ON BOUNDED SLOPE VARIATION AND

THE HELLINGER INTEGRAL

by

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CONTENTS

INTRODUCTION

This thesis is primarily concerned with the concept of a function having bounded slope variation with respect to an increasing function and the relationship between the Hellinger integral and the Lebesque integral of these functions.

In 1911, F. Riesz showed that a necessary and sufficient condition that F be the indefinite integral of a function of bounded variation on an interval [a,b] is that F have bounded slope variation with respect to the identity function, i.e. that

$$\sum_{p} \left| \frac{F(x_{i+1})-F(x_i)}{x_{i+1} - x_i} - \frac{F(x_i)-F(x_{i-1})}{x_i - x_{i-1}} \right|$$

be bounded over all partitions p of [a,b].

This concept (which was not given a name) apparently was not further investigated until J. R. Webb in 1967 generalized it to find necessary and sufficient conditions that F be a bounded linear functional on the Banach space of quasi-continuous functions. Webb gave this concept the name of bounded slope variation and was unaware of Riesz's work. The concept was systematically investigated by Huggins starting in 1971 who used it as a fundamental hypothesis for the existence of the Hellinger integral. Huggins also investigated the relationship between uniform Lipschitz conditions and bounded slope variation.

Chapter 1 of this thesis contains a survey of the literature, and Chapter 2 contains new results which (under certain assumptionS) equates the Hellinger integral to a Lebesque Stieltjes integral.

CHAPTER 1

BOUNDED SLOPE VARIATION AND THE HELLINGER INTEGRAL

Definition 1. f has bounded slope variation with respect to m on [a,b] means that m is strictly increasing and that there is a $B > 0$ such that if $p = \{t_0, t_1, \ldots, t_n\}$ is a partition of [a,b] with $n > 0$, then

$$\sum_p \left| \frac{f_{i+1}-f_i}{m_{i+1}-m_i} - \frac{f_i-f_{i-1}}{m_i-m_{i-1}} \right| \leq B \quad \left[\text{where } \begin{matrix} f_i = f(t_i) \\ m_i = m(t_i) \end{matrix}\right]$$

The smallest bound that will suffice is denoted by $V_a^b \frac{df}{dm}$.

Remark 1. The above sum is non-decreasing under refinement.

Definition 2. If m is strictly increasing, then

$$D_m^+ f(c) = \lim_{t \to c^+} \frac{f(t)-f(c)}{m(t)-m(c)}, \text{ where } c \in [a,b).$$

Similarly we define $D_m^- f(c)$ and $D_m f(c)$.

Theorem 1. If f has bounded slope variation with respect to m then $D_m^+ f(t)$ exists for $t \in [a,b)$, and $D_m f(t)$ exists except on a countable set E.

Lemma:

$$\sum_{p=1}^{n-1} \frac{k_{p+1}-k_p}{e_{p+1}} - \frac{k_p-k_{p-1}}{e_p} \geq \sum_{p=1}^{n-2} \frac{k_{p+1}-k_0}{\sum_{q=1}^{p+1}} - \frac{k_p-k_0}{\sum_{q=1}^{p} e_q}$$

$$+ \frac{1}{e_n} \left(\sum_{q=1}^{n} e_q\right) \left| \frac{k_n - k_0}{\sum\limits_{q=1}^{n} e_q} - \frac{k_{n-1} - k_0}{\sum\limits_{q=1}^{n-1} e_q} \right|$$

where $n \geq 2$ and $k_0, k_1, \ldots, k_n$ is a sequence of numbers and $e_1, e_2, \ldots, e_n$ is a sequence of positive real numbers.

For the case $n = 3$,

$$\left|\frac{k_3-k_2}{e_3} - \frac{k_2-k_1}{e_2}\right| + \left|\frac{k_2-k_1}{e_2} - \frac{k_1-k_0}{e_1}\right|$$

$$= \left|\frac{k_3-k_2}{e_3} - \frac{k_2-k_1}{e_2}\right| + \frac{e_1}{e_2}\left|\frac{k_2-k_0}{e_1+e_2} - \frac{k_1-k_0}{e_1}\right|$$

$$+ \left|\frac{k_2-k_0}{e_1+e_2} - \frac{k_1-k_0}{e_1}\right|$$

$$\geq \left|\frac{k_3-k_0}{e_3} - \frac{(k_2-k_0)(e_1+e_2+e_3)}{e_3(e_1+e_2)}\right| + \left|\frac{k_2-k_0}{e_1+e_2} - \frac{k_1-k_0}{e_1}\right|$$

Thus it may be seen that the conclusion is true for this case.

