Global Regularity Aspects of Equations in Hydrodynamics

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ABSTRACT

The question of the global regularity of the two-dimensional magnetohydrodynamics system without viscous dissipation is currently unknown. A challenging problem concerning the global regularity of the two-and-a-half and three-dimensional Hall-magnetohydrodynamics system is still open. The global regularity of two-dimensional and three-dimensional Kuramoto–Sivashinsky equations are not solved as well.

Chapter 2 explores some cancellations and bounds within the Hall term for both two-and-a-half dimensional and three-dimensional cases, as well as various regularity criteria. The two-a-half-dimensional Hall equation and Hall-magnetohydrodynamics system are also proved to be globally well-posed when magnetic dissipation is considered at the below-critical level in the horizontal direction and at the supercritical level in the vertical direction.

The purpose of Chapter 3 is to introduce and prove some global regularity criteria on the Sobolev and Besov spaces in dimensions two and three. In Chapter 4, a two-and-a-half-dimensional magnetohydrodynamics system is presented, and its global well-posedness is demonstrated. Furthermore, a magnetohydrodynamics system without viscous dissipation is introduced, and a global regularity criterion is derived.

CHAPTER 1
INTRODUCTION

The Navier-Stokes equations system is a fundamental mathematical model of incompressible viscous fluid flow, written as

$$\begin{aligned}&\frac{\partial u}{\partial t}+(u\cdot\nabla)u+\nabla\pi=-\nu(-\Delta)^{\alpha}u\\&\nabla\cdot u=0,\end{aligned}\tag{1.1}$$

where $u:\mathbb{R}^D\times\mathbb{R}_+\to\mathbb{R}^D$ is velocity, $\Pi:\mathbb{R}^D\times\mathbb{R}_+\to\mathbb{R}$ is pressure, $\nu\geq 0$ is viscosity, and $(-\Delta)^{\alpha}$ denotes the operator defined by its Fourier symbol $|\xi|^{2\alpha}$ for $\alpha>0$. The note is made that $\alpha=1$ yields to the classical Navier-Stokes equations. Let's focus on $\alpha=1$.

Definition 1.0.1. *A function u is called a weak solution of* (1.1) *satisfying $u_0\in L^2$, if*

(1) $u\in L^{\infty}(0,T;L^2)\cap L^2(0,T;H^1)$ for all $T>0$ and

(2) u satisfies the equation

$$\int_0^{\tau}-\langle u,\partial_t\phi\rangle+\int_0^{\tau}\langle\nabla u,\nabla\phi\rangle+\int_0^{\tau}\langle(u\cdot\nabla)u,\phi\rangle=\langle u_0,\phi(0)\rangle-\langle u(\tau),\phi(\tau)\rangle\tag{1.2}$$

for a.e. $\tau>0$, for a suitable class of divergence-free test functions ϕ.

In [99], Leray proved that for any $u_0\in L^2$, there exists a global in-time weak solution that additionally obeys the energy inequality

$$\frac{1}{2}\|u(t)\|_{L^2}^2+\nu\int_0^t\|\nabla u(\tau)\|_{L^2}^2d\tau\leq\|u(0)\|_{L^2}^2.\tag{1.3}$$

In a smooth bounded domain, Hopf in [66] produced a similar result for the equations with Dirichlet boundary conditions. There is an open question: whether Leray-Hopf weak solutions for the three-dimensional Navier-Stokes equations (3-D NSE) are unique or not. The regular solutions of 3-D NSE satisfy the energy inequality

$$\frac{1}{2}\|u(t)\|_{L^2}^2+\nu\int_0^t\|\nabla u(\tau)\|_{L^2}^2d\tau=\|u(0)\|_{L^2}^2\tag{1.4}$$

However, it remains open whether energy equality is valid or not for Leray-Hopf weak solutions.

It is still unknown whether such a system (1.1) (with $\alpha = 1$) always admits a unique global smooth solution for any given smooth initial data which can be informally stated as follows:

Navier–Stokes regularity problem. Do the equations have a (unique) smooth global solution for whatever smooth initial data u_0?

If $\alpha = \frac{5}{4}$, then solution of (1.1) is invariant under the scaling. The scaling is defined as $u_\lambda(x,t) = \lambda^{2\alpha-1}u(\lambda x, \lambda^{2\alpha}t)$ and $\Pi_\lambda(x,t) = \lambda^{4\alpha-2}\Pi(\lambda x, \lambda^{2\alpha}t)$ with $\lambda > 0$.

It is known that the 3D NSE enjoys the global weak solutions, the short-time existence of strong solutions, and the uniqueness of strong solutions.

If $\alpha \geq \frac{5}{4}$, then the 3-D NSE always has a global regularity solution corresponding to any smooth initial data. Does the global regularity remain valid when $\alpha < \frac{5}{4}$? Tao investigated the hyper-dissipative NSE in [106]. He replaced $(-\Delta)^{\frac{5}{4}}u$ by $\frac{(-\Delta)^{\frac{5}{4}}}{log^{\frac{1}{2}}(I-\Delta)}u$. still gives us a unique global solution. Very recently, Buckmaster and Vicol [13] proved the non-uniqueness of certain types of weak solutions to the 3-D NSE.

One can consider that the magnetohydrodynamics (MHD) system has more applications than the Navier-Stokes and Euler equations because it is not merely Navier-Stokes equations; it is formed by Navier-Stokes equations and Maxwell's equations. In addition, the widespread applications of the MHD system include geophysics, astrophysics, cosmology, and engineering.

Applications of the Hall-magnetohydrodynamics (Hall-MHD) system include modeling electronically conducting fluids, for example, plasmas or electrolytes. Magnetic reconnection is influenced by the Hall term; and its applications are observed in plasmas, solar flares, star formation, neutron stars, and among others.

Modeling reaction-diffusion systems, flame propagation, and viscous flow issues are among the applications of the Kuramoto-Sivashinsky equation (KSE). The equation is considered a prototype for generalized Burgers equations, which are composed of arbitrary linear parabolic parts and a quadratic nonlinear part. Physically, the two-dimensional KSE appears more appealing when considering the laminar flame front model. In contrast, the problem of global existence in 2-D and 3-D has been open for several decades. The main issue, mathematically, is the lack of conserved quantity for the 2-D and 3-D cases. For the 1-D KSE case, however, the global regularity, dissipativity, and global attractor are already established.

CHAPTER 2
MOTIVATION OF HALL-MAGNETOHYDRODYNAMICS FROM PHYSICS AND REAL-WORLD APPLICATIONS

For its physical significance and mathematical complexity, over the past few decades, the Hall-MHD system drew a substantial amount of attention from many physicists and mathematicians. According to Lighthill [83], the Hall-MHD system was introduced in 1960 by adding a term which is called a Hall term. It arises whenever the current density is written as the addition of the ohmic current and a Hall current. Hall-MHD system is important to study the plasmas, star formation, solar flares, neutron stars, magnetic re-connection, and turbulence. Concerning these applications, a physical review can be found in [93]. Moreover, we can refer to [5, 52, 15, 65, 86, 114] for details. Acheritogaray, Degond, Frouvelle, and Liu in [1] derived the Hall-MHD system by using two fluids models or kinetic models. The global weak solutions in 3-D Hall-MHD are investigated in [31].

We are considering the following mathematical model for the incompressible Hall-MHD system:

$$\partial_t u + (u \cdot \nabla)u + \nabla \pi = \nu \Delta u + (b \cdot \nabla)b \qquad t > 0, \tag{2.1a}$$

$$\partial_t b + (u \cdot \nabla)b + \epsilon \nabla \times (j \times b) = \eta \Delta b + (b \cdot \nabla)u \qquad t > 0, \tag{2.1b}$$

$$\nabla \cdot u = 0, \tag{2.1c}$$

where $u : \mathbb{R}^D \times \mathbb{R}_+ \to \mathbb{R}^3$ is velocity, $\pi : \mathbb{R}^D \times \mathbb{R}_+ \to \mathbb{R}$ is pressure, $b : \mathbb{R}^D \times \mathbb{R}_+ \to \mathbb{R}^3$ is the magnetic field, $\nu \geq 0$ is viscosity, $\eta \geq 0$ is magnetic resistivity, and ϵ is Hall parameter.

$$j = (j_1, j_2, j_3) = \nabla \times b = (\partial_2 b_3 - \partial_3 b_2, -\partial_1 b_3 + \partial_3 b_1, \partial_1 b_2 - \partial_2 b_1) \tag{2.2}$$

is current density field for 3-D Hall-MHD.

$$j = (j_1, j_2, j_3) = \nabla \times b = (\partial_2 b_3, -\partial_1 b_3, \partial_1 b_2 - \partial_2 b_1) \tag{2.3}$$

is current density field for $2\frac{1}{2}$-D Hall-MHD. In which the note is made that if $\nabla \cdot b_0 = 0$, then $\nabla \cdot b = 0$ remains true for all $t > 0$ (see [117, Equation (1)]). A nonlinear term with the highest derivative is called the Hall term, which makes the system a quasi-linear system. A quasi-linear system is usually more challenging than a semi-linear system.

From [85, Section 2.3.1], we are familiar with the definition of $2\frac{1}{2}$-D Euler equations and Navier-Stokes equations.

We also list some progresses for studying the Hall-MHD such as the temporal decay for the weak solutions and smooth solutions with small data, the well-posedness results, partial regularity results, Liouville theorem for stationary solutions, singularity formation without resistivity, referring to [31, 32, 46, 33, 34, 36, 35] for details.

