}$$

Upon substituting v_2 into (2.24), we have

$$T_\lambda(x) = x - \frac{1}{3!}\left[\left(-\frac{3}{2}+\frac{1}{2}\sqrt{1+4\lambda^2}\right)\left(\frac{3}{2}+\frac{1}{2}\sqrt{1+4\lambda^2}\right)\right]x^3$$
$$+\frac{1}{5!}\left[\left(-\frac{3}{2}+\frac{1}{2}\sqrt{1+4\lambda^2}\right)\left(\frac{3}{2}+\frac{1}{2}\sqrt{1+4\lambda^2}\right)\left(-\frac{7}{2}+\frac{1}{2}\sqrt{1+4\lambda^2}\right)\left(\frac{7}{2}+\frac{1}{2}\sqrt{1+4\lambda^2}\right)\right]x^5$$
$$-\frac{1}{7!}\left[\left(-\frac{3}{2}+\frac{1}{2}\sqrt{1+4\lambda^2}\right)\left(\frac{3}{2}+\frac{1}{2}\sqrt{1+4\lambda^2}\right)\left(-\frac{7}{2}+\frac{1}{2}\sqrt{1+4\lambda^2}\right)\left(\frac{7}{2}+\frac{1}{2}\sqrt{1+4\lambda^2}\right)\right.$$
$$\left.\left(-\frac{11}{2}+\frac{1}{2}\sqrt{1+4\lambda^2}\right)\left(\frac{11}{2}+\frac{1}{2}\sqrt{1+4\lambda^2}\right)\right]x^7+\ldots$$

$$T_\lambda(x) = x - \frac{1}{3!}\left[\lambda^2-2\right]x^3 + \frac{1}{5!}\left[(\lambda^2-2)(\lambda^2-12)\right]x^5$$
$$-\frac{1}{7!}\left[(\lambda^2-2)(\lambda^2-12)(\lambda^2-30)\right]x^7+\ldots$$
$$\therefore T_\lambda(x) = x + \frac{2-\lambda^2}{3!}x^3 + \frac{24-14\lambda^2+\lambda^4}{5!}x^5 + \frac{720-444\lambda^2+44\lambda^4-\lambda^6}{7!}x^7+\ldots \quad (2.27)$$

Hence, we have two solutions of (2.16). A calculation similar to that which led to (2.26) and (2.27) will give additional solutions by substituting v_2 into (2.23) and (2.24). However, these solutions will be the same as (2.26) and (2.27), as is apparent by comparing the form of v_1 and v_2.

Let us conclude this section by taking a brief look at a substitution which will be considered in more detail in the next section, namely, let $w(x) = 1 + x^2$. Upon substitution into (2.12), and rearranging the equation, we see

$$\frac{d}{dx}\left[(1+x^2)\frac{dv(x)}{dx}\right] = -\lambda^2 v(x) \quad . \qquad (2.28)$$

Then, make a change of variables, letting $x = i\eta$, hence $dx = id\eta$. After substituting into (2.28) we see

$$\frac{1}{i}\frac{d}{d\eta}\left[\frac{1-\eta^2}{i}\frac{d\,v(i\eta)}{d\eta}\right] = -\lambda^2 v(i\eta) \quad . \qquad (2.29)$$

Letting $\bar{v}(\eta) = v(i\eta)$ and simplifying, we are left with

$$\frac{d}{d\eta}\left[1-\eta^2\frac{d\,\hat{v}(\eta)}{d\eta}\right] = \lambda^2\,\hat{v}(\eta) \quad . \qquad (2.30)$$