For the final step in the induction we begin by noting that

$$\sum_{p=1}^{n} \left|\frac{k_{p+1}-k_p}{e_{p+1}} - \frac{k_p-k_{p-1}}{e_p}\right| \geq \sum_{p=1}^{n-1} \left|\frac{k_{p+1}-k_0}{\sum\limits_{q=1}^{p+1} e_q} - \frac{k_p-k_0}{\sum\limits_{q=1}^{p} e_q}\right|$$

$$+ An, \text{ where } An = \frac{1}{e_{n+1}}\left(\sum_{q=1}^{n+1} e_q\right)\left|\frac{k_{n+1}-k_0}{\sum\limits_{q=1}^{n+1} e_q} - \frac{k_n-k_0}{\sum\limits_{q=1}^{n} e_q}\right|$$

i.e.

$$\left|\frac{k_{n+1}-k_n}{e_{n+1}} - \frac{k_n-k_{n-1}}{e_n}\right| \geq \left(1-\frac{1}{e_n}\sum_{q=1}^{n} e_q\right)\left|\frac{k_n-k_0}{\sum_{q=1}^{n} e_q} - \frac{k_{n-1}-k_0}{\sum_{q=1}^{n-1} e_q}\right|$$

$$+ An = -\frac{1}{e_n}\left(\sum_{q=1}^{n-1} e_q\right)\left|\frac{k_n-k_0}{\sum_{q=1}^{n} e_q} - \frac{k_{n-1}-k_0}{\sum_{q=1}^{n-1} e_q}\right| + An$$

but

$$\left|\frac{k_{n+1}-k_n}{e_{n+1}} - \frac{k_n-k_{n-1}}{e_n}\right| + \frac{1}{e_n}\left(\sum_{q=1}^{n-1} e_q\right)\left|\frac{k_n-k_0}{\sum_{q=1}^{n} e_q} - \frac{k_{n-1}-k_0}{\sum_{q=1}^{n-1} e_q}\right|$$

$$\geq \left|\frac{(k_{n+1}-k_0)-(k_n-k_0)}{e_{n+1}} - \frac{(k_n-k_0)-(k_{n-1}-k_0)}{e_n}\right.$$

$$\left. + \frac{(k_n-k_0)\sum_{q=1}^{n-1} e_q}{e_n \sum_{q=1}^{n} e_q} - \frac{k_{n-1}-k_0}{e_n}\right|$$

$$= \left|\frac{k_{n+1}-k_0}{e_{n+1}} - \frac{(k_n-k_0)\sum_{q=1}^{n+1} e_q}{e_{n+1}\sum_{q=1}^{n} e_q}\right| = An$$

Thus each of the inequalities is true. Now, if $s_0, s_1, \ldots, s_n$ is an increasing real number sequence, and $e_p = s_p - s_{p-1}$, then

$$\sum_{p=1}^{n-1}\left|\frac{k_{p+1}-k_p}{s_{p+1}-s_p} - \frac{k_p-k_{p-1}}{s_p-s_{p-1}}\right| \geq \frac{s_n-s_0}{s_n-s_{n-1}}\sum_{p=1}^{n-1}\left|\frac{k_{p+1}-k_0}{s_{p+1}-s_0} - \frac{k_p-k_0}{s_p-s_0}\right|$$

$$\geq \sum_{p=1}^{n-1} \left| \frac{k_{p+1}-k_0}{s_{p+1}-s_0} - \frac{k_p-k_0}{s_p-s_0} \right| .$$

Proof of theorem: Suppose c is in [a,b) and $\lim_{t\to c^+} (f(t)-f(c))/(m(t)-m(c)$ does not exist. Then there exists a positive number ε such that if r is in (c,b) then there exists a number s in (c,r) for which

$$\left| \frac{f(r)-f(c)}{m(r)-m(c)} - \frac{f(s)-f(c)}{m(s)-m(c)} \right| \geq \varepsilon.$$

Then if n is an integer greater than 2 there exists an increasing number sequence $s_0, s_1, \ldots, s_n$ with $s_0=c$ and each term in [c,b], such that

$$\sum_{p=1}^{n-1} \left| \frac{f(s_{p+1})-f(c)}{m(s_{p+1})-m(c)} - \frac{f(s_p)-f(c)}{m(s_p)-m(c)} \right| \geq (n-1)\varepsilon.$$

But from this inequality and the inequality we got by induction, it follows that

$$\sum_{p=1}^{n-1} \left| \frac{f(s_{p+1})-f(s_p)}{m(s_{p+1})-m(s_p)} - \frac{f(s_p)-f(s_{p-1})}{m(s_p)-m(s_{p-1})} \right| \geq (n-1)\varepsilon.$$

Now clearly there is an n for which $(n-1)\varepsilon > V_a^b \frac{df}{dm}$, which is a contradiction. Hence $D_m^+ f(c)$ exists for each c in [a,b). Similarly $D_m^- f(c)$ exists for each c in (a,b].