The Hall equation consists of

$$\partial_t b + \epsilon \nabla \times (j \times b) + \eta(-\Delta)^{\delta} b = 0 \quad \text{for } t > 0, \tag{2.4}$$

The note is made that if $\nabla \cdot b_0 = 0$, then $\nabla \cdot b = 0$ remains true for all $t > 0$. The Hall equation possesses the scaling invariant solution i.e. if $b(t, x)$ solves (2.4), then so does

$$b_\lambda(t, x) = \lambda^{2\delta - 2} b(\lambda^{2\delta} t, \lambda x) \; \forall \, \lambda > 0. \tag{2.5}$$

However, the Hall-MHD system breaks the scaling invariant property because of the Hall term. When $\delta \geq 1 + \frac{d}{4}$; d is the dimension, we know that the Hall equation has a global regularity solution. To exploit the Hall term, we decompose the Hall term as follows:

$$(\nabla \times (j \times b)) = -(j \cdot \nabla) b + (b \cdot \nabla) j. \tag{2.6}$$

It is known that the global regularity issue related to 3-D and even $2\frac{1}{2}$-D Hall-MHD system remains completely open with standard Laplacian diffusion and standard Laplacian dissipation. To address the global regularity issues, we may take advantage of the blow-up criteria; discovered by Chae and Lee in [29]. In [29], the authors stated a regularity criterion for the 3-D Hall-MHD system for $u, \nabla b \in L_t^r L_x^\beta$ for $\frac{3}{\beta} + \frac{2}{r} \leq 1, \beta \in (3, \infty]$ (see [126]). Moreover, a blow-up criterion for strong solutions of the Hall equation was established in [29]. From these results, one can hypothesize that $2\frac{1}{2}$-D Hall-MHD system has a regular solution if the criterion holds:

$\nabla b \in L_t^r L_x^\beta$ for $\frac{2}{\beta} + \frac{2}{r} \leq 1, \beta \in (2, \infty]$.

One can refer to [30, 16, 72, 128] for the regularity results of NSE concerning one or two components, and many of these global regularity criteria results originated from [100, 94]. As we know that the structures of nonlinear terms between the NSE and MHD system are somewhat similar, and many results can be extended from the NSE to the MHD system (see [17, 118]). On the other hand, due to the complexity of the Hall term, no significant improvement can be made to explore the cancellation inside the Hall term and global regularity criteria with one or two components for $2\frac{1}{2}$-D, and 3-D respectively. In this study, we investigate the regularity criteria with one or two components and global regularity results.

For convenience, we list the result of the local existence of the strong solutions theorem. One can find the proof of this theorem in [31].

Theorem 1 (Local existence of strong solutions)**.** *Let* $(u_0, b_0) \in H^s(\mathbb{R}^3) \times H^s(\mathbb{R}^3)$, $s > \frac{5}{2}$ *is an integer, then there exists* $T = T(\|u_0\|_{H^s}, \|b_0\|_{H^s})$ *such that there exists a unique solution such that* $(u, b) \in L^\infty([0, T); H^s(\mathbb{R}^3)) \cap L^2(0, T; H^{s+1}(\mathbb{R}^3))$.

Based on two previous works, the statements of all Theorems and Propositions and their proofs are divided into two different sections.

2.1 Statement of the Theorems: 2-4

The Theorems 2-4 and their proofs come from a previous work [97] with Prof. Kazuo Yamazaki.

From [29], in the $2\frac{1}{2}$-D case, the blow-up criterion as follows:

$$\limsup_{t \nearrow T^*} (\|u(t)\|_{H^m}^2 + \|b(t)\|_{H^m}^2) = \infty \text{ if and only if } \int_0^{T^*} \|j\|_{BMO}^2 dt = \infty, \tag{2.7}$$

where $m > 2$ is an integer. To work on $2\frac{1}{2}$-D Hall-MHD system, we know $H^1(\mathbb{R}^2)$-bound on (u, b) is enough to show higher regularity. This is due to (2.7) and the following inequality

$$\|u\|_{BMO} \leq C\|u\|_{\dot{H}^{\frac{n}{2}}} \quad \forall u \in L^1_{loc}(\mathbb{R}^n) \cap \dot{H}^{\frac{n}{2}}(\mathbb{R}^n). \tag{2.8}$$

For details of this inequality, one can go with [9, Theorem 1.48]. For convenience, $\nu = \eta = \epsilon = 1$.

Theorem 2. *Suppose that* $(u_0, b_0) \in H^m(\mathbb{R}^2) \times H^m(\mathbb{R}^2)$ *where* $m > 2$ *is an integer and* $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$ *and* (u, b) *is a corresponding local regular solution to the* $2\frac{1}{2}$*-D Hall-MHD* (2.1) *on* $[0, T^*)$ *for some positive* $T^* < \infty$*. If*

$$\int_0^{T^*} \|\nabla b_3\|_{L^p}^r dt \approx \sum_{k=1}^{2} \int_0^{T^*} \|j_k\|_{L^p}^r dt < \infty \text{ where } \frac{2}{p} + \frac{2}{r} \le 1, \ \ p \in (2, \infty] \tag{2.9}$$

then

$$\limsup_{t \to T^*} (\|u(t)\|_{H^m}^2 + \|b(t)\|_{H^m}^2) < \infty. \tag{2.10}$$

Consequently, we can extend (u, b) *beyond* T^**.*

We obtain the following criterion, which might be similar to the works of [80, 121].

Theorem 3. *Suppose that* $(u_0, b_0) \in H^m(\mathbb{R}^2) \times H^m(\mathbb{R}^2)$ *where* $m > 2$ *is an integer and* $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$ *and* (u, b) *is a corresponding local regular solution to the* $2\frac{1}{2}$*-D Hall-MHD* (2.1) *on* $[0, T^*)$ *for some positive* $T^* < \infty$*. If*

$$\int_0^{T^*} \|j_3\|_{L^p}^r dt < \infty \text{ where } \frac{2}{p} + \frac{2}{r} \le 1, \ \ p \in (2, \infty],$$

then

$$\limsup_{t \to T^*} (\|u(t)\|_{H^m}^2 + \|b(t)\|_{H^m}^2) < \infty.$$

Therefore, we can extend (u, b) *beyond* T^**.*

Motivated by [124], we consider the following system:

$$\partial_t u + (u \cdot \nabla) u + \nabla \pi = \nu \Delta u + (b \cdot \nabla) b \qquad t > 0, \tag{2.11a}$$

$$\partial_t b + (u \cdot \nabla) b + \epsilon \nabla \times (j \times b) = -\eta_h \Lambda^3 b_h - \eta_v \Lambda^2 b_v + (b \cdot \nabla) u \quad t > 0, \tag{2.11b}$$

$$\nabla \cdot u = 0, \tag{2.11c}$$

where the note is made that if $\nabla \cdot b_0 = 0$, then $\nabla \cdot b = 0$ remains true for all $t > 0$, and where

$$b_h = \begin{pmatrix} b_1 & b_2 & 0 \end{pmatrix}^T \text{ and } b_v = \begin{pmatrix} 0 & 0 & b_3 \end{pmatrix}^T.$$

Theorem 4. *Suppose that* $(u_0, b_0) \in H^m(\mathbb{R}^2) \times H^m(\mathbb{R}^2)$ *where* $m > 2$ *is an integer and* $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. *Then there exists a unique solution*

$$u \in L^\infty((0,\infty); H^m(\mathbb{R}^2)) \cap L^2((0,\infty); H^{m+1}(\mathbb{R}^2)), b \in L^\infty((0,\infty); H^m(\mathbb{R}^2)) \quad (2.12)$$

such that

$$b_h \in L^2((0,\infty); H^{m+\frac{3}{2}}(\mathbb{R}^2)), b_v \in L^2((0,\infty); H^{m+1}(\mathbb{R}^2)) \quad (2.13)$$

to (2.11a)-(2.11c), *and* $(u, b)|_{t=0} = (u_0, b_0)$.

2.2 Proofs of Theorems 2-4

Since the proof of the theorem is available in previous work [97] with Prof. Kazuo Yamazaki, it is presented in a sketch for the convenience of readers.

2.2.1 Proof of Theorem 2

Taking $L^2(\mathbb{R}^2)$-inner products on (2.1a) with $-u$ and (2.1b) with $-b$. Adding them up to compute as follows:

$$(\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2) + 2\int_0^t (\|\nabla b\|_{L^2}^2 + \|\nabla u\|_{L^2}^2) d\tau = (\|b_0\|_{L^2}^2 + \|u_0\|_{L^2}^2). \quad (2.14)$$

Taking $L^2(\mathbb{R}^2)$-inner products on (2.1a) with $-\Delta u$ and (2.1b) with $-\Delta b$. Adding them up to compute as follows:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2 &= \int_{\mathbb{R}^2} (u\cdot\nabla)u \cdot \Delta u dx \\ &- \int_{\mathbb{R}^2} (b\cdot\nabla)b \cdot \Delta u \, dx + \int_{\mathbb{R}^2} (u\cdot\nabla)b \cdot \Delta b \, dx \\ &- \int_{\mathbb{R}^2} (b\cdot\nabla)u \cdot \Delta b \, dx + \int_{\mathbb{R}^2} \nabla \times (j \times b) \cdot \Delta b \, dx \\ &= \sum_{i=1}^5 I_i \qquad (2.15) \end{aligned}$$

Recall a vector calculus identity,

$$(v \times w) \cdot v = 0. \tag{2.16}$$

Recall a vector calculus identity,

$$\int_{\mathbb{R}^2} (\nabla \times f) \cdot g dx = \int_{\mathbb{R}^2} f \cdot (\nabla \times g) dx. \tag{2.17}$$

By applying (2.17), $\nabla \times b = j$, integration by parts, and (2.16), we compute I_5 as follows:

$$\begin{aligned}
I_5 &= \int_{\mathbb{R}^2} \nabla \times (j \times b) \cdot \Delta b \, dx = \int_{\mathbb{R}^2} \nabla \times (j \times b) \cdot (\partial_1^2 + \partial_2^2) b dx \\
&= \int_{\mathbb{R}^2} (j \times b) \cdot (\partial_1^2 + \partial_2^2)(\nabla \times b) dx \\
&= \int_{\mathbb{R}^2} (j \times b) \cdot \partial_1^2 j + \int_{\mathbb{R}^2} (j \times b) \cdot \partial_2^2 j \, dx \\
&= -\int_{\mathbb{R}^2} (\partial_1 j \times b) \cdot (\partial_1 j) dx - \int_{\mathbb{R}^2} (j \times \partial_1 b) \cdot (\partial_1 j) dx - \int_{\mathbb{R}^2} (\partial_2 j \times b) \cdot (\partial_2 j) dx \\
&- \int_{\mathbb{R}^2} (j \times \partial_2 b) \cdot (\partial_2 j) dx \\
&= -\int_{\mathbb{R}^2} (j \times \partial_1 b) \cdot (\partial_1 j) dx - \int_{\mathbb{R}^2} (j \times \partial_2 b) \cdot (\partial_2 j) dx \\
&= I_{5,1} + I_{5,2}
\end{aligned} \tag{2.18}$$

