BI-PERSPECTIVE OBSERVABILITY

by

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CHAPTER I

INTRODUCTION

A rigid body is in motion at constant linear and angular velocities. From a distance an individual with normal binocular vision is observing this body. Under what conditions is the individual able to distinguish its position? To answer this question we would like to formulate a mathematical realization for the process of human binocular vision. A realization of this type would be useful in the medical field to indicate and diagnose problems in an individual's eyesight.

An object is seen by an individual with normal binocular vision as long as one eye is able to distinguish the object's position. Both eyes distinguishing the body is not necessary. So when will at least one eye be able to observe the position of the moving body?

This is the question we hope to answer with the formulation of a mathematical binocular vision problem. To formulate this problem and gain some insight into its solution, we will begin in Chapter II by examining the case of monocular vision. If a body in constant motion is observed by a single visual device, can the position of the body be obtained and under what conditions is this possible? This problem was formulated and solved in [1]. The derivation of the monocular vision problem and its solution will be discussed. In Chapter III the binocular vision problem will be derived by the addition of a second visual device. The question will then become, when can the position of the object be distinguished by at least one of these devices? Chapter IV will submit a theorem which provides sufficient conditions for observability of the body using both points of observation. Since in normal binocular vision it is not necessary for both eyes to distinguish an objects position for the object to be seen, in Chapter V an example will be given where the first visual device in no way distinguishes the body and yet together both visual devices will be able to completely determine the its position. Another example will be given where neither device can distinguish the object. These examples will show that although the conditions in the theorem are sufficient to ensure observability of the body, they are not necessary. In Chapter VI we will summarize our findings.

The dynamics of human binocular vision is a complicated process which involves the use of parallel rays and the lenses of the eye to project an inverse image of the object in view onto the retina, located at the back of the eye. In order to devise a mathematical realization for human vision, this process will be simplified. Instead of two parallel rays, consider that each eye will view a single point on the body along a line formed by the point of position of the eye and the observed point on the body. In human vision the point on the body would then be projected by the lenses of the eye onto the retina. Although the retina is a concave surface, we will represent the retina as a plane. In the case of binocular vision the same plane will represent both retini. The projection on the observed point onto the plane is the method which will be used in the derivation of both the monocular and binocular vision problems.

CHAPTER II

BACKGROUND ON MONOCULAR VISION

Consider a rigid body moving in space at constant linear and angular velocities. Suppose a single visual device, such as a human eye, is observing the motion of one particular point on this object. Define an n-dimensional coordinate system so that the visual device is positioned at the origin, and label the point on the body as x_t. The visual device will then observe this point along the line formed by x_t and the origin. This line has an equation $z = \alpha x_t$, where α is a real number (Figure 2.1).

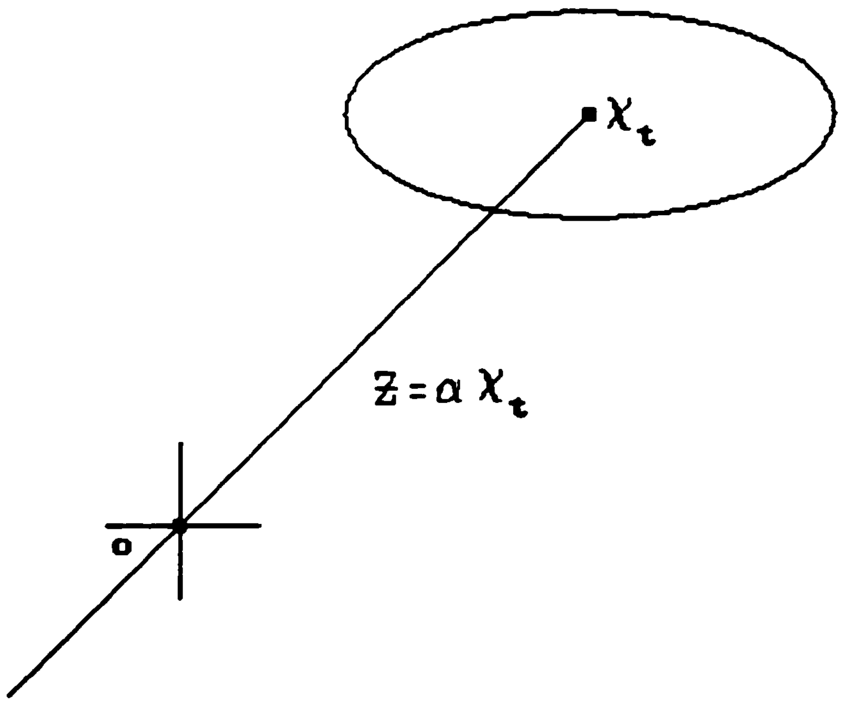

Figure 2.1: Monocular Vision Representation

Let $c_1 z = c_1 b$ be the equation of a plane oriented in the coordinate system so that the line, $z = \alpha x_t$, intersects this plane. Then the point of intersection of the plane and this line is the projection of x_t onto the plane $c_1 z = c_1 b$ (Figure 2.2).