Let E_1 denote the set of all numbers x in [a,b] such that $D_m^- f(x) < D_m^+ f(x)$ and E_2 denote the set of all number x in [a,b] such that $D_m^- f(x) > D_m^+ f(x)$. Let the rational numbers be arranged in a sequence $r_1, r_2, r_3, \ldots$ Then if c is a number in E_1, there is a smallest positive integer k such that

$$D_m^- f(c) < r_k < D_m^+ f(c).$$

There is a smallest positive integer h such that $r_h < c$ and

$$\frac{f(x)-f(c)}{m(x)-m(c)} < r_k$$

for $r_h < x < c$ and a smallest positive integer n such that $r_n > c$ and

$$\frac{f(x)-f(c)}{m(x)-m(c)} > r_k$$

for $c < x < r_n$. These two inequalities together give

$$(1) \qquad f(x)-f(c) > r_k[m(x)-m(c)]$$

for $r_h < x < r_n$, $x \neq c$. Thus to every c in E_1 there corresponds a unique triad (h,k,n) of positive integers. Suppose some two numbers x_1 and x_2 of E_1 correspond to the same triad (h,k,n). Then $r_h < x_1 < r_n$ and $r_h < x_2 < r_n$ and $x \neq c$. On putting $c = x_1$ and $x = x_2$ in (1), we have

$$f(x_2)-f(x_1) > r_k[m(x_2)-m(x_1)]$$

or

$$f(x_2)-f(x_1) < r_k[m(x_1)-m(x_2)].$$

This involves a contradiction. Therefore no two numbers of E_1 correspond to the same triad. Since the set of triads of positive integers is countable, it follows that E_1 is countable. A similar argument will show that E_2 is countable. Therefore $E = E_1 \cup E_2$ is countable.

Definition 3. If m is strictly increasing, then the Hellinger integral, $\int_a^b \frac{dgdf}{dm}$, is defined as the limit under refinement of

$$\sum_p \frac{(g_i-g_{i-1})(f_i-f_{i-1})}{m_i-m_{i-1}} .$$

(cf [9])

Remark 2. If f has bounded slope variation with respect to m and s is a step function, then $\int_a^b \frac{dsdf}{dm}$ exists. (cf [9])

Definition 4. f is said to be quasi-continuous on [a,b] if f has a left and right hand limit at each point of the interior of [a,b], and has a limit (from the interior) at the end points.

Theorem 2. If x is quasi-continuous and f has bounded slope variation with respect to m on [a,b], then

$\int_a^b \frac{dxdf}{dm}$ exists. (cf [9])

The main result that Webb proved is the following theorem characterizing the space of bounded linear functionals on the Banach space of quasi-continuous functions.

Theorem 3. If F is a bounded linear functional on the Banach space of quasi-continuous functions on $[a,b]$ (with supremam norm), then there exist two functions f and m with f having bounded slope variation with respect to m such that

$$F(x) = \int_a^b \frac{dxdf}{dm}$$

for each quasi-continuous function x.

Huggins was aware of Webb's and Riesz's results and in [2], [3], [4] and [5] he systematically investigated the concept of bounded slope variation. Some of his results are listed below.

Theorem 4. If f has bounded slope variation with respect to m on $[a,b]$, and m is continuous on the right (left) at a point c, then f is continuous on the right (left) at c.

Proof. Let $\varepsilon > 0$ and c is a number in $[a,b)$. Then $D_m^+ f(c)$ exists. Therefore there exists $\delta_1 > 0$ such that if $c < x < c + \delta_1$, then

$$\left|\frac{f(x)-f(c)}{m(x)-m(c)} - D_m^+ f(c)\right| < |$$

from which it follows that

$$|f(x)-f(c)| < [|D_m^+ f(c)| + 1]|m(x)-m(c)| \;.$$

Since m is continuous on the right at $(c,m(c))$, there exists $\delta_2 > 0$ such that if $c < x < c + \delta_2$, then

$$|m(x)-m(c)| < \varepsilon/[|D_m^+ f(c)| + 1].$$

Let $\delta = \min.[\delta_1, \delta_2]$. Then if $c < x < c + \delta$,

$$|f(x)-f(c)| < [|D^+_m f(c)| + 1]|m(x)-m(c)|$$
$$< [|D_m^+ f(c)| + 1]\cdot\varepsilon/[|D^+_m f(c)| + 1]$$
$$= \varepsilon .$$

Therefore f is continuous on the right at $(c,f(c))$.

Theorem 5. If f and g are continuous on [a,b], m is strictly increasing and $D_m f = D_m g$ except on a countable set, then $f(x) = g(x)-g(a) + f(a)$.

Huggins generalized Riesz's result of 1911 with the following result.

Theorem 6. Let m be continuous and strictly increasing on [a,b]. F has bounded slope variation with respect to m on [a,b], if and only if $F(x) = \int_a^x f(t)dm(t)$, where f is of bounded variation on [a,b].

Remark 3. If f has bounded slope variation with respect to m on [a,b] then f is of bounded variation on [a,b], and hence quasi-continuous on [a,b].