As in (2.16) and (2.17), this is again simply Legendre's differential equation with the identification

$$\bar{v}(\bar{v}+1) = -\lambda^2 \quad , \tag{2.31}$$

where the "bar" designation on $\bar{v}$ distinguishes (2.31) from (2.18). Using the quadratic formula as before, and solving for $\bar{v}$, we find

$$\bar{v} = \frac{-1 \pm \sqrt{1-4\lambda^2}}{2} \quad . \tag{2.32}$$

Make the following designations for clarity in future calculations:

$$\bar{v}_1 = \frac{-1 + \sqrt{1-4\lambda^2}}{2} \tag{2.33}$$

and

$$\bar{v}_2 = \frac{-1 - \sqrt{1-4\lambda^2}}{2} \quad . \tag{2.34}$$

Hence, by using the given solutions of Legendre's equation in (2.23) and (2.24), we are now in a position to find additional solutions of (2.30). Namely, upon substituting the above values of $\bar{v}_1$ and $\bar{v}_2$ into (2.23) and (2.24) and following similar calculations which led us to (2.26) and (2.27) we find

$$\bar{S}_\lambda(x) = 1 + \frac{\lambda^2}{2!}x^2 + \frac{6\lambda^2 + \lambda^4}{4!}x^4 + \frac{120\lambda^2 + 26\lambda^4 + \lambda^6}{6!}x^6 + \ldots \tag{2.35}$$

and

$$\bar{T}_\lambda(x) = x + \frac{2+\lambda^2}{3!}x^3 + \frac{24 + 14\lambda^2 + \lambda^4}{5!}x^5 + \frac{720 - 444\lambda^2 + 44\lambda^4 + \lambda^6}{7!}x^7 + \ldots \tag{2.36}$$

where again the "bar" designation on the $\bar{S}$ and $\bar{T}$ distinguishes (2.35) and (2.36) from (2.26) and (2.27).

CHAPTER III

REDUCTION OF ORDER

We have been considering

$$\frac{d}{dx}\left[w(x)\frac{dv}{dx}\right] = -\lambda^2 v(x) \tag{3.1}$$

in the special case

$$\frac{d}{d\xi}\left[(1+\xi^2)\frac{d\hat{v}(\xi)}{dx}\right] = -\hat{v}(\xi) \quad . \tag{3.2}$$

Let $i\eta = \xi$ to obtain

$$\frac{1}{i}\frac{d}{d\eta}\left[\frac{1-\eta^2}{i}\frac{d\,\bar{v}(\eta)}{d\eta}\right] = -\bar{v}(\eta) \tag{3.3}$$

or

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{d\,\bar{v}(\eta)}{d\eta}\right] = \bar{v}(\eta) \quad . \tag{3.4}$$

If we imagine $\bar{v}(\eta)$ to be transposed to the left side of (3.4), we obtain Legendre's differential equation with the identification

$$\nu(\nu+1) = -1 \tag{3.5}$$

which implies

$$\nu = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} = \exp\left(\pm\frac{2\pi i}{3}\right) \quad . \tag{3.6}$$

If ν_1 is defined using the plus sign, and ν_2 is defined using the minus sign, it is interesting to observe

$$\nu_1 + 1 = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + 1 = \frac{1}{2} + i\frac{\sqrt{3}}{2} = -\nu_2$$

and

$$\nu_2 + 1 = \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) + 1 = \frac{1}{2} - i\frac{\sqrt{3}}{2} = -\nu_1 \quad . \tag{3.7}$$

As (3.4) has a regular point at $x = 0$, the power series method provides a solution of (3.4) in a neighborhood of this point [22]. Hence, let

$$\bar{v}(\eta) = \sum_{k=0}^{\infty} c_k \eta^k \quad . \tag{3.8}$$

Then, suppress subscripts on v to obtain

$$c_{k+2} = \frac{k(k+1) - v(v+1)}{(k+1)(k+2)} c_k \qquad \text{for} \quad 0 \le k \le \infty \quad . \tag{3.9}$$

It should be noted that if v is an integer, (3.8) reduces to a polynomial. However, in the case where v is not an integer, whether (3.8) is an acceptable solution, and therefore of interest, depends upon its convergence properties. The ratio test for convergence says that if in a series of positive numbers the ratio of the (n+1)th term to the nth term approaches a limit L as n increases without limit, and if L is less than one, the series converges. That is, convergence requires

$$\lim_{k \to \infty} \frac{|c_{k+1}|}{|c_k|} = L \qquad \text{where} \qquad L < 1 \tag{3.10}$$