If the linear and angular velocities of the object in motion are known, then we can describe this motion by a dynamical system, $\dot{x} = Ax$, where $x(0) = x_0$,

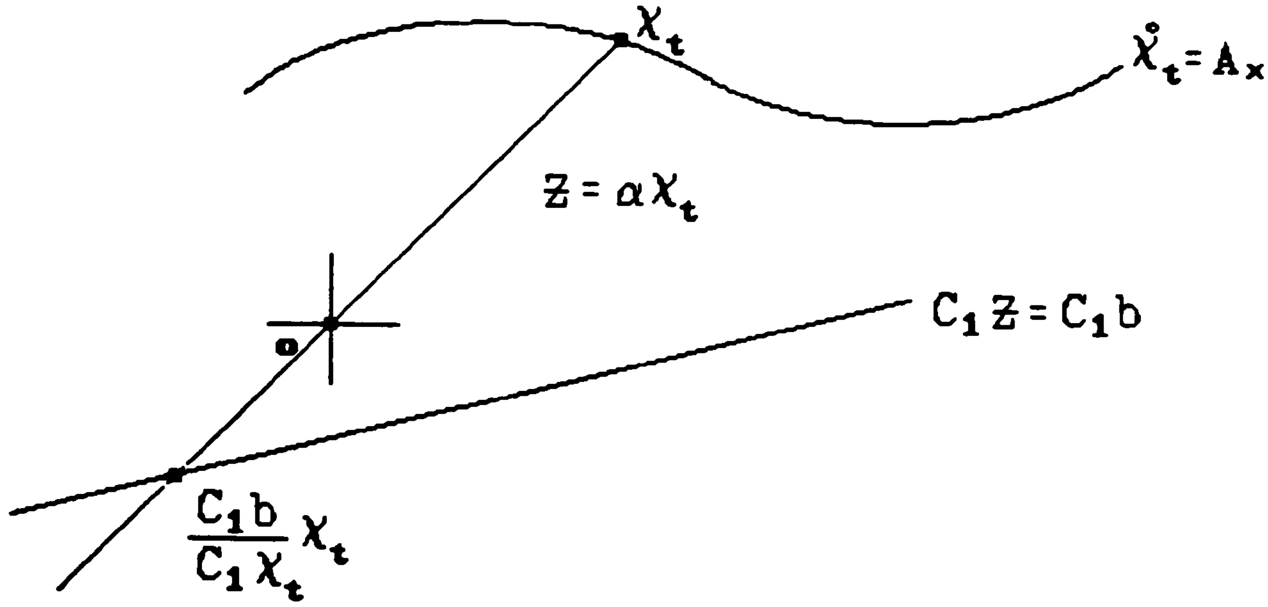

Figure 2.2: Monocular Vision Projection

$x_0 \neq 0$, is the initial position of the point on the body. We specify $x_0 \neq 0$ so that the motion of the body does not originate at the same point where the visual device is positioned. The projection of the point x_t onto the plane defines an observation function of the form $\breve{y}(z) = c_2 z$. Specifically, at the point of intersection of the plane $c_1 z = c_1 b$ and the line $z = \alpha x_t$, we have

$$c_1 \alpha x_t = c_1 b.$$

This implies

$$\alpha = \frac{c_1 b}{c_1 x_t},$$

and the point of intersection is then $c_1 b \left[\frac{x_t}{c_1 x_t}\right]$. The observation function can then be specifically defined as

$$y(x) = c_1 b \left[\frac{c_2 x}{c_1 x}\right].$$

For generality let us assume $c_1 b = 1$. The observed system is then

$$\dot{x} = Ax, \quad x(0) = x_0, \quad x_0 \neq 0 \tag{2.1}$$

$$y_1(x) = \frac{c_2 x}{c_1 x}. \tag{2.2}$$

The dynamical system $\dot{x} = Ax$, $x(0) = x_0$ has the solution $x(t) = e^{At}x_0$. Our question then becomes, if the observation function $y_1(x)$ is known for all $t \geq 0$, can we recover the initial position of the body in motion?

This is a question of system observability. Given a dynamical system with an observation function, where the observation function is known for all $t \geq 0$, when is the system observable? In other words, when can the initial data, $x(0) = x_0$ be recovered?

Consider a system

$$\dot{x}_i = Ax_i,$$

with an observation function

$$y = h(x_i),$$

and initial data

$$x_1(0) = x_0, \quad x_2(0) = x_1.$$

If we define the mappings O_0 and O_1 such that

$$O_0 : x_0 \rightarrow y_0(t) \text{ and } O_1 : x_1 \rightarrow y_1(t),$$

then

$$y_0(t) = h(e^{At}x_0) \text{ and } y_1(t) = h(e^{At}x_0).$$

By definition this system is observable if and only if

$$y_0(t) \equiv y_1(t) \text{ implies } x_0 \equiv x_1.$$

Applying this definition, the system

$$\dot{x} = Ax, \quad x(0) = x_0, \quad x_0 \neq 0$$

$$y_1(x) = \frac{c_2 x}{c_1 x},$$

is not observable. This can be easily shown by considering initial data $x_0 = a$ and $x_1 = \alpha a$, where α is a real number.

From the initial data the observation functions y_0 and y_1 become

$$y_0(t) = \frac{c_1 e^{At} a}{c_2 e^{At} a} \text{ and } y_1(t) = \frac{c_1 e^{At} \alpha a}{c_2 e^{At} \alpha a}.$$

Since α in the equation for $y_1(t)$ is a scalar quantity and cancels, we have $y_0(t) = y_1(t)$ for all values of t. This system is therefore not observable since $x_0(t) \neq x_1(t)$ for all values of α. Notice from our choice of initial data that the observation function $y_1(x)$ does not distinguish points on the ray $z = \alpha a$. This leads us to another question. Is the system observable up to rays? In other words, can we distinguish the ray through the origin on which the initial data lies? This is the concept of perspective observability. A system is perspectively observable provided it is observable up to rays.