Let's compute $I_{5,1}$ as follows:

$$\begin{aligned}
I_{5,1} &= -\int_{\mathbb{R}^2} \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ j_1 & j_2 & j_3 \\ \partial_1 b_1 & \partial_1 b_2 & \partial_1 b_3 \end{pmatrix} \cdot \begin{pmatrix} \partial_1 j_1 \\ \partial_1 j_2 \\ \partial_1 j_3 \end{pmatrix} dx \\
&= -\int_{\mathbb{R}^2} \begin{pmatrix} j_2 \partial_1 b_3 - j_3 \partial_1 b_2 \\ -j_1 \partial_1 b_3 + j_3 \partial_1 b_1 \\ j_1 \partial_1 b_2 - j_2 \partial_1 b_1 \end{pmatrix} \cdot \begin{pmatrix} \partial_1 j_1 \\ \partial_1 j_2 \\ \partial_1 j_3 \end{pmatrix} dx \\
&= -\int_{\mathbb{R}^2} j_2 \partial_1 b_3 \partial_1 j_1 - j_3 \partial_1 b_2 \partial_1 j_1 - j_1 \partial_1 b_3 \partial_1 j_2 + j_3 \partial_1 b_1 \partial_1 j_2 + j_1 \partial_1 b_2 \partial_1 j_3
\end{aligned}$$

$$
\begin{aligned}
&- j_2\partial_1 b_1 \partial_1 j_3 \; dx \\
&= \sum_{i=1}^{6} I_{5,1,i} \qquad (2.19)
\end{aligned}
$$

If we pair up $I_{5,1,1}$ and $I_{5,1,3}$ and integrate by parts, then we can find a cancellation as follows:

$$
\begin{aligned}
I_{5,1,1} + I_{5,1,3} &= -\int_{\mathbb{R}^2} j_2\partial_1 b_3 \partial_1 j_1 - j_1 \partial_1 b_3 \partial_1 j_2 dx \\
&= -\int_{\mathbb{R}^2} -\partial_1 b_3 \partial_1 b_3 \partial_1 \partial_2 b_3 + \partial_2 b_3 \partial_1 b_3 \partial_1 \partial_1 b_3 dx \\
&= \int_{\mathbb{R}^2} \partial_1 b_3 \partial_1 b_3 \partial_1 \partial_2 b_3 - \partial_2 b_3 \partial_1 b_3 \partial_1 \partial_1 b_3 dx \\
&= \int_{\mathbb{R}^2} -\partial_1 b_3 \frac{1}{2}\partial_2 |\partial_1 b_3|^2 + \partial_2 b_3 \frac{1}{2} \partial_1 |\partial_1 b_3|^2 dx \\
&= \int_{\mathbb{R}^2} \frac{1}{2}\partial_1\partial_2 b_3 |\partial_1 b_3|^2 - \frac{1}{2}\partial_1 \partial_2 b_3 |\partial_1 b_3|^2 dx = 0. \qquad (2.20)
\end{aligned}
$$

To estimate $I_{5,1,2}, I_{5,1,4}, I_{5,1,5},$ *and* $I_{5,1,6}$ more concisely, we pair up them and apply integration by parts as follows:

$$
I_{5,1,2} + I_{5,1,4} + I_{5,1,5} + I_{5,1,6} \lesssim \int_{\mathbb{R}^2} (|j_1| + |j_2|)|\nabla b||\nabla^2 b| dx \qquad (2.21)
$$

Let's compute $I_{5,2}$ as follows:

$$
\begin{aligned}
I_{5,2} = &-\int_{\mathbb{R}^2} \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ j_1 & j_2 & j_3 \\ \partial_2 b_1 & \partial_2 b_2 & \partial_2 b_3 \end{pmatrix} \cdot \begin{pmatrix} \partial_2 j_1 \\ \partial_2 j_2 \\ \partial_2 j_3 \end{pmatrix} dx \\
= &-\int_{\mathbb{R}^2} \begin{pmatrix} j_2 \partial_2 b_3 - j_3 \partial_2 b_2 \\ -j_1 \partial_2 b_3 + j_3 \partial_2 b_1 \\ j_1 \partial_2 b_2 - j_2 \partial_2 b_1 \end{pmatrix} \cdot \begin{pmatrix} \partial_2 j_1 \\ \partial_2 j_2 \\ \partial_2 j_3 \end{pmatrix} dx \\
= &-\int_{\mathbb{R}^2} j_2 \partial_2 b_3 \partial_2 j_1 - j_3 \partial_2 b_2 \partial_2 j_1 - j_1 \partial_2 b_3 \partial_2 j_2 + j_3 \partial_2 b_1 \partial_2 j_2 + j_1 \partial_2 b_2 \partial_2 j_3 \\
&- j_2 \partial_2 b_1 \partial_2 j_3 \; dx
\end{aligned}
$$

$$= \sum_{i=1}^{6} I_{5,2,i} \tag{2.22}$$

If we pair up $I_{5,2,1}$ and $I_{5,2,3}$ and integrate by parts, then we can find a cancellation as follows:

$$\begin{aligned} I_{5,2,1} + I_{5,2,3} &= -\int_{\mathbb{R}^2} j_2 \partial_2 b_3 \partial_2 j_1 - j_1 \partial_2 b_3 \partial_2 j_2 dx \\ &= -\int_{\mathbb{R}^2} -\partial_1 b_3 \partial_2 b_3 \partial_2 \partial_2 b_3 + \partial_2 b_3 \partial_2 b_3 \partial_2 \partial_1 b_3 dx \\ &= \int_{\mathbb{R}^2} \partial_1 b_3 \partial_2 b_3 \partial_2 \partial_2 b_3 - \partial_2 b_3 \partial_2 b_3 \partial_2 \partial_1 b_3 dx \\ &= \int_{\mathbb{R}^2} -\partial_1 b_3 \frac{1}{2} \partial_2 |\partial_2 b_3|^2 + \partial_2 b_3 \frac{1}{2} \partial_1 |\partial_2 b_3|^2 dx \\ &= \int_{\mathbb{R}^2} \frac{1}{2} \partial_1 \partial_2 b_3 |\partial_2 b_3|^2 - \frac{1}{2} \partial_1 \partial_2 b_3 |\partial_2 b_3|^2 dx = 0. \end{aligned} \tag{2.23}$$

To estimate $I_{5,2,2}, I_{5,2,4}, I_{5,2,5},$ *and* $I_{5,2,6}$ more concisely, we pair up them and apply integration by parts as follows:

$$I_{5,2,2} + I_{5,2,4} + I_{5,2,5} + I_{5,2,6} \lesssim \int_{\mathbb{R}^2} (|j_1| + |j_2|) |\nabla b| |\nabla^2 b| dx. \tag{2.24}$$

Consequently, we can estimate I_5 more completely as follows:

$$\begin{aligned} I_5 &\lesssim \int_{\mathbb{R}^2} (|j_1| + |j_2|) |\nabla b| |\nabla^2 b| dx \\ &\approx \int_{\mathbb{R}^2} |\nabla b| |\nabla^2 b| \sum_{l=1}^{2} |j_l| dx \lesssim \|\nabla b\|_{L^{\frac{2p}{p-2}}} \|\Delta b\|_{L^2} \sum_{l=1}^{2} \|j_l\|_{L^p} \\ &\leq C \sum_{l=1}^{2} \|j_l\|_{L^p}^{\frac{2p}{p-2}} \|\nabla b\|_{L^2}^2, + \frac{1}{4} \|\Delta b\|_{L^2}^2, \end{aligned} \tag{2.25}$$

in which we applied Hölder's, Gagliardo-Nirenberg interpolation, and Young's inequalities.

To estimate I_1-I_4, we pair up them as follows:

$$I_1 + I_2 + I_3 + I_4 \lesssim \|\nabla u\|_{L^2} \|\Delta u\|_{L^2} \|\nabla u\|_{L^2} + \|\nabla u\|_{L^2} \|\Delta b\|_{L^2} \|\nabla b\|_{L^2}$$

$$\leq C(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2)\|\nabla u\|_{L^2}^2 + \frac{1}{4}\|\Delta u\|_{L^2}^2 + \frac{1}{4}\|\Delta b\|_{L^2}^2, \tag{2.26}$$

in which we applied integration by parts, Hölder's inequality, Gagliardo-Nirenberg interpolation, and Young's inequalities.

Combining with these estimates (2.25), (2.26), we have from (2.15) as follows:

$$\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2 \leq \frac{1}{2}\|\Delta b\|_{L^2}^2 + \frac{1}{4}\|\Delta u\|_{L^2}^2 + C(\|\nabla u\|_{L^2}^2 + \sum_{l=1}^{2}\|j_l\|_{L^p}^{\frac{2p}{p-2}})(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) \tag{2.27}$$

Using Grönwall's inequality, we attain the following bound:

$$(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) \leq e^{\int_o^t C(\sum_{l=1}^2 \|j_l\|_{L^p}^{\frac{2p}{p-2}} + \|\nabla u\|_{L^2}^2)ds}(\|\nabla u_0\|_{L^2}^2 + \|\nabla b_0\|_{L^2}^2), \tag{2.28}$$

which yields the proof of the Theorem 2 with applying the hypothesis and (2.14).

Since the proof of the Theorem 3 is available in previous work [97] with Prof. Kazuo Yamazaki, it is presented in a sketch for the convenience of readers.

2.2.2 Proof of Theorem 3

In order to prove Theorem 3, we must rely on Theorem 2. This is because we have all necessary computations in our hands due to the proof of the Theorem 2.