Remark 4. If f has bounded slope variation with respect to m on [a,b] and p is a positive integer, then f^p has bounded slope variation with respect to m on [a,b]. This result is not true if p is a rational number.

Example 1. $I^{1/2}$ does not have bounded slope variation with respect to I over [0,1].

$$\sum_{i=1}^{n-1} \left| \frac{\sqrt{x_{i+1}}-\sqrt{x_i}}{x_{i+1}-x_i} - \frac{\sqrt{x_i}-\sqrt{x_{i-1}}}{x_i-x_{i-1}} \right|$$

$$= \sum_{i=1}^{n-1} \left| \frac{\sqrt{x_{i-1}} - \sqrt{x_{i+1}}}{(\sqrt{x_{i+1}} + \sqrt{x_i})(\sqrt{x_i} + \sqrt{x_{i-1}})} \right| \geq \frac{\varepsilon}{(5/2\ \varepsilon)(3/2\ \varepsilon)} = \frac{4}{15}\,\frac{1}{\varepsilon}$$

(where $\sqrt{x_{i-1}} = \varepsilon/2$, $\sqrt{x_{i+1}} = 3\varepsilon/2$ $\sqrt{x_i}=\varepsilon$) is not bounded.

Remark 5. The Hellinger integral $\int_a^b \frac{(df)^2}{dm}$ has already been defined and it will be important to note that the approximating sums

$\sum_p \frac{(f_i-f_{i-1})^2}{(m_i-m_{i-1})}$ are non-decreasing under refinement. Theorem 2 effectively shows that the integral

$\int_a^b \frac{(df)^2}{dm}$ exists when f has bounded slope variation with respect to m.

Proof. To show the approximating sums

$\sum_p \frac{(f_i-f_{i-1})^2}{(m_i-m_{i-1})}$ are non-decreasing under refinement: Let

$p = \{x_0, x_1, x_2, \ldots x_i, x_{i+1}, \ldots, \ldots x_n\}$, $c \in (x_i, x_{i+1})$ and

and $p_0 = p \cup \{c\}$.

If

$$\frac{(f_i-f_{i-1})^2}{m_i-m_{i-1}} \leq \frac{(f_i-f(c))^2}{m_i-m(c)} + \frac{(f(c)-f_{i-1})^2}{m(c)-m_{i-1}}$$

then

$$\sum_{p} \frac{(f_i - f_{i-1})^2}{m_i - m_{i-1}} \leq \sum_{p_0} \frac{(f_i - f_{i-1})^2}{m_i - m_{i-1}} .$$

Let $a = f_i - f(c)$ $b = m(x_i) - m(c)$

$c = f(c) - f_{i-1}$ $d = m(c) - m_{i-1}$ $b, d > 0$

i.e.

$$\frac{(a+b)^2}{c+d} \leq \frac{a^2}{c} + \frac{b^2}{d}$$

but

$$\frac{a^2}{c} + \frac{b^2}{d} - \frac{(a+b)^2}{c+d} = \frac{(ad-bc)^2}{cd(c+d)} \geq 0.$$

Example 2. Let $x_0 = 0$, $x_1 = 1/4$, and for each positive integer $n \geq 2$ let $x_n = 1 - 1/n$. Let f be the function defined, for each x in [0,1], by $f(x_{2n}) = 0$, $f(x_{2n-1}) = 1/n$, $f(0) = f(1) = 0$, and let f be linear in $[x_{n-1}, x_n]$ $F(x) = \int_0^1 f(x)dt$ for each $x \in [0,1]$, then

$\int_0^1 \frac{(dF)^2}{dI}$ exists. But F does not have bounded slope variation with respect to I over [0,1].

Remark 6. The existence of $\int_a^b \frac{(df)^2}{dm}$ does not imply that f has bounded slope variation with respect to m; however, it does imply that if m is continuous on the right (left) then so if f. Also, it implies that f is of bounded variation on [a,b].

Theorem 7. If f has bounded slope variation with respect to m on [a,b], then f is absolutely continuous with respect to m on [a,b].

Proof. f is absolutely continuous with respect to m on [a,b] means that for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $\{[a_i b_i]\}_{i=1}^{n}$ is a collection of non-overlapping subintervals of [a,b] and

$$\sum_{i=1}^{n} |m(b_i)-m(a_i)| < \delta,$$

then

$$\sum_{i=1}^{n} |f(b_i)-f(a_i)| < \varepsilon.$$

Now if $\{[a_i b_i]\}_{i=1}^{h}$ is a collection of non-overlapping subintervals of [a,b] and

$$\sum_{i=1}^{n} |m(b_i)-m(a_i)| < \delta = \frac{\varepsilon}{\int_a^b \left(\frac{df}{dm}\right)^2}$$

for given $\varepsilon > 0$, then

$$\left[\sum_{i=1}^{n} |f(b_i)-f(a_i)|\right]^2 = \left[\sum_{i=1}^{n} \frac{|f(b_i)-f(a_i)|}{(m(b_i)-m(a_i))^{1/2}} (m(b_i)-m(a_i))^{1/2}\right]^2$$