A generalization of the perspective observability of the system (2.1), (2.2) was considered and solved in [1]. Let F be a field of real or complex numbers, and let A and C be nxn and mxn matrices, respectively, with $2 \leq m \leq n$. Consider the dynamical system

$$\dot{x} = Ax, \quad x(0) = x_0, \tag{2.3}$$

together with the observation function

$$Z : F^n - B \to FP^{m-1}, \tag{2.4}$$

where Z is given by

$$x \to [Cx].$$

Here $B = \{x \in F^n : Cx = 0\}$, and $[Cx]$ represents the homogeneous coordinates of Cx as an element of FP^{m-1}, the $m - 1$ dimensional projective space of all homogeneous lines in F^m. If $[Cx(t)]$ is known for all $t \geq 0$, we have the following theorem.

Theorem 1 *The dynamical system (2.3), (2.4) is perspectively observable over the field of complex numbers if and only if*

$$\text{Rank} \begin{bmatrix} (A - sI)(A - wI) \\ C \end{bmatrix} = n,$$

for every pair of eigenvalues s, w of A. If the set of all real numbers is the field then these conditions are sufficient for perspective observability.

If we return to the system

$$\dot{x} = Ax; \;\; x(0) = x_0, \;\; x_0 \neq 0$$

with the observation function

$$y_1(x) = \frac{c_2 x}{c_1 x},$$

the above theorem tells us that this system is perspectively observable provided

$$\text{rank} \begin{bmatrix} (A - sI)(A - wI) \\ c_1 \\ c_2 \end{bmatrix} = n.$$

If we examine the observation function

$$y_1(x) = \frac{c_2 x}{c_1 x},$$

a question which may occur is what if $c_2 x = 0$ and $c_1 x = 0$? Utilizing our initial data this implies $c_2 e^{At} x_0 = c_1 e^{At} x_0 = 0$ for all values of t. If this occurs then by taking n derivatives and letting $t = 0$ these equations become $c_2 A^n x_0 = c_1 A^n x_0 = 0$. Let $M = \begin{bmatrix} (A - sI)(\bar{A} - wI) \\ c_1 \\ c_2 \end{bmatrix}$ and consider the set

$$\{x_0, Ax_0, A^2 x_0, \ldots\}.$$

There exists an eigenvector $y \in \text{span}\{x_0, Ax_0, A^2 x_0, \ldots\}$,

$$y = \sum_{i=1}^{n} \alpha_i A^i x_0,$$

and an eigenvalue w_0 such that $Ay = w_0 y$. Since $c_1 x_0 = c_2 x_0 = 0$, we have $My = 0$. This implies the rank condition fails and by theorem 1 the system is not perspectively observable. Geometrically, the intersection of the null spaces of $c_1 x_0$ and $c_2 x_0$ is a subspace of R^n or C^n with dimension 2. The trajectory of the moving body may pass through this 2-dimensional subspace but it will never remain. If it does remain the rank condition will fail, and the system will not be perspectively observable.

The perspective observability problem was originally formulated by problems in the area of computer vision. A rigid body moving at constant angular and linear velocity was observed by a system of lenses, perhaps a camera. Rather than a single point, a particular feature on the object is projected by the system of lenses onto a screen. This feature could be a single point, a line or perhaps a polynomial curve of fixed degree. This projected feature is parameterized as points on some feature space which has the structure of a manifold. The motion of the object is described by the dynamical system and the observation function is defined on the manifold by the projection of the feature onto the screen. The problem then becomes recovering the initial position of the object provided the observation function is known for all $t \geq 0$. The result of [1] was to give sufficient conditions for perspective observability of the initial data. (Ref. [1], [2], [3])

CHAPTER III
BINOCULAR VISION

In Chapter II a rigid body in motion at known constant velocities was observed by a single visual device positioned at the origin of a defined n-dimensional coordinate system. The observed point on the body, x_t, was then projected along a line through the origin onto a plane. This projection was used to define a single observation function. The motion of the body was described by the dynamical system $\dot{x} = Ax$, $x(0) = x_0$, $x_0 \neq 0$. Theorem 1 then gave conditions for perspective observability of the initial position of the body given the observation function was known at all points in time.

Suppose we add a second visual device positioned at a point a in the coordinate system. The observed point x_t is now observed by two visual devices (Figure 3.1).