Let's recall the two cancellations from (2.20) and (2.23). Using these two cancellations, we need only work to bound $I_{5,1,2}, I_{5,1,4}, I_{5,1,5}, I_{5,1,6}, I_{5,2,2}, I_{5,2,4}, I_{5,2,5}, \textit{ and } I_{5,2,6}$.

Using (2.19) and integration by parts, we compute as follows:

$$I_{5,1,2} + I_{5,1,4} + I_{5,1,5} + I_{5,1,6} \lesssim \int_{\mathbb{R}^2} |j_3||\nabla b||\nabla^2 b|dx. \tag{2.29}$$

Using (2.22) and integration by parts, we compute as follows:

$$I_{5,2,2} + I_{5,2,4} + I_{5,2,5} + I_{5,2,6} \lesssim \int_{\mathbb{R}^2} |j_3||\nabla b||\nabla^2 b|dx. \tag{2.30}$$

Similarly to the proof of the Theorem 2, we can prove the Theorem 3. The detailed strategies for proving the theorem are left in [97].

Since the proof of the Theorem 4 is available in previous work [97] with Prof. Kazuo Yamazaki, it is presented in a sketch for the convenience of readers.

2.2.3 Proof of Theorem 4

Taking $L^2(\mathbb{R}^2)$-inner products on (2.11a) with $-u$ and (2.11b) with $-b$. Adding them up to compute as follows:

$$(\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2) + 2\int_0^t (\|\nabla u\|_{L^2}^2 + \|\Lambda^{\frac{3}{2}} b_h\|_{L^2}^2 + \|\nabla b_v\|_{L^2}^2) d\tau = (\|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2). \tag{2.31}$$

To prove the Theorem 4, we need to depend on the Theorem 2. This is because we have all necessary computations in our hands due to the proof of the Theorem 2.

Taking $L^2(\mathbb{R}^2)$-inner products on (2.11a) with $-\Delta u$ and (2.11b) with $-\Delta b$. Adding them up to compute as follows:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \|\Delta u\|_{L^2}^2 + \|\Lambda^{\frac{5}{2}} b_h\|_{L^2}^2 + \|\Delta b_v\|_{L^2}^2 &= \int_{\mathbb{R}^2} (u\cdot\nabla)u\cdot\Delta u dx \\ &- \int_{\mathbb{R}^2} (b\cdot\nabla)b\cdot(\Delta u)\, dx + \int_{\mathbb{R}^2} (u\cdot\nabla)b\cdot\Delta b\, dx \\ &- \int_{\mathbb{R}^2} (b\cdot\nabla)u\cdot(\Delta b)\, dx + \int_{\mathbb{R}^2} \nabla\times(j\times b)\cdot\Delta b\, dx \\ &= \sum_{i=1}^{6} I_i \end{aligned} \tag{2.32}$$

Using (2.19) and integration by parts, we compute as follows:

$$\begin{aligned} I_{5,1,2} = \int_{\mathbb{R}^2} (\partial_1 b_2 - \partial_2 b_1)\partial_1 b_2 \partial_1\partial_2 b_3 dx &= -\int_{\mathbb{R}^2} \partial_1[(\partial_1 b_2 - \partial_2 b_1)\partial_1 b_2]\partial_2 b_3 dx \\ &\lesssim \int_{\mathbb{R}^2} |\nabla b_h||\nabla^2 b_h||\nabla b| dx \end{aligned} \tag{2.33}$$

Leaving details in [97], using (2.19) and integration by parts, we compute as follows:

$$I_{5,1,4} + I_{5,1,5} + I_{5,1,6} = \int_{\mathbb{R}^2} j_3\partial_1 b_1 \partial_1 j_2 dx \lesssim \int_{\mathbb{R}^2} |\nabla b_h||\nabla^2 b_h||\nabla b| dx, \tag{2.34}$$

Leaving details in [97], using (2.22) and integration by parts, we compute as follows:

$$I_{5,2,2} + I_{5,2,4} + I_{5,2,5} + I_{5,2,6} \lesssim \int_{\mathbb{R}^2} |\nabla b_h||\nabla^2 b_h||\nabla b| dx. \tag{2.35}$$

As a consequence of (2.33), (2.34), (2.35), (2.20), and (2.23), we can precisely estimate the Hall term as follows:

$$\begin{aligned} I_5 \lesssim & \|\nabla b_h\|_{L^4} \|\nabla^2 b_h\|_{L^4} \|\nabla b\|_{L^2} \\ \lesssim & \|\Lambda^{\frac{3}{2}} b_h\|_{L^2} \|\Lambda^{\frac{5}{2}} b_h\|_{L^2} \|\nabla b\|_{L^2} \leq \frac{1}{2} \|\Lambda^{\frac{5}{2}} b_h\|_{L^2}^2 + C \|\Lambda^{\frac{3}{2}} b_h\|_{L^2}^2 \|\nabla b\|_{L^2}^2, \end{aligned} \tag{2.36}$$

in which we applied Hölder's inequality, Sobolev embedding $\dot{H}^{\frac{1}{2}}(\mathbb{R}^2) \hookrightarrow L^4(\mathbb{R}^2)$, and Young's inequality.

To estimate of $I_1, I_2, I_3,$ and I_4 in (2.32) , we just sum up them and compute as follows:

$$\begin{aligned} I_1 + I_2 + I_3 + I_4 \leq & C(1 + \|\nabla b\|_{L^2}^2 + \|\nabla u\|_{L^2}^2)(\|\Lambda^{\frac{3}{2}} b_h\|_{L^2}^2 + \|\nabla u\|_{L^2}^2) + \frac{1}{4} \|\Delta b_v\|_{L^2}^2 \\ & + \frac{1}{4} \|\Delta u\|_{L^2}^2, \end{aligned} \tag{2.37}$$

in which we applied Hölder's inequality, Sobolev embedding $\dot{H}^{\frac{1}{2}}(\mathbb{R}^2) \hookrightarrow L^4(\mathbb{R}^2)$, and Young's inequality.

Combining with these estimates (2.36), (2.37), we have from (2.32) as follows:

$$\begin{aligned} & \frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \|\Delta u\|_{L^2}^2 + \|\Lambda^{\frac{5}{2}} b_h\|_{L^2}^2 + \|\Delta b_v\|_{L^2}^2 \\ & \qquad \lesssim (\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 + 1)(\|\nabla u\|_{L^2}^2 + \|\Lambda^{\frac{3}{2}} b_h\|_{L^2}^2). \end{aligned} \tag{2.38}$$

By Grönwall's inequality, we have

$$(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) \leq e^{\int_o^t (1 + \|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2) ds} (\|\nabla u_0\|_{L^2}^2 + \|\nabla b_0\|_{L^2}^2), \tag{2.39}$$

which yields the proof of the Theorem 4 with using (2.31). Reference for detailed information can be found in [97].

Let's turn our attention to investigating a global regularity criterion of the 3-D Hall equation and 3-D Hall-MHD system. Another Blow-up criterion from [29]:

$$\limsup_{t\nearrow T^*}(\|u(t)\|_{H^m}^2+\|b(t)\|_{H^m}^2)=\infty \text{ if and only if } \int_0^{T^*}(\|u\|_{BMO}^2+\|\nabla b\|_{BMO}^2)dt=\infty, \tag{2.40}$$

when $m>\frac{5}{2}$ is an integer for the 3-D case.

It is worthy of emphasizing, to work on a global regularity criterion for the 3-D Hall-MHD system, we need $H^2(\mathbb{R}^3)$ bound because $H^1(\mathbb{R}^3)$-bound may not be enough to show the higher regularity in the 3-D case. The reasoning for higher regularity is based on (2.7) and inequality in (2.8).

Consider the Hall equation which consists of

$$\partial_t b+\nabla\times(j\times b)+(-\Delta)^{\frac{3}{2}}b_h+(-\Delta)^{\alpha}b_v=0 \quad \text{for } t>0, \tag{2.41}$$

The note is made: if $\nabla\cdot b_0=0$, then $\nabla\cdot b=0$ remains true for all the $t>0$.

2.3 Statement of the Theorems 5-8

The Theorems 5-7 and their proofs come from a previous work [98] with Prof. Kazuo Yamazaki.

Theorem 5. *Suppose that* $(u_0,b_0)\in H^m(\mathbb{R}^3)\times H^m(\mathbb{R}^3)$ *where* $m>\frac{5}{2}$ *is an integer and* $\nabla\cdot u_0=\nabla\cdot b_0=0$ *and* (u,b) *is a corresponding local strong solution to the* 3*-D Hall-MHD* (2.1) *on* $[0,T^*)$ *for some positive* $T^*<\infty$*. If*

$$u_h\in L_t^{r_1}L_x^{p_1} \text{ where } \frac{3}{p_1}+\frac{2}{r_1}\leq 1, 3<p_1\leq\infty \tag{2.42}$$

and

$$\nabla^2 b_h\in L_t^{r_2}L_x^{p_2} \text{ where} \frac{3}{p_2}+\frac{2}{r_2}\leq 2, 2\leq p_2\leq 3 \tag{2.43}$$

then

$$\limsup_{t\to T^*}(\|u(t)\|_{H^m}^2+\|b(t)\|_{H^m}^2)<\infty.$$

Consequently, we can extend (u,b) *beyond* T^**.*

Theorem 6. *Let $\alpha \in (\frac{1}{2}, 1)$. Suppose that $b_0 \in H^3(\mathbb{R}^2)$ and $\nabla \cdot b_0 = 0$. Then, there exists a unique solution b such that*

$$b \in L^\infty((0,\infty); H^3(\mathbb{R}^2)),\ \ b_h \in L^2((0,\infty); H^{\frac{9}{2}}(\mathbb{R}^2)),\ \ b_v \in L^2((0,\infty); H^{3+\alpha}(\mathbb{R}^2))$$

to (2.41) and $b|_{t=0} = b_0$.