$$\leq \sum_{i=1}^{n} \frac{(f(b_i)-f(a_i))^2}{m(b_i)-m(a_i)} \sum_{i=1}^{n} (m(b_i)-m(a_i))$$

$$\leq \int_a^b \frac{(df)^2}{dm} \sum_{i=1}^{n} (m(b_i)-m(a_i)) < \int_a^b \frac{(df)^2}{dm} \frac{\varepsilon}{\int_a^b \frac{(df)^2}{dm}} = \varepsilon$$

which shows that f is absolutely continuous with respect to m on [a,b] and also yields that

$$V_a^b f \leq [\int_a^b \frac{(df)^2}{dm} (m(b)-m(a)]^{1/2}$$

Remark 7. If m is continuous on the right (left) at c and $D_m f(c)$ exists, then f is continuous on the right (left).

Proof. Since $D_m f(c)$ exists, there exists $\delta_0 > 0$ such that

$$\left|\frac{f(x)-f(c)}{m(x)-m(c)}\right| < |D_m f(c)| + 1 = M$$

for all $x \in (c-\delta_0, c+\delta_0)$.

Since m is continuous on the right at (c, m(c)), for $\varepsilon > 0$, there exists $\delta < \delta_0$ such that $|m(x)-m(c)| < \varepsilon/M$, if $c < x < c + \delta$.

$$|f(x)-f(c)| = \left|\frac{f(x)-f(c)}{m(x)-m(c)}\right| |m(x)-m(c)| < M \frac{\varepsilon}{M} = \varepsilon$$

if $c < x < c + \delta$. That is f is continuous on the right.

Remark 8. If $D_m f(c)$ exists and f has bounded slope variation with respect to m, then it is not necessarily true that f or m have a derivative at c.

Example 3. Let $f = I^{2/3}$, $m = I^{1/3}$, then f has bounded slope variation with respect to m over [-1,1] and $D_m f(0)=0$.

But neither f nor m has a derivative at (0,0).

Theorem 8. (Mean Value Theorem) If f and m are continuous on [a,b], m is strictly increasing and $D_m f$ exists on [a,b], then there is a c in (a,b) such that

$$D_m f(c) = \frac{f(b)-f(a)}{m(b)-m(a)} .$$

Remark 9. If f has bounded slope variation with respect to m on [a,b], then $D_m^+ f$ is of bounded variation on [a,b] and $V_a^b\,(D_m^+ f) \le V_a^b\,\frac{df}{dm}$. Here $D_m^+ f(b)$ is taken as $D_m^- f(b)$.

Proof. Let $\varepsilon > 0$ and P be a partition of [a,b] $p = \{x_0, x_1, \ldots, x_n\}$. For each x_i there exists $t_i\,(x_i < t_i < x_{i+1})$ such that

$$\left|\frac{f(t_i)-f(x_i)}{m(t_i)-m(x_i)} - D_m f(x_i)\right| < \frac{\varepsilon}{2n}$$

Now consider

$$\Sigma\,\left|D_m^+ f(x_i) - Dm^+ f(x_{i-1})\right|$$

$$\le \Sigma\,\left|\frac{f(t_i)-f(x_i)}{m(t_i)-m(x_i)} - \frac{f(t_{i-1})-f(x_{i-1})}{m(t_{i-1})-m(t_{i-1})}\right| + \varepsilon$$

$$\le \Sigma\,\left|\frac{f(t_i)-f(x_i)}{m(t_i)-m(x_i)} - \frac{f(x_i)-f(t_{i-1})}{m(x_i)-m(t_{i-1})}\right| +$$

$$\left|\frac{f(x_i)-f(t_{i-1})}{m(x_i)-m(t_{i-1})} - \frac{f(t_{i-1})-f(x_{i-1})}{m(t_{i-1})-m(x_{i-1})}\right| + \varepsilon$$

$$\le V_a^b\,\frac{df}{dm} + \varepsilon$$

hence $V_a^b\, D_m f$ exists and $V_a^b\, D_m f \le V_a^b \frac{df}{dm}$.

Theorem 9. Let m be continuous and strictly increasing on [a,b], and $D_m f$ exists on [a,b]. Then f has bounded slope variation with respect to m on [a,b], if and only if $D_m f$ is of bounded variation on [a,b]. Moreover,

$$V_a^b \frac{df}{dm} = V_a^b\, (D_m f) .$$

Proof. By Remark 9, we have that if f has bounded slope variation with respect to m on [a,b], then $V_a^b\, D_m f$ exists and $V_a^b\, D_m f \le V_a^b \frac{df}{dm}$.