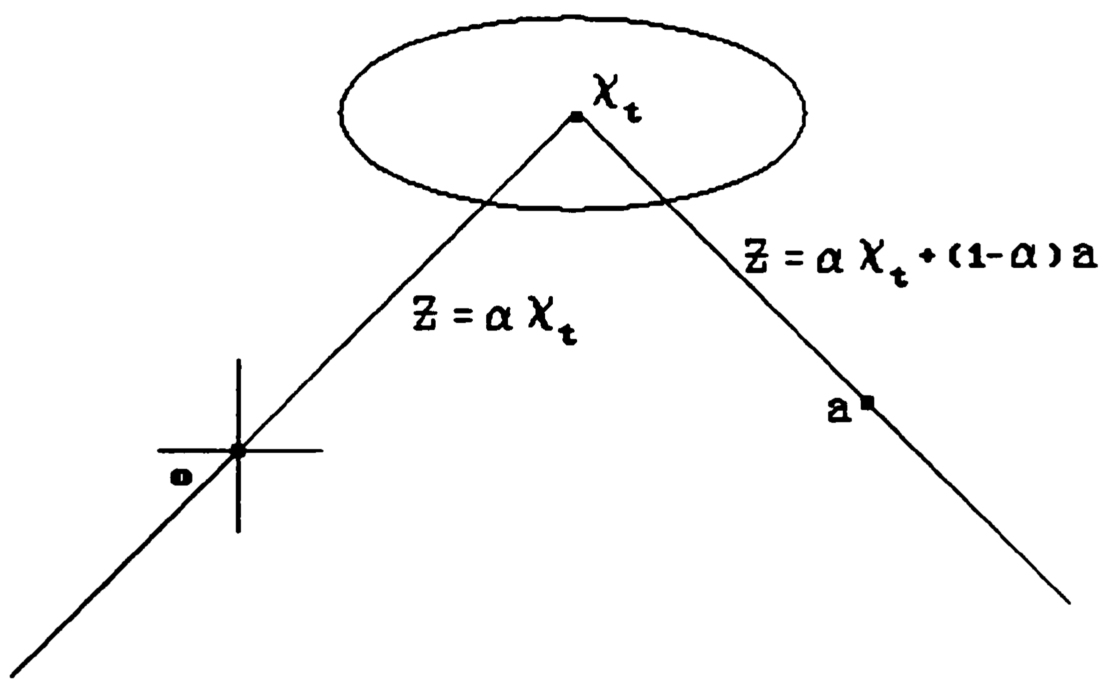

Figure 3.1: Binocular Vision Representation

The visual device positioned at the origin will observe the point x_t along the line $z = \alpha x_t$, where α is a real number. The second device will observe x_t along the line joining x_t and a, which gives this line the equation $z = \alpha x_t + (1 - \alpha)a$.

As in the monocular vision problem position the plane $c_1 z = c_1 b$ in the coordinate system so that the lines $z = \alpha x_t$ and $z = \alpha x_t + (1 - \alpha)a$ intersect the plane (Figure 3.2).

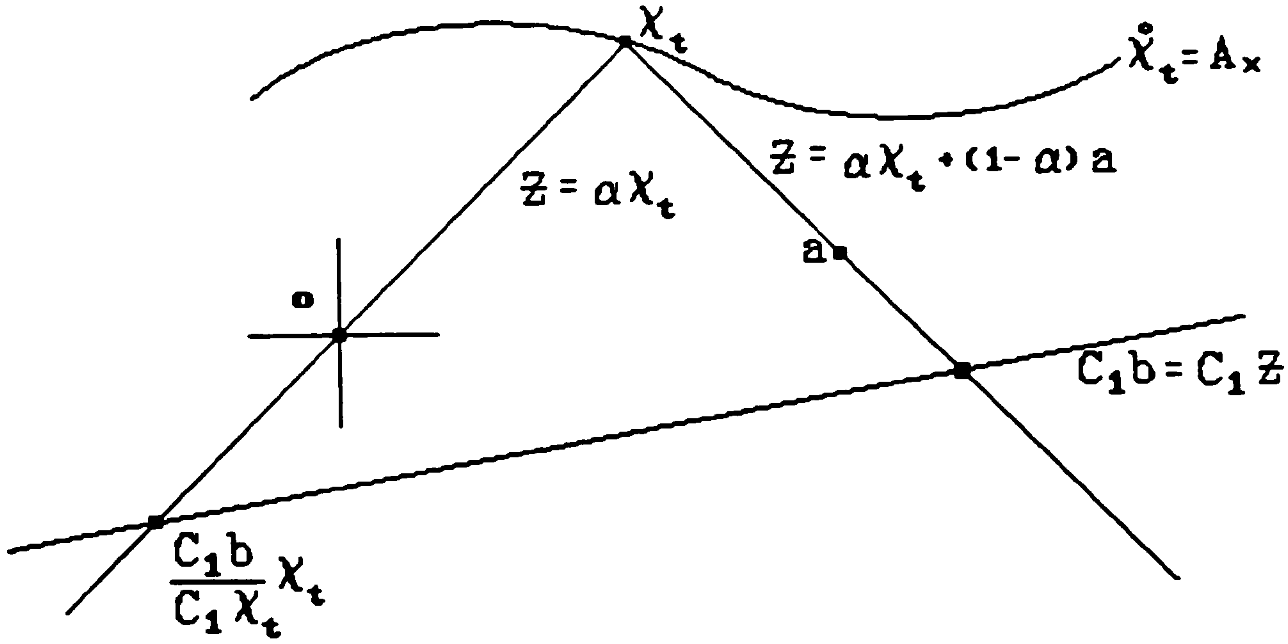

Figure 3.2: Binocular Vision and Plane Position

The point x_t is now projected onto the plane as two points rather than one in the case of monocular vision. These two projections will define two observation functions of the form

$$y(z) = c_2 z.$$

From the derivation of the monocular vision problem the projection of the point x_t through the origin will define the same observation function as in the monocular vision problem. This observation function is

$$y_1(x) = \frac{c_2 x}{c_1 x}.$$

Recall the system $\dot{x} = Ax$, $x(0) = x_0$, $x_0 \neq 0$ together with the observation function $y_1(x)$ is not observable, but can be perspectively observable provided the rank condition stated in theorem 1 holds.