It is evident that in equations (2.23) and (2.24) the ratio is

$$\frac{|c_{k+2}|}{|c_k|} x^2 \tag{3.11}$$

but from equation (3.9) it is readily seen that

$$\lim_{k \to \infty} \frac{|c_{k+2}|}{|c_k|} = \lim_{k \to \infty} \frac{\left| \frac{k(k+1) - v(v+1)}{(k+1)(k+2)} c_k \right|}{c_k} = \lim_{k \to \infty} \left| \frac{k}{k+2} - \frac{v(v+1)}{(k+1)(k+2)} \right| = 1 \quad . \tag{3.12}$$

Hence, the condition that equations (2.23) and (2.24) converge is that x^2 be less than 1, and this is true only if $|x| < 1$. Hence, for values of x in the range $-1 < x < 1$, the series solutions are valid.

Alternatively, consider (3.1) in the special case

$$\frac{d}{dx}\left[(1-x^2)\frac{dv(x)}{dx}\right] = -\lambda^2 v(x) \quad , \tag{3.13}$$

where the normalization replacing x by ξ using $\xi = \lambda x$ has been dropped, and $w(x)$ has a minus sign leading directly to Legendre's equation in the form

$$\frac{d}{dx}\left[(1-x^2)\frac{dv(x)}{dx}\right] + \lambda^2 v(x) = 0 \quad , \tag{3.14}$$

where we identify

$$v(v+1) = v^2 + v = \lambda^2 \quad . \tag{3.15}$$

Solve (3.15) for v in the form

$$v = \frac{-1 \pm \sqrt{1+4\lambda^2}}{2} \quad . \tag{3.16}$$

Overcome the tendency to imagine some restrictions on choices of λ , and consider a sequence of values of λ defined by

$$\begin{aligned} 1+4\lambda^2 &= (2v+1)^2 \\ 4\lambda^2 &= 4v^2 + 4v \quad , \\ \lambda^2 &= v^2 + v \qquad\qquad \text{and} \\ \lambda &= \pm\sqrt{v^2+v} \end{aligned} \tag{3.17}$$

where v takes the successive values in $S_1 \equiv \{0,\ 1,\ 2,\ \ldots\ \}$ corresponding to the positive sign in (3.16). If the values in (3.14) were used with the negative signs in (3.16) the successive values of v would lie in $S_2 \equiv \{-1,-2,-3\ \ldots\ \}$. The sum of each element in S_1 and the corresponding element in S_2 is -1. Then (3.14) becomes

$$\frac{d}{dx}\left[(1-x^2)\frac{dv(x)}{dx}\right] + n(\ n+1)v(x) = 0 \quad \text{where} \quad n = v\ \in S_1 \tag{3.18}$$

which has the Legendre polynomial given by Rodrigue's formula in

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n\left[(x^2-1)^n\right]}{dx^n} \quad \text{where } n = v\ \in S_1 \ , \tag{3.19}$$

as one of its solutions.

If we let $v_{1,n} = P_n$, a second solution of (3.18) can be obtained by reduction of order to obtain $v_{2,n} = q_n P_n$ using

$$\begin{aligned}
&-n(n+1)q_n(x)P_n(x) \\
&= \frac{d}{dx}\left\{(1-x^2)\frac{d}{dx}\left[q_n(x)P_n(x)\right]\right\} \\
&= \frac{d}{dx}\left\{(1-x^2)\left[P_n(x)\frac{dq_n(x)}{dx} + q_n(x)\frac{dP_n(x)}{dx}\right]\right\} \\
&= \frac{d}{dx}\left[(1-x^2)P_n(x)\frac{dq_n(x)}{dx}\right] + \frac{d}{dx}\left[(1-x^2)q_n(x)\frac{dP_n(x)}{dx}\right] \\
&= \frac{d}{dx}\left[(1-x^2)P_n(x)\frac{dq_n(x)}{dx}\right] \\
&+(1-x^2)\frac{dP_n(x)}{dx}\frac{dq_n(x)}{dx} + \frac{d}{dx}\left[(1-x^2)\frac{dP_n(x)}{dx}\right]q_n(x) \quad , \qquad (3.20)
\end{aligned}$$