The converse will be shown true in the following:

Let $p = \{x_0, x_1, \ldots, x_n\}$ be a partition of [a,b] and for $[x_i, x_{i+1}]$ there exist t_i in (x_i, x_{i+1}) such that

$$D_m f(t_i) = \frac{f(x_{i+1}) - f(x_i)}{m(x_{i+1}) - m(x_i)}$$

Now

$$\sum_{i=1}^{n-1} \left| \frac{f(x_{i+1}) - f(x_i)}{m(x_{i+1}) - m(x_i)} - \frac{f(x_i) - f(x_{i-1})}{m(x_i) - m(x_{i-1})} \right|$$

$$= \sum_{i=1}^{n-1} \left| D_m f(t_{i+1}) - D_m f(t_i) \right| \le V_a^b\, D_m f$$

hence $V_a^b \frac{df}{dm}$ exists and $V_a^b \frac{df}{dm} \le V_a^b\, D_m f$. Moreover

$$V_a^b \frac{df}{dm} = V_a^b\, D_m f.$$

Theorem 10. If f has bounded slope variation with respect to m over [a,b], then f satisfies a uniform Lipschitz condition of order 1 with respect to m on [a,b].

Proof. f satisfies a uniform Lipschitz condition of order p with respect to m on [a,b] means that m is increasing on [a,b], p is a positive number, and there exists a $k \geq 0$ such that if x_1 and x_2 are in [a,b], then

$$|f(x_1)-f(x_2) \leq K. \; |m(x_1)-m(x_2)|^p \; .$$

Case 1. If $a = x_1$ and $b = x_2$, then

$$\left|\frac{f(x_2)-f(x_1)}{m(x_2)-m(x_1)}\right| = \left|\frac{f(b)-f(a)}{m(b)-m(a)}\right| = K_1$$

Case 2. If $x_2 \neq b$, then for x_2, there exists x_s such that

$x_2 < x_3 < b$ and

$$\left|\frac{f(b)-f(x_3)}{m(b)-m(x_3)}\right| \leq |D_m^- f(b)| + 1$$

$$V_a^b \frac{df}{dm} \geq \left|\frac{f(x_3)-f(x_2)}{m(x_3)-m(x_2)} - \frac{f(x_2)-f(x_1)}{m(x_2)-f(x_1)}\right| + \left|\frac{f(b)-f(x_3)}{m(b)-m(x_3)} - \frac{f(x_3)-f(x_2)}{m(x_3)-m(x_2)}\right|$$

$$\geq \left|\frac{f(x_2)-f(x_1)}{m(x_2)-f(x_1)}\right| - \left|\frac{f(b)-f(x_3)}{m(b)-m(x_3)}\right|$$

hence $\left|\frac{f(x_2)-f(x_1)}{m(x_2)-f(x_1)}\right| \leq V_a^b \frac{df}{dm} + |D_m^- f(b)| + 1 = K_2$

Case 3. If $x_1 \neq a$, then similarly we can get

$$\left|\frac{f(x_2)-f(x_1)}{m(x_2)-m(x_1)}\right| \leq V_a^b \frac{df}{dm} + |D_m^+ f(a)| + 1 = K_3$$

Let $K = \max\{k_1, k_2, k_3\}$, then

$$|f(x_1)-f(x_2)| = \left|\frac{f(x_2)-f(x_1)}{m(x_2)-m(x_1)}\right| \, |m(x_1)-m(x_2)|$$

$$\leq k \, |m(x_1)-m(x_2)|$$

for any x_1, x_2 in [a,b].

The converse of Theorem 10 is not true. It is also of interest to note that if f satisfies a uniform Lipschitz condition of order ≥ 1 with respect to m on [a,b], then $\int_a^b \frac{(df)^2}{dm}$ exists; however, the existence of this integral does not imply that f satisfies a uniform Lipschitz condition of order ≥ 1 with respect to m on [a,b], but it does imply that f satisfies a uniform Lipschitz condition of order 1/2 with respect to m on [a,b].

CHAPTER 2

THE EQUIVALENCE OF THE HELLINGER INTEGRAL AND THE LEBESQUE INTEGRAL

In attempting to write the Hellinger integral as a Lebesque integral, it is convenient to use the following definition of differentiation of a function with respect to a function. Cohen has some remarks about this derivative in [1]. Throughout this chapter we will suppose m to be strictly increasing and continuous from the right.

Definition 5. If m is strictly increasing, then $d_m f(c)$ exists means that if $\varepsilon > 0$, then there exists $\delta > 0$ such that

$$\left|\frac{f(y)-f(x)}{m(y)-m(x)} - d_m f(c)\right| < \varepsilon \text{ if } c-\delta < x < c < y < c + \delta.$$

Remark 10. If $D_m f(c)$ exists, then $d_m f(c)$ exists and $d_m f(x) = D_m f(x)$.

Proof. We will show that $\left|\frac{f(\beta_n)-f(\alpha_n)}{m(\beta_n)-m(\alpha_n)} - D_m f(x)\right|$ goes to zero as n gets large, where $\alpha_n < x < \beta_n$, $\alpha_n(\beta_n)$ goes to x from right (left).