When the line through the point a intersects the plane we have

$$c_1\alpha(x_t - a) + c_1 a = c_1 b,$$

which implies

$$\alpha = \frac{c_1(b-a)}{c_1(x_t - a)}.$$

This intersection point is then found to be

$$c_1(b-a)\left[\frac{x_t - a}{c_1(x_t - a)}\right] + a \text{ (Figure 3.3).}$$

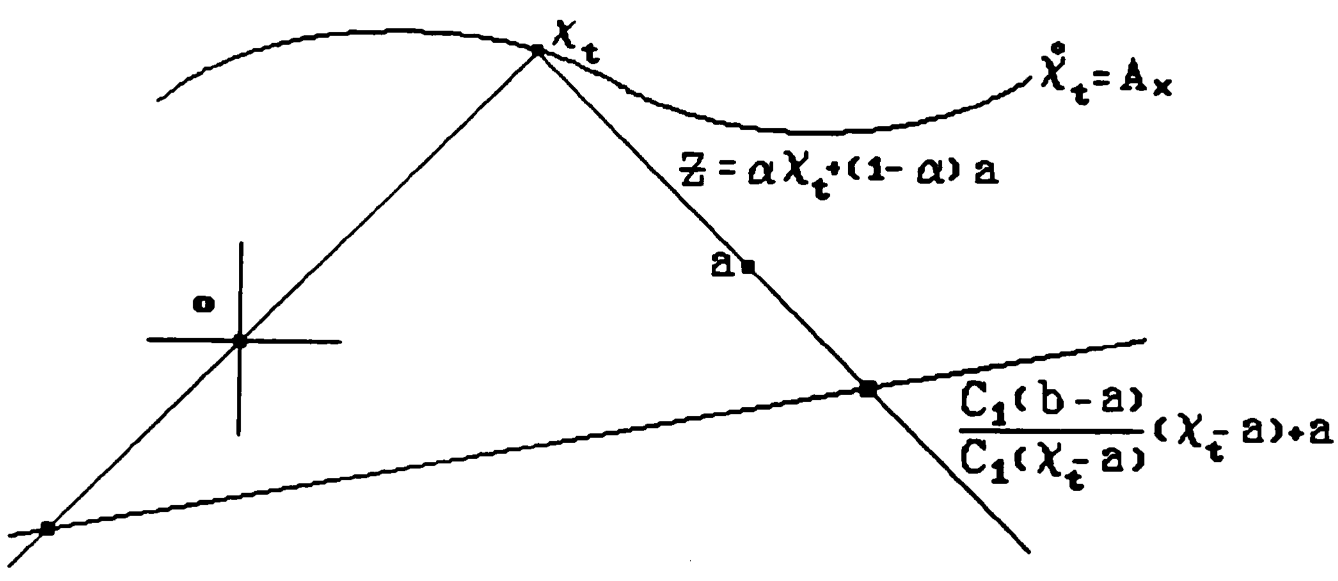

Figure 3.3: Binocular Vision Projection

A second observation function can then be defined as

$$y_2(x) = c_1(b-a)\left[\frac{c_2(x-a)}{c_1(x-a)}\right] + c_2 a.$$

The problem we would like to solve is given the dynamical system

$$\dot{x} = Ax, \quad x(0) = x_0,$$

together with the observation function

$$Y(x) = (y_1(x), y_2(x)),$$

under what conditions is this system observable? We know the dynamical system with the observation function $y_1(x)$ is not observable, but can be perspectively observable. What can be said about the observability system $\dot{x} = Ax$, $x(0) = x_0$, $x_0 \neq 0$ if we use the observation function $Y(x)$?

CHAPTER IV
MAIN THEOREM

In Chapter III we derived the following binocular vision problem. A rigid body is in motion at known constant linear and angular velocities. Two visual devices which are positioned apart are observing a single point on this body. We would like to find conditions which ensure that the position of the body is distinguishable by at least one of the visual devices.

We obtained a mathematical representation for this system by first describing the motion of the body by a dynamical system $\dot{x} = Ax$, $x(0) \neq 0$. The two visual devices were positioned at the origin and at a point a of an n-dimensional coordinate system. Two observation functions were then defined by projecting the point on the body as two separate points, first through the origin and then through the point a. These projections have been obtained since the initial movement of the body, so that both observation functions are known for all points in time. Since from the solution of the dynamical system we have $x(t) = e^{At}x_0$, can the initial position of the point on the body be recovered from the information obtained by these observation functions?

Specifically, the motion of the body is described by the dynamical system

$$\dot{x} = Ax, \quad x(0) = x_0, \quad x_0 \neq 0$$

and we have defined an observation function Y with two components.

$$Y(x) = (y_1(x), y_2(x)), \text{ where}$$

$$y_1(x) = \frac{c_2 x}{c_1 x} \text{ and } y_2(x) = c_1(b-a)\left[\frac{c_2(x-a)}{c_1(x-a)}\right] + c_2 a.$$

We know from the solution of the monocular vision problem stated by theorem 1 that if the rank condition specified in the theorem holds then the observation function defined for the visual device positioned at the origin, $y_1(x)$, can distinguish the ray through the origin on which the initial position of the object lies. If we assume this rank condition holds then the following theorem will give the condition under which the precise initial position of the point on the body can be determined using both observation functions.