where $n = \nu \in S_1$. Set

$$z_n = \frac{dq_n(x)}{dx} \qquad (3.21)$$

and use P_n in (3.18) to work with (3.20) as follows:

$$\begin{aligned}
0 &= \left\{n(n+1)P_n(x) + \frac{d}{dx}\left[(1-x^2)\frac{dP_n(x)}{dx}\right]\right\}q_n(x) \\
&+ \frac{d}{dx}\left[(1-x^2)P_n(x)\frac{dq_n(x)}{dx}\right] + (1-x^2)\frac{dP_n(x)}{dx}\frac{dq_n(x)}{dx} \\
&= \frac{d}{dx}\left[(1-x^2)P_n(x)z_n(x)\right] + (1-x^2)\frac{dP_n(x)}{dx}z_n(x) \\
&= (1-x^2)P_n(x)\frac{dz_n(x)}{dx} + \frac{d}{dx}\left[(1-x^2)P_n(x)\right]z_n(x) \\
&+(1-x^2)\frac{dP_n(x)}{dx}z_n(x) \quad . \qquad (3.22)
\end{aligned}$$

Thus,

$$-(1-x^2)P_n(x)\frac{dz_n(x)}{dx} = \left\{\frac{d}{dx}\left[(1-x^2)P_n(x)\right] + (1-x^2)\frac{dP_n(x)}{dx}\right\}z_n(x) \quad , \qquad (3.23)$$

and

$$-\frac{1}{z_n(x)}\frac{dz_n(x)}{dx} = \frac{1}{(1-x^2)P_n(x)}\frac{d}{dx}\left[(1-x^2)P_n(x)\right] + \frac{1}{P_n(x)}\frac{dP_n(x)}{dx} \quad . \qquad (3.24)$$

This may be integrated to obtain

$$\ln\left[(1-x^2)P_n(x)\right]+\ln P_n(x)+\ln z_n(x)$$

$$=\ln\left[(1-x^2)P_n^2(x)z_n(x)\right]=0 \quad . \tag{3.25}$$

Hence, by (3.25) we see

$$1=(1-x^2)P_n^2(x)z_n(x)$$

or

$$z_n(x)=\frac{1}{(1-x^2)P_n^2(x)} \quad . \tag{3.26}$$

Thus, we know by (3.21) that

$$q_n(x)=\int\frac{dx}{(1-x^2)P_n^2(x)} \tag{3.27}$$

and

$$v_{2,n}=P_n(x)\int\frac{dx}{(1-x^2)P_n^2(x)} \tag{3.28}$$

where $n=\nu\in S_1$.

Perhaps the complexities in the above computations suggest checking the solution in (3.28) against Legendre's equation. Compute

$$\frac{dv_{2,n}(x)}{dx}=\frac{dP_n(x)}{dx}q_n(x)+P_n(x)\frac{dq_n(x)}{dx}$$

$$=\frac{dP_n(x)}{dx}q_n(x)+\frac{1}{(1-x^2)P_n(x)} \quad , \tag{3.29}$$

and $$\frac{d}{dx}\left[(1-x^2)\frac{dv_{2,n}(x)}{dx}\right]=\frac{d}{dx}\left\{(1-x^2)\frac{d\left[P_n(x)q_n(x)\right]}{dx}\right\}$$

$$=\frac{d}{dx}\left\{(1-x^2)\left[\frac{dP_n(x)}{dx}q_n(x)+\frac{dq_n(x)}{dx}P_n(x)\right]\right\}$$

$$=\frac{d}{dx}\left[(1-x^2)\frac{dP_n(x)}{dx}q_n(x)\right]+\frac{d}{dx}\left[(1-x^2)\frac{dq_n(x)}{dx}P_n(x)\right]$$