Theorem 7. *Let $\alpha \in (\frac{1}{2}, 1)$. Suppose that $u_0, b_0 \in H^3(\mathbb{R}^2)$ and $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Then, there exists a unique solution (u, b) such that*

$$\begin{aligned} &u \in L^\infty((0,\infty); H^3(\mathbb{R}^2)), \quad u_h \in L^2((0,\infty); H^{3+\alpha}(\mathbb{R}^2)), \quad u_v \in L^2((0,\infty); H^4(\mathbb{R}^2)), \\ &b \in L^\infty((0,\infty); H^3(\mathbb{R}^2)), \quad b_h \in L^2((0,\infty); H^{\frac{9}{2}}(\mathbb{R}^2)), \quad b_v \in L^2((0,\infty); H^{3+\alpha}(\mathbb{R}^2)) \end{aligned}$$

to

$$\partial_t u + (u \cdot \nabla)u + \nabla \pi + (-\Delta)^\alpha u_h - \Delta u_v = (b \cdot \nabla)b \qquad t > 0, \qquad (2.44a)$$
$$\partial_t b + (u \cdot \nabla)b + \nabla \times (j \times b) + (-\Delta)^{\frac{3}{2}} b_h + (-\Delta)^\alpha b_v = (b \cdot \nabla)u \qquad t > 0, \qquad (2.44b)$$
$$\nabla \cdot u = 0 \qquad t > 0, \qquad (2.44c)$$

and $(u, b)|_{t=0} = (u_0, b_0)$.

It is noted that if $\nabla \cdot b_0 = 0$, then $\nabla \cdot b = 0$ remains true for all the $t > 0$.

Let's recall the Hall equation written as

$$\partial_t b + \nabla \times ((\nabla \times b) \times b) = \Delta b \qquad t > 0 \qquad (2.45)$$

It is noted that if $\nabla \cdot b_0 = 0$, then $\nabla \cdot b = 0$ remains true for all the $t > 0$, and the decomposition of the Hall term can be seen again as $(\nabla \times (j \times b)) = -(j \cdot \nabla)b + (b \cdot \nabla)j$.

Theorem 8. *Suppose that $b_0 \in H^s(\mathbb{R}^3)$ where $s > \frac{5}{2}$ is an integer and $\nabla \cdot b_0 = 0$ and b is a corresponding local strong solution to the 3-D Hall equation (2.45) on $[0, T^*)$ for some positive $T^* < \infty$. If $\nabla b_h \in L_t^r L_x^p$ where $\frac{3}{p} + \frac{2}{r} \le 1$, $3 < p \le \infty$, and $\nabla_h b_3 \in L_t^n L_x^m$ where $\frac{3}{m} + \frac{2}{n} \le 1$, $3 < m \le \infty$, then*

$$\limsup_{t \to T^*} \|b(t)\|_{H^s}^2 < \infty.$$

Consequently, we can extend b beyond T^*.

2.4 The bounds and the cancellation within the Hall term

Since the proof of the following proposition is available in previous work [98] with Prof. Kazuo Yamazaki, it is presented in a sketch for the convenience of readers.

The following proposition plays an important role because we applied these bounds to prove the Theorems 5-7.

Proposition 9.

1. *Suppose that $b(x) = (b_1, b_2, b_3)(x_1, x_2, x_3)$ is smooth. Then it satisfies*

$$\int_{\mathbb{R}^3} \Delta\nabla \times (j \times b) \cdot \Delta b dx \lesssim \int_{\mathbb{R}^3} |\nabla^2 b_h|(|\nabla b||\nabla^3 b| + |\nabla^2 b_v||\nabla^2 b|)dx. \tag{2.46}$$

2. *Suppose that $b(x) = (b_1, b_2, b_3)(x_1, x_2)$ is smooth and $\nabla \cdot b = 0$. Then it satisfies*

$$\int_{\mathbb{R}^2} \Delta\nabla \times (j \times b) \cdot \Delta b dx \lesssim \int_{\mathbb{R}^2} (|\nabla b_h||\nabla^3 b_h| + |\nabla^2 b_h|^2)|\nabla^2 b_v| dx. \tag{2.47}$$

Proof of Proposition 9. Let's recall

$$j = (j_1, j_2, j_3) = (\partial_2 b_3 - \partial_3 b_2, -\partial_1 b_3 + \partial_3 b_1, \partial_1 b_2 - \partial_2 b_1) \tag{2.48a}$$

$$j = (j_1, j_2, j_3) = (\partial_2 b_3, -\partial_1 b_3, \partial_1 b_2 - \partial_2 b_1) \tag{2.48b}$$

Applying (2.48), we compute $\int_{\mathbb{R}^d} \Delta\nabla \times (j \times b) \cdot \Delta b dx$ (where d is dimension) as follows:

$$\begin{aligned}
\int_{\mathbb{R}^d} \Delta\nabla \times (j \times b) \cdot \Delta b dx &= \int_{\mathbb{R}^d} \sum_{k=1}^{d} \partial_k^2 (j \times b) \cdot \sum_{l=1}^{d} \partial_l^2 j dx \\
&= 2 \sum_{k,l=1}^{d} \int_{\mathbb{R}^d} (\partial_k j \times \partial_k b) \cdot \partial_l^2 j dx + \sum_{k,l=1}^{d} \int_{\mathbb{R}^d} (j \times \partial_k^2 b) \cdot \partial_l^2 j dx \\
&= I + II
\end{aligned} \tag{2.49}$$

We simplify I as follows:

$$I_1 = 2\sum_{k,l=1}^{d}\int_{\mathbb{R}^d}\partial_k j_2\partial_k b_3\partial_l^2 j_1 dx, \quad I_2 = -2\sum_{k,l=1}^{d}\int_{\mathbb{R}^d}\partial_k j_3\partial_k b_2\partial_l^2 j_1 dx \tag{2.50a}$$

$$I_3 = -2\sum_{k,l=1}^{d}\int_{\mathbb{R}^d}\partial_k j_1\partial_k b_3\partial_l^2 j_2 dx, \quad I_4 = 2\sum_{k,l=1}^{d}\int_{\mathbb{R}^d}\partial_k j_3\partial_k b_1\partial_l^2 j_2 dx \tag{2.50b}$$

$$I_5 = 2\sum_{k,l=1}^{d}\int_{\mathbb{R}^d}\partial_k j_1\partial_k b_2\partial_l^2 j_3 dx, \quad I_6 = -2\sum_{k,l=1}^{d}\int_{\mathbb{R}^d}\partial_k j_2\partial_k b_1\partial_l^2 j_3 dx \tag{2.50c}$$

Summing up I_1 and I_2 simplifies to compute I_1 and I_3, as follows:

$$\begin{aligned} I_1 + I_3 =& 2\sum_{k,l=1}^{d}\int_{\mathbb{R}^d}\partial_k b_3\partial_k j_2\partial_l^2 j_1 dx - \int_{\mathbb{R}^d}\partial_k b_3\partial_k j_1\partial_l^2 j_2) dx \\ =& \sum_{p=1}^{8} I_{1,3,p} \end{aligned} \tag{2.51}$$

We can simplify (2.51)

$$I_{1,3,1} = -2\sum_{k,l=1}^{d}\int_{\mathbb{R}^d}\partial_k b_3\partial_k\partial_1 b_3\partial_l^2\partial_2 b_3 dx, \; I_{1,3,2} = 2\sum_{k,l=1}^{d}\int_{\mathbb{R}^d}\partial_k b_3\partial_k\partial_1 b_3\partial_l^2\partial_3 b_2 dx, \tag{2.52a}$$

$$I_{1,3,3} = 2\sum_{k,l=1}^{d}\int_{\mathbb{R}^d}\partial_k b_3\partial_k\partial_3 b_1\partial_l^2\partial_2 b_3 dx, \quad I_{1,3,4} = -2\sum_{k,l=1}^{d}\int_{\mathbb{R}^d}\partial_k b_3\partial_k\partial_3 b_1\partial_l^2\partial_3 b_2 dx, \tag{2.52b}$$

$$I_{1,3,5} = 2\sum_{k,l=1}^{d}\int_{\mathbb{R}^d}\partial_k b_3\partial_k\partial_2 b_3\partial_l^2\partial_1 b_3 dx, \quad I_{1,3,6} = -2\sum_{k,l=1}^{d}\int_{\mathbb{R}^d}\partial_k b_3\partial_k\partial_2 b_3\partial_l^2\partial_3 b_1 dx, \tag{2.52c}$$

$$I_{1,3,7} = -2\sum_{k,l=1}^{d}\int_{\mathbb{R}^d}\partial_k b_3\partial_k\partial_3 b_2\partial_l^2\partial_1 b_3 dx, \quad I_{1,3,8} = 2\sum_{k,l=1}^{d}\int_{\mathbb{R}^d}\partial_k b_3\partial_k\partial_3 b_2\partial_l^2\partial_3 b_1 dx. \tag{2.52d}$$

We find a cancellation by summing up $I_{1,3,1}$ and $I_{1,3,5}$:

$$\begin{aligned} I_{1,3,1} + I_{1,3,5} =& 2\sum_{k,l=1}^{d} \int_{\mathbb{R}^d} -\partial_k b_3 \partial_k \partial_1 b_3 \partial_l^2 \partial_2 b_3 dx + \partial_k b_3 \partial_k \partial_2 b_3 \partial_l^2 \partial_1 b_3 dx \\ =& 2\sum_{k,l=1}^{d} \int_{\mathbb{R}^d} \frac{1}{2}\partial_1 |\partial_k b_3|^2 \partial_l^2 \partial_2 b_3 - \frac{1}{2}\partial_2 |\partial_k b_3|^2 \partial_l^2 \partial_1 b_3 dx \\ =& \sum_{k,l=1}^{d} \int_{\mathbb{R}^d} |\partial_k b_3|^2 \partial_l^2 \partial_1 \partial_2 b_3 - |\partial_k b_3|^2 \partial_l^2 \partial_2 \partial_1 b_3 dx = 0. \end{aligned} \tag{2.53}$$

Summing up I_2 and I_5 simplifies the computations in computing I_2 and I_5, and also paves the path to finding a cancellation. We left the details in [98]

Next, summing up I_4 and I_6 simplifies to compute I_4 and I_6. It also paves the path to find a cancellation. We left the details in [98].

Leaving details in [98], we can similarly compute $II = \sum_{k,l=1}^{d} \int_{\mathbb{R}^d} (j \times \partial_k^2 b) \cdot \partial_l^2 j dx$.