Theorem 2 *Consider the dynamical system*

$$\dot{x} = Ax, \tag{4.1}$$

with initial data

$$x(0) = x_0, \quad x_0 \neq 0$$

together with the observation function

$$Y(x) = (y_1(x), y_2(x)), \tag{4.2}$$

$$y_1(x) = \frac{c_2 x}{c_1 x} \text{ and } y_2(x) = c_1(b-a)\left[\frac{c_2(x-a)}{c_1(x-a)}\right] + c_2 a,$$

where $x \in F^n$, *the field of real or complex numbers. Let the rank* $\begin{bmatrix} (A-sI)(A-wI) \\ c_1 \\ c_2 \end{bmatrix} = n$ *for every pair* s, w *of eigenvalues of* A. *The system is not observable if and only if there exists an eigenvector,* x_0, *of* A *such that*

$$\frac{c_1 x_0}{c_2 x_0} = \frac{c_1 a}{c_2 a}.$$

PROOF

Assume the rank condition above holds, and let x_0 and αx_0 be two initial points of the dynamical system such that $x_0 \neq 0$. Then from the solution of the differential equation we have

$$x(t) = e^{At}x_0 \text{ and } x_\alpha(t) = \alpha e^{At}x_0.$$

The observation functions are now

$$y_2(t) = c_1(b-a)\left[\frac{c_2 e^{At}x_0 - a)}{c_1(e^{At}x_0 - a)}\right] + c_2 a,$$

and

$$y_{2\alpha}(t) = c_1(b-a)\left[\frac{c_2(\alpha e^{At}x_0 - a)}{c_1(\alpha e^{At}x_0 - a)}\right] + c_2 a.$$

PART 1

First, we will assume $y_{2\alpha}(t) = y_2(t)$, then

$$\frac{c_2(\alpha e^{At}x_0 - a)}{c_1(\alpha e^{At}x_0 - a)} = \frac{c_2(e^{At}x_0 - a)}{c_1(e^{At}x_0 - a)}.$$

If $c_1(e^{At}x_0 - a) = 0$, then $c_1(\alpha e^{At}x_0 - a) = 0$, or vice versa. So $c_1 e^{At}x_0 - c_1 a = \alpha c_1 e^{At}x_0 - c_1 a$, which implies

$$x_0 = \alpha x_0 \text{ or } \alpha = 1.$$

But then $y_2(t) = y_{2\alpha}(t) \Rightarrow x_0 = \alpha x_0$ and the system is therefore observable. If neither $c_1(e^{At}x_0 - a) = 0$ nor $c_1(\alpha e^{At}x_0 - a) = 0$ then we can cross multiply and

$$(\alpha c_2 e^{At}x_0 - c_2 a)(c_1 e^{At}x_0 - c_1 a) = (\alpha c_1 d^{At}x_0 - c_1 a)(c_2 e^{At}x_0 - c_2 a),$$

which implies

$$(\alpha - 1)c_1 a c_2 e^{At}x_0 = (\alpha - 1)c_2 a c_1 e^{At}x_0.$$

We then have

$$\frac{c_2 a}{c_1 a} = \frac{c_2 e^{At}x_0}{c_1 e^{At}x_0}.$$

Since $\frac{c_2 a}{c_1 a}$ is constant, let

$$\frac{c_2 a}{c_1 a} = \frac{c_2 e^{At}x_0}{c_1 e^{At}x_0} = k. \tag{4.3}$$

So when is $\frac{c_2 e^{At}x_0}{c_1 e^{At}x_0} = k = \frac{c_2 x_0}{c_1 x_0}$?
$\frac{c_2 e^{At}x_0}{c_1 e^{At}x_0} = k$ implies $c_2 e^{At}x_0 - k c_1 e^{At}x_0 = 0$, then

$$\frac{d}{dt}(c_2 e^{At}x_0 - k c_1 e^{At}x_0) = 0.$$

$$c_2 e^{At}x_0 - k c_1 e^{At}Ax_0 = 0.$$

This implies

$$\frac{c_2 e^{At}Ax_0}{c_1 e^{At}Ax_0} = k = \frac{c_2 e^{At}x_0}{c_1 e^{At}x_0}.$$

Since the rank of $\begin{bmatrix} A \\ c_1 \\ c_2 \end{bmatrix} = n$, we see that

$$Ax_0 = wx_0, \text{ where } w \text{ is a constant.}$$

This implies x_0 is an eigenvector of A, and by equation (4.3) above

$$\frac{c_2 a}{c_1 a} = \frac{c_2 e^{At}x_0}{c_1 e^{At}x_0} = \frac{c_2 x_0}{c_1 x_0}.$$

So we have if the system (4.1) (4.2) is not observable, then x_0 is an eigenvector of the matrix A, and $\frac{c_2 a}{c_1 a} = \frac{c_2 x_0}{c_1 x_0}$.

PART 2

Once again assuming the rank condition holds, let x_0 be an eigenvector of the matrix A with w as its corresponding eigenvalue, and suppose

$$\frac{c_1 x_0}{c_2 x_0} = \frac{c_2 a}{c_1 a}.$$

This implies there exists a constant $k = \frac{c_2 a}{c_1 a}$, and by part 1 eq. (4.3)

$$k = \frac{c_2 a}{c_1 a} = \frac{c_2 e^{At} x_0}{c_1 e^{At} x_0} = \frac{c_2 x_0}{c_1 x_0}.$$

Since

$$\frac{c_2 a}{c_1 a} = \frac{c_2 e^{At} x_0}{c_1 e^{At} x_0},$$

then

$$(\alpha - 1) c_1 a c_2 e^{At} x_0 = (\alpha - 1) c_2 a c_1 e^{At} x_0,$$

which implies the following.

$$c_2 a c_1 e^{At} x_0 + \alpha c_1 a c_2 e^{At} x_0 = c_1 a c_2 e^{At} x_0 + \alpha c_2 a c_1 e^{At} x_0$$

$$(\alpha c_2 e^{At} x_0 - c_2 a)(c_1 e^{At} x_0 - c_1 a) = (\alpha c_1 e^{At} x_0 - c_1 a)(c_2 e^{At} x_0 - c_2 a)$$

$$\frac{c_2(\alpha e^{At} x_0 - a)}{c_1(\alpha e^{At} x_0 - a)} = \frac{c_2(e^{At} x_0 - a)}{(c_1 e^{At} x_0 - a)}.$$

We then have $y_{2\alpha}(t) = y_2(t)$, which implies the system (4.1), (4.2) is not observable. This concludes the proof.