Let $\lambda_n = \frac{m(\beta_n)-m(x)}{m(\beta_n)-m(\alpha_n)}$, then $0 < \lambda_n < 1$ and

$$\lambda_n\left(\frac{f(\beta_n)-f(x)}{m(\beta_n)-m(x)} - D_m f(x)\right) + (1-\lambda_n)\left(\frac{f(\alpha_n)-f(x)}{m(\alpha_n)-m(x)} - D_m f(x)\right)$$

$$= \frac{f(\beta_n)-f(\alpha_n)}{m(\beta_n)-m(\alpha_n)} - D_m f(x)$$

The left side goes to zero, since $D_m^+ f(x)$, $D_m^- f(x)$ exist and λ_n is bounded, i.e. $d_m f(x)$ exists and $d_m f(x) = D_m f(x)$.

Example 4. Let $f(x) = |x|$, $x \in [-1,1]$ and

$$m(x) = \begin{cases} x & -1 \leq x \leq 0 \\ 1+x & 0 < x \leq 1 \end{cases}$$

then $d_m f(0) = 0$, $D_m^+ f(0) = 0$, $D_m^- f(0) = -1$. So $D_m f(0)$ does not exist.

Remark 11. If f has bounded slope variation with respect to m on [a,b], then $d_m f$ exists except on a countable set, which is a subset of the countable set where $D_m f$ fails to exist.

Example 5. Let $f(x) = |x|$ $x \in [-1,1]$; $m(x) = x$ $x \in [-1,1]$, then f has bounded slope variation with respect to m on [-1,1]; $V_a^b \frac{df}{dm} = 2$ and $d_m f(0)$ does not exist.

Theorem 11. If f has bounded slope variation with respect to m on [a,b], then $d_m f$ exists except on a countable set E' with $\mu_m(E') = 0$.

Proof. Recall μ_m is the (Borel) measure generated by the right-continuous non-decreasing function m, $\mu_m([a,b]) = m(b) - m(a)$ and $\mu_m(\{c\}) = m(c) - m(c-)$. Hence if m is continuous at c, then $\mu_m(\{c\}) = 0$.

If c is a point of discontinuity of m, $d_m f(c)$ exists. Since f is quasicontinuous then

$$d_m f(c) = \frac{f(c+)-f(c-)}{m(c+)-m(c-)}$$

exists. So $d_m f(x)$ exists except for a countable subset E' of the set of points of continuity of m. i.e. $\mu_m(E') = 0$.

Theorem 12. If f has bounded slope variation with respect to m on [a,b], then $d_m f$ is of bounded variation on [a,b]-E' and

$$V_{[a,b]-E'}(d_m f) \leq V_a^b \left(\frac{df}{dm}\right)$$

In particular $d_m f$ is finite μ_m almost everywhere.

Proof. P is any partition of [a,b] such that $P \cap E' = \emptyset$. Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$, for every x_i there exists $z_{i-1} < y_i < x_i < z_i < y_{i+1}$ $(y_0 = x_0,\ z_n = x_n)$ such that

$$\left| d_m f(x_i) - \frac{f(z_i)-f(y_i)}{m(z_i)-m(y_i)} \right| < \frac{\varepsilon}{2n}$$

where ε is any fixed positive number. Consider

$$P' = \{y_i\}_{i=0}^{n} \cup \{z_i\}_{i=0}^{n}$$

$$\sum_{P} |d_m f(x_{i+1}) - d_m f(x_i)|$$

$$\leq \sum_{P'} \left| \frac{f(z_{i+1})-f(y_{i+1})}{m(z_{i+1})-m(y_{i+1})} - \frac{f(z_i)-f(y_i)}{m(z_i)-m(y_i)} \right| + \varepsilon$$

$$= \sum_{p'} \left| \frac{f(z_{i+1})-f(y_{i+1})}{m(z_{i+1})-m(y_{i+1})} - \frac{f(y_{i+1})-f(z_i)}{m(y_{i+1})-m(z_i)} + \frac{f(y_{i+1})-f(z_i)}{m(y_{i+1})-m(z_i)} - \frac{f(z_i)-f(y_i)}{m(z_i)-m(y_i)} \right| + \varepsilon$$

$$\leq \sum_{p'} \left| \frac{f(z_{i+1})-f(y_{i+1})}{m(z_{i+1})-m(y_{i+1})} - \frac{f(y_{i+1})-f(z_i)}{m(y_{i+1})-m(z_i)} \right| + \left| \frac{f(y_{i+1})-f(z_i)}{m(y_{i+1})-m(z_i)} - \frac{f(z_i)-f(y_i)}{m(z_i)-m(y_i)} \right| + \varepsilon$$

$$\leq V_a^b \frac{df}{dm} + \varepsilon$$

hence $V_{[a,b]-E'}\, d_m f \leq V_a^b \frac{df}{dm}$.