Combining all the estimates, we can show that

$$\int_{\mathbb{R}^3} \Delta \nabla \times (j \times b) \cdot \Delta b dx \lesssim \int_{\mathbb{R}^3} |\nabla^2 b_h| (|\nabla b| |\nabla^3 b| + |\nabla^2 b_v| |\nabla^2 b|) dx. \tag{2.54}$$

$$\int_{\mathbb{R}^2} \Delta \nabla \times (j \times b) \cdot \Delta b dx \lesssim \int_{\mathbb{R}^2} (|\nabla b_h| |\nabla^3 b_h| + |\nabla^2 b_h|^2) |\nabla^2 b_v| dx. \tag{2.55}$$

The details can be found in [98]. □

2.5 Proof of Theorem 5

Since the proof of the Theorem 5 is available in previous work [98] with Prof. Kazuo Yamazaki, thus one can find the detailed proof there. Here it is presented in a sketch for the convenience of readers.

Proposition 10. *Under the hypothesis of Theorem 5, suppose that* (u, b) *is a local strong solution to the 3-D Hall-MHD system* (2.1) *over* $[0, T)$. *Then*

$$u, b \in L_t^\infty H_x^1 \cap L_t^2 H_x^2.$$

Proof of Proposition 10. Taking $L^2(\mathbb{R}^3)$-inner products on (2.1a) with $-\Delta u$ and (2.1b) with $-\Delta b$. Adding them up to compute as follows:

$$\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)+\|\Delta u\|_{L^2}^2+\|\Delta b\|_{L^2}^2=-\sum_{m=1}^{3}\int_{\mathbb{R}^3}(\partial_m u\cdot\nabla)u\cdot\partial_m u dx$$
$$-\sum_{m=1}^{3}\int_{\mathbb{R}^3}(\partial_m u\cdot\nabla)b\cdot\partial_m b dx+\sum_{m=1}^{3}\int_{\mathbb{R}^3}(\partial_m b\cdot\nabla)b\cdot\partial_m u dx$$
$$+\sum_{m=1}^{3}\int_{\mathbb{R}^3}(\partial_m b\cdot\nabla)u\cdot\partial_m b dx+\int_{\mathbb{R}^3}\nabla\times(j\times b)\cdot\Delta b dx=\sum_{l=1}^{5}I_l \quad (2.56)$$

We compute I_1 as follows:

$$I_1=\sum_{n=1}^{2}\sum_{m=1}^{3}\int_{\mathbb{R}^3}\partial_m u_n\partial_n u_n\partial_m u_n dx+\sum_{n=1}^{2}\sum_{m=1}^{3}\int_{\mathbb{R}^3}\partial_m u_3\partial_3 u_n\partial_m u_n dx$$
$$+\sum_{m=1}^{3}\int_{\mathbb{R}^3}\partial_m u_3\partial_3 u_3\partial_m u_3 dx \quad (2.57)$$

As a consequence of (2.57), we can bound I_1 as follows:

$$I_1\lesssim\int_{\mathbb{R}^3}|u_h||\nabla u||\nabla^2 u|dx\lesssim\|u_h\|_{L^{p_1}}\|\nabla u\|_{L^{\frac{2p_1}{p_1-2}}}\|\Delta u\|_{L^2}\leq C\|u_h\|_{L^{p_1}}^{\frac{2p_1}{p_1-3}}\|\nabla u\|_{L^2}^2$$
$$+\frac{1}{8}\|\Delta u\|_{L^2}^2, \quad (2.58)$$

in which we applied the divergence-free condition, integration by parts, Hölder's inequality, Gagliardo-Nirenberg interpolation, and Young's inequalities.

We can simplify and estimate I_2 as follows:

$$I_2\lesssim\|u_h\|_{L^{p_1}}\|\nabla b\|_{L^{\frac{2p_1}{p_1-2}}}\|\Delta b\|_{L^2}+\|b_h\|_{L^\infty}(\|\Delta u\|_{L^2}+\|\Delta b\|_{L^2})(\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2})$$
$$\leq C(\|u_h\|_{L^{p_1}}^{\frac{2p_1}{p_1-3}}+\|\nabla^2 b_h\|_{L^{p_2}}^{\frac{6p_2}{7p_2-6}})(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)+\frac{1}{8}(\|\Delta u\|_{L^2}^2+\|\Delta b\|_{L^2}^2), \quad (2.59)$$

in which we applied integration by parts, Hölder's inequality, Gagliardo-Nirenberg interpolation, and Young's inequalities.

Similarly, we can estimate I_3 and I_4, as follows:

$$I_3+I_4 \leq \frac{1}{8}(\|\Delta u\|_{L^2}^2+\|\Delta b\|_{L^2}^2)+C(\|u_h\|_{L^{p_1}}^{\frac{2p_1}{p_1-3}}+\|\nabla^2 b_h\|_{L^{p_2}}^{\frac{6p_2}{7p_2-6}})(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2). \quad (2.60)$$

Leaving details in [98], using (2.54), and almost similar computations give a bound I_5, as follows:

$$I_5 \lesssim \|\nabla b\|_{L^6}\|\nabla b_h\|_{L^{\frac{6p_2}{5p_2-6}}}\|\nabla^2 b_h\|_{L^{p_2}} \leq \frac{1}{6}\|\Delta b\|_{L^2}^2 + C\|\nabla b\|_{L^2}^2\|\nabla^2 b_h\|_{L^{p_2}}^{\frac{2p_2}{2p_2-3}}. \quad (2.61)$$

Combining these estimates (2.58), (2.59), (2.60), and (2.61), from (2.56) we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\|\nabla b\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 \quad (2.62)$$

$$\leq \frac{1}{2}(\|\Delta b\|_{L^2}^2 + \|\Delta u\|_{L^2}^2)$$

$$+ C(\|u_h\|_{L^{p_1}}^{\frac{2p_1}{p_1-3}} + \|\nabla^2 b_h\|_{L^{p_2}}^{\frac{2p_2}{2p_2-3}} + \|\nabla^2 b_h\|_{L^{p_2}}^{\frac{6p_2}{7p_2-6}})(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2). \quad (2.63)$$

In order to complete the proof, we use Grönwall's inequality and the assumption of the theorem. The details are left in [98]. □

Proposition 11. *Under the hypothesis of Theorem 5, suppose that* (u, b) *is a local strong solution to the 3-D Hall-MHD system* (2.1) *over* $[0, T)$. *Then*

$$u, b \in L_t^\infty H_x^2 \cap L_t^2 H_x^3.$$

Proof of Proposition 11. Applying Δ on the Hall-MHD system (2.1) and taking $L^2(\mathbb{R}^3)$-inner products on (2.1a) with Δu and (2.1b) with Δb. Adding them up to compute as follows:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\Delta u\|_{L^2}^2+\|\Delta b\|_{L^2}^2) + \|\Delta\nabla u\|_{L^2}^2 + \|\Delta\nabla b\|_{L^2}^2 = &-\int_{\mathbb{R}^3}\Delta[(u\cdot\nabla)u]\cdot\Delta u dx\\ &+\int_{\mathbb{R}^3}\Delta[(b\cdot\nabla)b]\cdot\Delta u dx - \int_{\mathbb{R}^3}\Delta[(u\cdot\nabla)b]\cdot\Delta b dx\\ &+\int_{\mathbb{R}^3}\Delta[(b\cdot\nabla)u]\cdot\Delta b dx - \int_{\mathbb{R}^3}\Delta\nabla\times(j\times b)\cdot\Delta b dx. \quad (2.64)\end{aligned}$$

Using Hölder's inequality, the Sobolev embedding $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, the Kato-Ponce commutator estimate [73], and Young's inequality, we have as follows:

$$\frac{d}{dt}(\|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2) + \|\Delta \nabla u\|_{L^2}^2 + \|\Delta \nabla b\|_{L^2}^2$$
$$\leq C(\|\Delta u\|_{L^2} + \|\Delta b\|_{L^2} + \|\nabla^2 b_h\|_{L^{p_2}}^{\frac{2p_2}{2p_2-3}})(\|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2). \tag{2.65}$$

Thus one can prove the theorem with Grönwall's inequality and the assumption, and the details are left in [98]. □

2.6 Proof of Theorem 6

As the proof of the Theorem 6 can be found in [98] with Prof. Kazuo Yamazaki, it is presented in a sketch for the convenience of readers.

Proposition 12. *Under the hypothesis of Theorem 6, suppose that b is a local strong solution to the Hall equation* (2.41) *over* $[0, T)$*. Then*

$$b \in L_t^\infty H_x^2, \quad b_h \in L_t^2 H_x^{\frac{7}{2}}, \quad b_v \in L_t^2 H_x^{2+\alpha}. \tag{2.66}$$

Proof of Proposition 12 . Taking $L^2(\mathbb{R}^2)$-inner products on (2.41) with $-b$ implies as follows:

$$\sup_{t\in[0,T]} \|b(t)\|_{L^2} + \int_0^t \|\Lambda^{\frac{3}{2}} b_h\|_{L^2}^2 + \|\Lambda^\alpha b_v\|_{L^2}^2 d\tau \leq \|b_0\|_{L^2}^2. \tag{2.67}$$

Taking $L^2(\mathbb{R}^2)$-inner products on (2.41) with $-\Delta b$ give us identical bound to (2.68), which is as follows:

$$\int_{\mathbb{R}^2} \Delta \nabla \times (j \times b) \cdot \Delta b dx \lesssim \int_{\mathbb{R}^2} (|\nabla b_h||\nabla^3 b_h| + |\nabla^2 b_h|^2)|\nabla^2 b_v| dx. \tag{2.68}$$

Applying Δ to (2.41), and taking $L^2(\mathbb{R}^2)$-inner products on (2.41) with Δb.