But what do the conditions for observability mean from a geometrical standpoint? We can interpret these conditions in terms of the positions of the visual devices and the trajectory of the moving body. If $\frac{c_2 x_0}{c_1 x_0} = \frac{c_2 a}{c_1 a}$, then the initial position of the point on the body, x_0, lies on the ray through the origin and the point a, the two positions of the visual devices. The origin, a, x_0, are all collinear. If x_0 is an eigenvector of A, then the trajectory of the point on the moving body is along the line approaching the eyes. So the system is not observable if the trajectory of the moving body is a ray through both eyes (Figure 4.1).

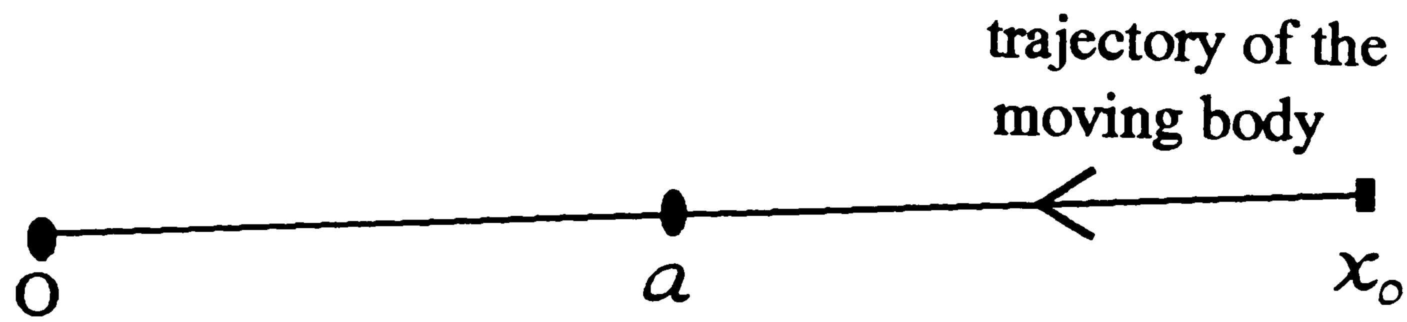

Figure 4.1: Geometric Conditions of Theorem 2

CHAPTER V

EXAMPLES

In Chapter II we considered the situation where a single visual device is observing a rigid body in constant motion. The question was asked, can the position of the body be determined by the visual device? And under what conditions is this possible? The visual device was positioned at the origin of a n-dimensional coordinate system and an observation function $y_1(x)$ was defined by projecting the point on the body through the origin and onto a plane. The motion of the body was described by the dynamical system $\dot{x} = Ax$ with initial data $x(0) = x_0$, $x_0 \neq 0$. The monocular vision problem was then posed. If the observation function $y_1(x)$ is known at all points in time, can the initial position of the point be recovered? The answer to this question was stated in theorem 1. The system was perspectively observable provided a specified rank condition held. That is, the ray on which the initial data was found could be determined, but not the precise coordinates.

In Chapter III another visual device was added to the system. The point on the body was now observed by two visual devices. The question was again asked, can the position of the point on the body be recovered? And under what conditions is this possible for either visual device. The additional visual device was positioned at a point a. The point on the body was then projected through the origin and onto a plane, so that $y_1(x)$ was defined as in the monocular vision problem. The addition came when the point on the body was also projected through the point a onto the plane. This motion of the body continued to be described by the dynamical system $\dot{x} = Ax$, but we then had the observation function $Y(x)$ with two perspective points of view, $Y(x) = (y_1(x), y_2(x))$. The binocular vision problem is then if we have all of the data produced by $Y(x)$ since the initial movement of the body, can the initial position be obtained and under what conditions is this possible? If the initial position of the body can be found then from the solution of the dynamical system the position of the body at any point in time can be determined.

The result of the theorem in Chapter IV was to give a condition for the recovery of the initial position of the point on the body by $Y(x)$, provided the rank condition stated in the theorem for monocular vision was satisfied. In other words, the observation function $Y(x)$ could recover the initial position if the first component,

$y_1(x)$ could determine the ray on which the initial data was found. These conditions are sufficient for recovery of the initial data, but it can be shown that it is not necessary. It is possible for the initial position of the body to be recovered by the observation function $Y(x)$ without perspective observability by $y_1(x)$. It is also possible that the system is completely unobservable using $Y(x)$.