Theorem 13. If f has bounded slope variation with respect to m on [a,b], then $d_m f$ and $(d_m f)^2$ are Lebesque-Stieltjes integrable.

Proof. Recall that if $\{f_n\}$ is a sequence of measurable function, then $\lim_{n\to\infty} f_n$ is measurable. If U, V are measurable then so are $U \cdot V$, U/V $(V(x)\neq 0)$. If f is measurable and f = g a.e., then g is measurable. Define

$$g_n(x) = \frac{f(x+1/n) - f(x-1/n)}{m(x+1/n) - m(x-1/n)}, \quad x \in [a,b]-E'$$

then $g_n(x)$ are Borel measurable, so is $d_m f(x)$ $(= \lim_{n\to\infty} g_n$ a.e.) i.e. $d_m f$ and $(d_m f)^2$ are Borel measurable.

Again recall if f is a bounded measurable function defined on a measurable set E with finite measure, then f is

Lebesque-Stieltjes integrable relative to the Borel measure μ_m.

Theorem 14. If f has bounded slope variation with respect to m on [a,b], m is strictly increasing and continuous from the right, then the Hellinger integral $\int_a^b \frac{(df)^2}{dm}$ exists and the Lebesque integral $\int_a^b (d_m f)^2 \, d\mu_m$ exists and the two are equal.

Proof. Let $P_1, P_2, \ldots$ be a sequence of partitions of [a,b] with P_n a refinement of P_{n-1}, and

$$\sum_{P_n} \frac{[f(x_i)-f(x_{i-1})]^2}{m(x_i)-m(x_{i-1})} \text{ goes to } H\int_a^b \frac{(df)^2}{dm}$$

and the mesh of P_n goes to zero.

$$\text{Let } g_n(x) = \sum_{P_n} \left[\frac{f(x_i)-f(x_{i-1})}{m(x_i)-m(x_{i-1})}\right]^2 X_{(x_{i-1},x_i]}(x) ,$$

where $x_{i-1}, x_i \in P_n$.

$$\text{Now } L\int_a^b g_n d\mu_m = \sum_{P_n} \left[\frac{f(x_i)-f(x_{i-1})}{m(x_i)-m(x_{i-1})}\right]^2 \mu_m(x_{i-1},x_i]$$

$$= \sum_{P_n} \left[\frac{f(x_i)-f(x_{i-1})}{m(x_i)-m(x_{i-1})}\right]^2 [m(x_i)-m(x_{i-1})]$$

$$= \sum_{P_n} \frac{[f(x_i)-f(x_{i-1})]^2}{m(x_i)-m(x_{i-1})}$$

Thus we have L $\int_a^b g_n \, d\mu_m$ goes monotonically increasing to

H $\int_a^b \frac{(df)^2}{dm}$.

Let E be the subset of [a,b] where $D_m f$ fails to exist and E' be the subset of [a,b] where $d_m f$ fails to exist. We have that E is countable, $E' \subset E$ and $\mu_m(E') = 0$.

We now consider the pointwise limit of g_n and show that it converges to $(d_m f)^2$ a.e. relative to μ_m.

(i) If $x \in \cup P_n$ and $x \notin E$, then clearly $g_n(x)$ goes to $(D_m^- f(x))^2 = (d_m f(x))^2$.

(ii) If $x \notin \cup P_n$ and $x \notin E'$, then clearly $g_n^{(x)}$ goes to $(d_m f(x))^2$.

(iii) If $x \in \cup P_n$ and $x \in (E-E')$, then $g_n(x)$ goes to $(D_m^- f(x))^2$ on (a,b] (the value at {a} is immaterial since $\mu_m\{a\} = 0$).

Now if $t < x$, then we consider two subcases:

(a) if x is a point of continuity of m, then

$$\mu_m\{x\} = \lim_{t \to x^-} \mu_m(t,x] = m(x)-m(t) = 0$$

and the value of $D_m^- f(x)$ is immaterial since this set is countable and has μ_m measure zero.

(b) If x is a point of discontinuity of m, then since m is continuous from the right we know that f is continuous from the right (Theorem 4), so that

$$g_n(x) = [\frac{f(x)-f(t)}{m(x)-m(t)}]^2 \quad \text{goes to}$$

$$[\frac{f(x)-f(x-)}{m(x)-m(x-)}]^2 = [\frac{f(x+)-f(x-)}{m(x+)-m(x-)}]^2 = [d_m f(x)]^2$$

Hence, by the Lebesque Bounded Convergence Theorem, we have that

$L \int_a^b g_n d\mu_m$ goes to $L \int_a^b (d_m f)^2 \, d\mu_m$.

Remark 12. We see that under the above hypothesis in Theorem 14, if $F(x) = \int_a^x \frac{(df)^2}{dm}$ then F is absolutely continuous with respect to m which answers a question raised by Huggins in [3].

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