Next, using (2.68), Hölder's inequality, $\dot{H}^{\frac{1}{2}}(\mathbb{R}^2) \hookrightarrow L^4(\mathbb{R}^2)$, Gagliardo-Nirenberg interpolation and Young's inequalities help to have a following bound

$$\frac{1}{2}\frac{d}{dt}\|\Delta b\|_{L^2}^2 + \|\Lambda^{\frac{7}{2}} b_h\|_{L^2}^2 + \|\Lambda^{2+\alpha} b_v\|_{L^2}^2 \lesssim \|\nabla b_h\|_{L^4}\|\nabla^3 b_h\|_{L^4}\|\Delta b\|_{L^2} + \|\nabla^2 b_h\|_{L^4}^2 \|\Delta b\|_{L^2}$$
$$\lesssim \|\Lambda^{\frac{3}{2}} b_h\|_{L^2}\|\Lambda^{\frac{7}{2}} b_h\|_{L^2}\|\Delta b\|_{L^2} \leq C\|\Lambda^{\frac{3}{2}} b_h\|_{L^2}^2 \|\Delta b\|_{L^2}^2 + \frac{1}{2}\|\Lambda^{\frac{7}{2}} b_h\|_{L^2}^2. \tag{2.69}$$

The proof is completed by leaving the details in [98]. □

Since we just finished the proof of $H^3(\mathbb{R}^2)$-bound, thus we need the $H^3(\mathbb{R}^2)$-bound of the solution b to the electron MHD system (2.41). The following computations show that how we attain $H^3(\mathbb{R}^2)$-bound:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|b\|_{\dot{H}^3}^2 + \|\Lambda^{\frac{3}{2}}b_h\|_{\dot{H}^3}^2 + \|\Lambda^{\alpha}b_v\|_{\dot{H}^3}^2 &= -\sum_{p,k,l=1}^{2}\int_{\mathbb{R}^2}\partial_p\partial_k\partial_l(j\times b)\cdot\partial_p\partial_k\partial_l j\,dx \\ &= \sum_{p,k,l=1}^{2}\int_{\mathbb{R}^2}[\partial_k\partial_l j\times\partial_p b + \partial_p\partial_l j\times\partial_k b + \partial_p\partial_k j\times\partial_l b + j\times\partial_p\partial_k\partial_l b]\cdot\partial_p\partial_k\partial_l j\,dx \\ &- \sum_{p,k,l=1}^{2}\int_{\mathbb{R}^2}[\partial_l j\times\partial_p\partial_k b + \partial_k j\times\partial_p\partial_l b + \partial_p j\times\partial_k\partial_l b]\cdot\partial_p\partial_k\partial_l j\,dx. \end{aligned} \tag{2.70}$$

Using a standard inequality, Hölder's and Gagliardo-Nirenberg interpolation inequalities, various Sobolev embedding complete the proof of Theorem 6 due to Proposition 12.

2.7 Proof of Theorem 7

The proof of the theorem can be found in [97] with Prof. Kazuo Yamazaki, thus one can find the detailed proof there. Here it is presented in a sketch for the convenience of readers.

Proof. Taking $L^2(\mathbb{R}^2)$-inner products on (2.44a) with $-u$ and (2.44b) with $-b$. Adding them up to compute as follows:

$$\sup_{t\in[0,T]}(\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2) + \int_0^t \|u_h\|_{\dot{H}^\alpha}^2 + \|u_v\|_{\dot{H}^1}^2 + \|b_h\|_{\dot{H}^{\frac{3}{2}}}^2 + \|b_v\|_{\dot{H}^\alpha}^2 d\tau \le \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2. \tag{2.71}$$

Proposition 13. *Under the hypothesis of Theorem 7, suppose that* (u, b) *is a local strong solution to* (2.44) *over* $[0, T)$. *Then*

$$u \in L_t^\infty \dot{H}_x^2,\ u_h \in L_t^2 H_x^{2+\alpha},\ u_v \in L_t^2 \dot{H}_x^3, \tag{2.72a}$$

$$b \in L_t^\infty \dot{H}_x^2,\ b_h \in L_t^2 H_x^{\frac{7}{2}},\quad b_v \in L_t^2 \dot{H}_x^{2+\alpha}. \tag{2.72b}$$

Recall the following identity:

$$\nabla \times (j \times b) = \nabla \times \left[(b \cdot \nabla) b - \nabla (\frac{|b|^2}{2}) \right] = \nabla \times ((b \cdot \nabla) b). \tag{2.73}$$

In [98], we can see the following cancellation:

$$(\omega \cdot \nabla) u_3 = \omega_1 \partial_1 u_3 + \omega_2 \partial_2 u_3 = \partial_2 u_3 \partial_1 u_3 + (-\partial_1 u_3) \partial_2 u_3 = 0. \tag{2.74}$$

Assuming the third component of $z = \omega + b$, third component of vorticity formulation of (2.44a), and using (2.73) (2.74), the equation can be written as follows:

$$\partial_t z_3 + (u \cdot \nabla) z_3 - (b \cdot \nabla) u_3 + (-\Delta)^\alpha z_3 = 0. \tag{2.75}$$

Taking $L^2(\mathbb{R}^2)$-inner products on (2.75) with z_3 implies the following bound:

$$\frac{d}{dt} \|z_3\|_{L^2}^2 + 2 \|z_3\|_{\dot{H}^\alpha}^2 \lesssim (\|b_h\|_{H^{\frac{3}{2}}}^2 + \|\nabla u_3\|_{L^2}^2) \|z_3\|_{L^2}, \tag{2.76}$$

in which we applied Hölder's inequality, the Sobolev embedding of $H^{\frac{3}{2}}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$, and Young's inequality. Since $\omega_v = \nabla \times u_h$, (2.76) implies $u_h \in L_t^\infty H_x^1 \cap L_t^2 H_x^{1+\alpha}$. For the rest of the parts (more precisely, H^1, H^2, and H^3-bound) if we compute similarly to Theorem 7, we can prove the theorem by leaving details in [98]. □

2.8 Proof of Theorem 8

Proof. We take the $L^2(\mathbb{R}^3)$-inner products on (2.45) with $-\Delta b$ imply that

$$\frac{1}{2} \frac{d}{dt} \|\nabla b\|_{L^2}^2 + \|\Delta b\|_{L^2}^2 = \int_{\mathbb{R}^3} \nabla \times (j \times b) \cdot \Delta b dx$$

$$= \int_{\mathbb{R}^3} (b\cdot\nabla)j\cdot\Delta b dx - \int_{\mathbb{R}^3} (j\cdot\nabla)b\cdot\Delta b dx$$
$$= I_1 + I_2 \tag{2.77}$$

We simplify I_1 in (2.77) and apply integration by parts to compute as follows:

$$\begin{aligned} I_1 = \int_{\mathbb{R}^3} (b\cdot\nabla)j\cdot\Delta b dx &= \int_{\mathbb{R}^3} (b\cdot\nabla)j\cdot\Delta b dx \\ &= -\sum_{k=1}^{3}\int_{\mathbb{R}^3} (\partial_k b\cdot\nabla)j\cdot\partial_k b dx - \sum_{k=1}^{3}\int_{\mathbb{R}^3}(b\cdot\nabla)\partial_k j\cdot\partial_k b dx \\ &= I_{1,1} + I_{1,2} \end{aligned} \tag{2.78}$$

We apply Hölder's inequality, Gagliardo-Nirenberg interpolation, and Young's inequalities to estimate $I_{1,1}$ in (2.78),

$$\begin{aligned} I_{1,1} &= -\sum_{k=1}^{3}\int_{\mathbb{R}^3} (\partial_k b_h\cdot\nabla_h)j\cdot\partial_k b + \partial_k b_3(\partial_3 j_h\cdot\partial_k b_h) + \partial_k b_3\partial_3 j_3\partial_k b_3 dx \\ &= -\sum_{k=1}^{3}\int_{\mathbb{R}^3} (\partial_k b_h\cdot\nabla_h)j\cdot\partial_k b + \partial_k b_3(\partial_3 j_1\partial_k b_1 + \partial_3 j_2\partial_k b_2) + \partial_k b_3\partial_3 j_3\partial_k b_3 dx \\ &\le \|\nabla b_h\|_{L^p}\|\nabla b\|_{L^{\frac{2p}{p-2}}}\|\nabla j\|_{L^2} - \sum_{k=1}^{3}\int_{\mathbb{R}^3}\partial_k\partial_3 b_3 j_3\partial_k b_3 dx \\ &= \|\nabla b_h\|_{L^p}\|\nabla b\|_{L^{\frac{2p}{p-2}}}\|\nabla j\|_{L^2} - \sum_{k=1}^{3}\int_{\mathbb{R}^3}\partial_k\partial_3 b_3(\partial_1 b_2 - \partial_2 b_1)\partial_k b_3 dx \\ &= \|\nabla b_h\|_{L^p}\|\nabla b\|_{L^{\frac{2p}{p-2}}}\|\nabla j\|_{L^2} + \|\nabla b_h\|_{L^p}\|\nabla b\|_{L^{\frac{2p}{p-2}}}\|\Delta b\|_{L^2} \\ &\lesssim \|\nabla b_h\|_{L^p}\|\nabla b\|_{L^2}^{\frac{p-3}{p}}\|\nabla j\|_{L^2}^{\frac{3+p}{p}} \\ &\lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}}\|\nabla b\|_{L^2}^2 + \frac{1}{8}\|\nabla j\|_{L^2}^2 \end{aligned} \tag{2.79}$$