$$=\frac{d}{dx}\left[(1-x^2)\frac{dP_n(x)}{dx}\right]q_n(x)+(1-x^2)\frac{dP_n(x)}{dx}\frac{dq_n(x)}{dx}+\frac{d}{dx}\left[\frac{1}{P_n(x)}\right]$$

$$= -n(n+1)P_n(x)q_n(x) + \frac{1}{[P_n(x)]}\frac{dP_n(x)}{dx} + \frac{d}{dx}\left[\frac{1}{P_n(x)}\right]$$

$$= -n(n+1)P_n(x)q_n(x) = -n(n+1)v_{2,n}(x) \quad , \tag{3.30}$$

and $v_{2,n}(x)$ satisfies Legendre's equation.

To continue, use q_1 from (3.27) to compute

$$q_1(x) = \int \frac{dx}{(1-x^2)P_1^2(x)} = \int \frac{dx}{(1-x^2)x^2}$$

$$= \int \left\{ \frac{1}{2}\left[\frac{1}{1-x} + \frac{1}{1+x}\right] + \frac{1}{x^2} \right\} dx$$

$$= \ln\sqrt{\left|\frac{1+x}{1-x}\right|} - \frac{1}{x}, \tag{3.31}$$

which has singularities for $x \in \{-1, 0, 1\}$. However, use $v_{2,1}$ from (3.28) to compute

$$v_{2,1}(x) = P_1(x)q_1(x) = xq_1(x) = x\ln\sqrt{\left|\frac{1+x}{1-x}\right|} - 1 \quad , \tag{3.32}$$

which has singularities for $x \in \{-1, 1\}$.

As an example, note that Legendre's polynomial for $n = 1$ compares with (2.24) as shown in

$$P_1 = T_1(x) = x \quad . \tag{3.33}$$

To develop $v_{2,1}$ from (3.32) for comparison with (3.33) recall

$$\ln|1+x| = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots \tag{3.34}$$

Although unnecessary, write

$$\ln|1-x| = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \ldots \tag{3.35}$$

for clarity in using (3.32) to compute

$$v_{2,1}(x)+1 = x\ln\sqrt{\left|\frac{1+x}{1-x}\right|} = \frac{x}{2}\ln\frac{1+x}{1-x} = \frac{x}{2}\left[\ln|1+x| - \ln|1-x|\right] \tag{3.36}$$

$$= \frac{x}{2}\left[x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots\right]$$

$$-\frac{x}{2}\left[x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \ldots\right]$$

$$= x\left[x + \frac{1}{3}x^3 + \ldots\right] = x^2 + \frac{1}{3}x^4 + \ldots \quad . \tag{3.37}$$

Thus,

$$v_{2,1}(x) = -1 + x^2 + \frac{1}{3}x^4 + \ldots = -S_{\sqrt{2}}(x) \quad . \tag{3.38}$$

CHAPTER IV
ADDITIONAL SOLUTIONS

Recall, from the variation of parameters method, a second solution to Legendre's equation may be found as

$$v_{2,n} = P_n(x)q_n(x) \tag{4.1}$$

where $P_n(x)$ are the associated Legendre polynomials and

$$q_n(x) = \int \frac{dx}{(1-x^2)P_n^2(x)} \quad . \tag{4.2}$$

Let us find a few more solutions to (3.15), given the following Legendre polynomials [22]:

$$\left.\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2}(3x^2-1) \\ P_3(x) &= \frac{1}{2}(5x^3-3x) \\ P_4(x) &= \frac{1}{8}(35x^4-30x^2+3) \end{aligned}\right\} \tag{4.3}$$

From (4.2), we see

$$q_0(x) = \int \frac{dx}{(1-x^2)} \tag{4.4}$$

or

$$q_0(x) = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| \quad . \tag{4.5}$$

Hence, by (4.1)

$$v_{2,0} = P_0(x)q_0(x) = (1)\left(\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|\right) = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| = \ln\sqrt{\left|\frac{1+x}{1-x}\right|} \quad . \tag{4.6}$$