Suppose A is the 3×3 zero matrix and the vectors $c_1 = c_2 = (1, 0, 0)$. The rank condition of theorem 1 clearly fails since the rank of $\begin{bmatrix} A^2 \\ c_1 \\ c_2 \end{bmatrix} = 1$. Let $x(0) = (p_1, p_2, p_3)^T \neq 0$ be the initial position of the observed point on the body, then we have

$$x(t) = e^{At} x_0 = x_0,$$

as the solution of the differential equation. Recall,

$$y_1(x) = \frac{c_2 x}{c_1 x} \text{ and}$$

$$y_2(x) = c_1(b-a)\left[\frac{c_2(x-a)}{c_1(x-a)}\right] + c_2 a,$$

so we have

$$y_1(t) = \frac{p_1}{p_1} = 1 \text{ and } y_2(t) = (b_1 - a_1)\left[\frac{(p_1 - a_1)}{(p_1 - a_1)}\right] + a_1 = b_1.$$

This system is clearly not observable, since the output from the observation functions give us no information which would help to determine the coordinates of $x(0)$. This is the case where the system $\dot{x} = Ax$ together with the observation function $Y(x)$ is completely unobservable, as neither component of $Y(x)$ can recover the initial data.

Next let $A = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix}$, $c_1 = (0,0,1)$, and $c_2 = (1,0,0)$. The rank condition fails, since the rank of $\begin{vmatrix} A^2 \\ c_1 \\ c_2 \end{vmatrix}$ is two.

$$x(t) = e^{At}x_0 = \begin{vmatrix} 1 & t & t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{vmatrix} (p_1, p_2, p_3)^T = \begin{vmatrix} p_1 + p_2 + \left(\frac{p_3t^2}{2}\right) \\ p_2 + p_3 \\ p_3 \end{vmatrix}.$$

So we have

$$y_1(t) = \frac{p_1 + p_2t + \frac{p_3t^2}{2}}{p_3} = \frac{p_1}{p_3} + \frac{p_2t}{p_3} + \frac{t^2}{2},$$

and

$$y_2(t) = \frac{(b_3 - a_3)(p_1 + p_2t + \frac{p_3t^2}{2} - a_1)}{(p_3 - a_3)} - a_1.$$

Let $r_0 = \frac{p_1}{p_3}$, $r_1 = \frac{p_2}{p_3}$, $r_2 = \frac{p_1 - a_1 - a_1(p_3 - a_3)}{(p_3 - a_3)}$, $r_3 = \frac{p_2}{(p_3 - a_3)}$, and $r_4 = \frac{p_3}{2(p_3 - a_3)}$. All of the r_i, for $i = 1, \ldots, 4$, are measure values obtained for the observation functions. We would like to place conditions on the values of a_1, a_2, a_3 which would allow us to determine the coordinates of $x(0)$, p_1, p_2, p_3.

We have the system of equations:

(1) $p_1 = p_3r_0$,

(2) $p_2 = p_3r_1$,

(3) $r_3p_3 - p_2 = r_3a_3$,

(4) $(1 - 2r_4)p_3 = -2a_3r_4$,

(5) $(p_3 - a_3)r_2 = p_1 - a_1$.

Using equations 2 and 3 consider the matrix equation

$$\begin{vmatrix} 1 & -r_1 \\ -1 & r_3 \end{vmatrix} (p_2, p_3)^T = (0, r_3a_3)^T.$$

The equation indicates that p_2 and p_3 can be recovered provided $r_1 \neq r_3$ and $r_1 \neq r_3$ provided $a_3 \neq 0$. If $a_3 \neq 0$ then $r_4 = \frac{p_3}{2(p_3 - a_3)} \neq \frac{1}{2}$. So by equation (4) $p_3 = \frac{-2a_3r_4}{1-2r_4}$, equation (1) $p_1 = p_3r_0 = \frac{-2a_3r_4r_0}{1-2r_4}$, and equation (2) $p_2 = p_3r_1 = \frac{-2a_3r_4r_1}{1-2r_4}$. So p_1, p_2, p_3 are all defined provided $r_4 \neq \frac{1}{2}$. The three coordinates of $x(0)$ are recoverable. The initial position of the point on the body can therefore be determined provided $a_3 \neq 0$. This example illustrates that the observation function $Y(x)$ can completely determine the initial data despite the fact that the system is not perspectively observable by the observation function $y_1(x)$.

The two examples given thus show that the rank condition necessary for perspective observability in the monocular vision problem can lead to a sufficient condition for binocular observability, but it is not necessary.

CHAPTER VI

CONCLUSION

We have been considering two visual devices observing from a distance a rigid body in constant motion. We would like to determine conditions which ensure that the body will be observed by at least one of these devices. In Chapter II the monocular vision problem was discussed. If a single optical device observes the motion of the moving body, when can its initial position be recovered? It was found in [1] that the initial data is perspectively observable, or the initial data is distinguishable only up to the ray on which it lies. The actual initial position cannot be recovered. A second visual device was then added in Chapter III to create a binocular visual effect. We found and proved in Chapter IV that if we assume the first visual device can distinguish the initial position of the body up to rays, then the system will be observable provided the position of the 2 visual devices and the initial position of the point on the body do not all lie on a common ray, with the trajectory of the body moving along this ray toward the visual devices. We therefore have a sufficient condition for observability of the body, but these conditions are not necessary. If the initial position of the body is not perspectively observable by the first visual device, then both visual devices together may or may not be able to distinguish the initial position. Examples were given in Chapter V to illustrate this fact. This realization agrees with the process of normal human binocular vision. An object is seen by an individual as long as one eye is able to observe it.

In order to produce a more precise mathematical realization for the process of human binocular vision necessary and sufficient conditions need to be found which ensure that the body is observable by at least one visual device. Further research is needed to find the complete solution for the binocular vision problem.

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