We have already calculated a solution associated with $P_1(x)$. For comparative purposes it will be restated here, that is

$$v_{2,1}(x) = P_1(x)q_1(x) = xq_1(x) = x\ln\sqrt{\left|\frac{1+x}{1-x}\right|} - 1 \quad . \tag{4.7}$$

To find $v_{2,2}$ we have

$$q_2(x) = \int \frac{dx}{(1-x^2)\left[\frac{1}{2}(3x^2-1)\right]^2} \tag{4.8}$$

or, by partial fractions

$$q_2(x) = \int \left[\frac{\frac{1}{2}}{(1-x)} + \frac{\frac{1}{2}}{(1+x)} + \frac{6}{(3x^2-1)^2} + \frac{3}{(3x^2-1)} \right] dx \quad . \tag{4.9}$$

Hence,

$$q_2(x) = \ln\sqrt{\left|\frac{1+x}{1-x}\right|} - \frac{3x}{(3x^2-1)} \tag{4.10}$$

and

$$v_{2,2} = P_2(x)q_2(x) = \frac{1}{2}(3x^2-1)\left[\ln\sqrt{\left|\frac{1+x}{1-x}\right|} - \frac{3x}{(3x^2-1)}\right] \quad . \tag{4.11}$$

Hence,

$$v_{2,2} = P_2(x)q_2(x) = \frac{1}{2}(3x^2-1)\ln\sqrt{\left|\frac{1+x}{1-x}\right|} - \frac{3x}{2} \tag{4.12}$$

Similar calculations will show that

$$q_3(x) = \int \frac{dx}{(1-x^2)\left[\frac{1}{2}x(5x^2-3)\right]^2} \tag{4.13}$$

or

$$q_3(x) = \ln\sqrt{\left|\frac{1+x}{1-x}\right|} - \frac{4}{9x} - \frac{25x}{9(5x^2-3)} \tag{4.14}$$

Again by (4.1)

$$v_{2,3} = P_3(x)q_3(x) = \frac{1}{2}(5x^3 - 3x)\left[\ln\sqrt{\left|\frac{1+x}{1-x}\right|} - \frac{4}{9x} - \frac{25x}{9(5x^2-3)}\right] . \tag{4.15}$$

Hence,

$$v_{2,3} = \frac{1}{2}(5x^3 - 3x)\ln\sqrt{\left|\frac{1+x}{1-x}\right|} - \frac{5}{2}x^2 + \frac{2}{3} . \tag{4.16}$$

Continued calculations in a manner similar to those above would generate an infinite number of solutions to (3.14). It is interesting to note that each solution is dependent on a combination of a logarithmic term times the nth degree Legendre polynomial plus linear combinations of lower order polynomials. Perhaps further computations would yield a simple algorithm for computing a particular order solution without the necessary integrations.

CHAPTER V
CONCLUSION

In this thesis, we have studied the classical and weighted heat equations, with an emphasis on the weighted equation

$$\frac{\partial}{\partial x}\left[w(x)\frac{\partial u}{\partial x}\right]=\frac{\partial u}{\partial t} \quad . \tag{5.1}$$

By assuming the solution is of the form

$$u(x,t)=v(x)z(t) \tag{5.2}$$

and applying the method of separation of variables, we were able to reduce the problem to one that resembled Legendre's differential equation as shown in Chapter II. Solutions were then computed by substitution into well-known solutions for Legendre's equation.

Furthermore, it was shown in Chapter III, that an infinite number of solutions could be generated for (5.1) when $w(x)=1-x^2$ by using the method of reduction of order. In this particular case, the additional solutions were given by the formula

$$v_{2,n}=P_n(x)q_n(x) \tag{5.3}$$

where

$$q_n(x)=\int\frac{dx}{(1-x^2)P_n^2(x)} \tag{5.4}$$

and $P_n(x)$ are the Legendre polynomials. Several solutions were calculated using this formula, and it was noted that successive solutions were combinations of preceding solutions. Perhaps further investigation would yield a simple algorithm which would allow one to forego the prohibitive integrations necessary to compute additional solutions. I feel the best step in this direction could be made with careful considerations of inductive arguments.

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