We simplify $I_{1,2}$ in (2.78),

$$I_{1,2} = -\sum_{k=1}^{3}\int_{\mathbb{R}^3}(b\cdot\nabla)\partial_k j\cdot\partial_k b dx$$

$$
\begin{aligned}
&= -\sum_{k=1}^{3}\int_{\mathbb{R}^3}(b\cdot\nabla)(\nabla\times\partial_k b)\cdot\partial_k b dx\\
&= -\sum_{k=1}^{3}\int_{\mathbb{R}^3}\left((b_1\partial_1+b_2\partial_2+b_3\partial_3)\begin{pmatrix}\partial_k\partial_2 b_3-\partial_k\partial_3 b_2\\ -\partial_k\partial_1 b_3+\partial_k\partial_3 b_1\\ \partial_k\partial_1 b_2-\partial_k\partial_2 b_1\end{pmatrix}\right)\cdot\begin{pmatrix}\partial_k b_1\\ \partial_k b_2\\ \partial_k b_3\end{pmatrix}dx\\
&= -\sum_{k=1}^{3}\int_{\mathbb{R}^3}\begin{pmatrix}(b_1\partial_1+b_2\partial_2+b_3\partial_3)(\partial_k\partial_2 b_3-\partial_k\partial_3 b_2)\\ (b_1\partial_1+b_2\partial_2+b_3\partial_3)(-\partial_k\partial_1 b_3+\partial_k\partial_3 b_1)\\ (b_1\partial_1+b_2\partial_2+b_3\partial_3)(\partial_k\partial_1 b_2-\partial_k\partial_2 b_1)\end{pmatrix}\cdot\begin{pmatrix}\partial_k b_1\\ \partial_k b_2\\ \partial_k b_3\end{pmatrix}dx\\
&= -\sum_{k=1}^{3}\int_{\mathbb{R}^3}(b_1\partial_1\partial_k\partial_2 b_3\partial_k b_1+b_2\partial_2\partial_k\partial_2 b_3\partial_k b_1+b_3\partial_3\partial_k\partial_2 b_3\partial_k b_1-b_1\partial_1\partial_k\partial_3 b_2\partial_k b_1\\
&-b_2\partial_2\partial_k\partial_3 b_2\partial_k b_1-b_3\partial_3\partial_k\partial_3 b_2\partial_k b_1-b_1\partial_1\partial_k\partial_1 b_3\partial_k b_2-b_2\partial_2\partial_k\partial_1 b_3\partial_k b_2\\
&-b_3\partial_3\partial_k\partial_1 b_3\partial_k b_2+b_1\partial_1\partial_k\partial_3 b_1\partial_k b_2+b_2\partial_2\partial_k\partial_3 b_1\partial_k b_2+b_3\partial_3\partial_k\partial_3 b_1\partial_k b_2\\
&+b_1\partial_1\partial_k\partial_1 b_2\partial_k b_3+b_2\partial_2\partial_k\partial_1 b_2\partial_k b_3+b_3\partial_3\partial_k\partial_1 b_2\partial_k b_3-b_1\partial_1\partial_k\partial_2 b_1\partial_k b_3\\
&-b_2\partial_2\partial_k\partial_2 b_1\partial_k b_3-b_3\partial_3\partial_k\partial_2 b_1\partial_k b_3)dx\\
&=\sum_{i=1}^{18} I_{1,2,i} \qquad (2.80)
\end{aligned}
$$

The strategy is to pair up the two terms $I_{1,2,1}, I_{1,2,16}$ in (2.80) and integrate by parts to deduce

$$
\begin{aligned}
&I_{1,2,1}+I_{1,2,16} = -\sum_{k=1}^{3}\int_{\mathbb{R}^3} b_1\partial_1\partial_k\partial_2 b_3\partial_k b_1 - b_1\partial_1\partial_k\partial_2 b_1\partial_k b_3 dx\\
&= -\sum_{k=1}^{3}\int_{\mathbb{R}^3} b_1\partial_1\partial_k\partial_2 b_3\partial_k b_1+\partial_1 b_1\partial_k\partial_2 b_1\partial_k b_3+b_1\partial_k\partial_2 b_1\partial_1\partial_k b_3 dx\\
&= -\sum_{k=1}^{3}\int_{\mathbb{R}^3} b_1\partial_1\partial_k\partial_2 b_3\partial_k b_1+\partial_1 b_1\partial_k\partial_2 b_1\partial_k b_3-\partial_2 b_1\partial_k b_1\partial_1\partial_k b_3-b_1\partial_k b_1\partial_1\partial_2\partial_k b_3 dx\\
&=\sum_{i=1}^{4} I_{1,2,1,16,i} \qquad (2.81)
\end{aligned}
$$

We apply Hölder's inequality, Gagliardo-Nirenberg interpolation, and Young's inequalities to estimate the terms $I_{1,2,1,16,2}, I_{1,2,1,16,3}$ in (2.81)

$$\begin{aligned}
&I_{1,2,1,16,2} + I_{1,2,1,16,3} = \sum_{k=1}^{3} \int_{\mathbb{R}^3} \partial_1 b_1 \partial_k \partial_2 b_1 \partial_k b_3 - \partial_2 b_1 \partial_k b_1 \partial_1 \partial_k b_3 dx \\
&\lesssim \|\nabla b_h\|_{L^p} \|\nabla b\|_{L^{\frac{2p}{p-2}}} \|\nabla j\|_{L^2} \\
&\approx \|\nabla b_h\|_{L^p} \|\nabla b\|_{L^2}^{\frac{p-3}{p}} \|\nabla j\|_{L^2}^{\frac{3+p}{p}} \\
&\lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} \|\nabla b\|_{L^2}^2 + \frac{1}{32} \|\nabla j\|_{L^2}^2. \qquad (2.82)
\end{aligned}$$

We found a cancellation by paring up these two terms $I_{1,2,1,16,1}, I_{1,2,1,16,4}$ in (2.81)

$$I_{1,2,1,16,1} + I_{1,2,1,16,4} = -\sum_{k=1}^{3} \int_{\mathbb{R}^3} b_1 \partial_1 \partial_k \partial_2 b_3 \partial_k b_1 - b_1 \partial_k b_1 \partial_1 \partial_2 \partial_k b_3 dx = 0 \qquad (2.83)$$

The strategy is to pair up the two terms $I_{1,2,2}, I_{1,2,17}$ in (2.80) and integrate by parts to deduce

$$\begin{aligned}
&I_{1,2,2} + I_{1,2,17} = -\sum_{k=1}^{3} \int_{\mathbb{R}^3} b_2 \partial_2 \partial_k \partial_2 b_3 \partial_k b_1 - b_2 \partial_2 \partial_k \partial_2 b_1 \partial_k b_3 dx \\
&= -\sum_{k=1}^{3} \int_{\mathbb{R}^3} b_2 \partial_2 \partial_k \partial_2 b_3 \partial_k b_1 + \partial_2 b_2 \partial_k \partial_2 b_1 \partial_k b_3 + b_2 \partial_k \partial_2 b_1 \partial_2 \partial_k b_3 dx \\
&= -\sum_{k=1}^{3} \int_{\mathbb{R}^3} b_2 \partial_2 \partial_k \partial_2 b_3 \partial_k b_1 + \partial_2 b_2 \partial_k \partial_2 b_1 \partial_k b_3 - \partial_2 b_2 \partial_k b_1 \partial_2 \partial_k b_3 - b_2 \partial_k b_1 \partial_2 \partial_k \partial_2 b_3 dx \\
&= \sum_{i=1}^{4} I_{1,2,2,i} \qquad (2.84)
\end{aligned}$$

Apply Hölder's inequality, Gagliardo-Nirenberg interpolation, and Young's inequalities in order to estimate the terms $I_{1,2,2,2}, I_{1,2,2,3}$ in (2.84)

$$\begin{aligned}
I_{1,2,2,2} + I_{1,2,2,3} &= -\sum_{k=1}^{3} \int_{\mathbb{R}^3} \partial_2 b_2 \partial_k \partial_2 b_1 \partial_k b_3 - \partial_2 b_2 \partial_k b_1 \partial_2 \partial_k b_3 dx \\
&\lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} \|\nabla b\|_{L^2}^2 + \frac{1}{32} \|\nabla j\|_{L^2}^2 \qquad (2.85)
\end{aligned}$$

We find a cancellation by paring up these terms $I_{1,2,2,1}, I_{1,2,2,4}$ in (2.84)

$$I_{1,2,2,1} + I_{1,2,2,4} = -\sum_{k=1}^{3} \int_{\mathbb{R}^3} b_2\partial_2\partial_k\partial_2 b_3\partial_k b_1 - b_2\partial_k b_1\partial_2\partial_k\partial_2 b_3 dx = 0 \tag{2.86}$$

The strategy is to pair up the two terms $I_{1,2,3}, I_{1,2,18}$ in (2.80) and integrate by parts to deduce

$$\begin{aligned}
&I_{1,2,3} + I_{1,2,18} = -\sum_{k=1}^{3} \int_{\mathbb{R}^3} b_3\partial_3\partial_k\partial_2 b_3\partial_k b_1 - b_3\partial_3\partial_k\partial_2 b_1\partial_k b_3 dx \\
&= -\sum_{k=1}^{3} \int_{\mathbb{R}^3} b_3\partial_3\partial_k\partial_2 b_3\partial_k b_1 + \partial_3 b_3\partial_k\partial_2 b_1\partial_k b_3 + b_3\partial_k\partial_2 b_1\partial_k\partial_3 b_3 dx \\
&= -\sum_{k=1}^{3} \int_{\mathbb{R}^3} b_3\partial_3\partial_k\partial_2 b_3\partial_k b_1 + \partial_3 b_3\partial_k\partial_2 b_1\partial_k b_3 - \partial_2 b_3\partial_k b_1\partial_k\partial_3 b_3 - b_3\partial_k b_1\partial_k\partial_2\partial_3 b_3 dx \\
&= \sum_{i=1}^{4} I_{1,2,3,18,i}
\end{aligned} \tag{2.87}$$

We apply the divergence-free condition, Hölder's, Gagliardo-Nirenberg interpolation, and Young's inequalities to estimate the terms $I_{1,2,3,18}, I_{1,2,2,3}$ in (2.87)

$$\begin{aligned}
I_{1,2,3,18,2} + I_{1,2,3,18,3} &= -\sum_{k=1}^{3} \int_{\mathbb{R}^3} \partial_3 b_3\partial_k\partial_2 b_1\partial_k b_3 - \partial_2 b_3\partial_k b_1\partial_k\partial_3 b_3 dx \\
&= -\sum_{k=1}^{3} \int_{\mathbb{R}^3} (-\partial_1 b_1 - \partial_2 b_2)\partial_k\partial_2 b_1\partial_k b_3 - \partial_2 b_3\partial_k b_1\partial_k\partial_3 b_3 dx \\
&\lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} \|\nabla b\|_{L^2}^2 + \frac{1}{32}\|\nabla j\|_{L^2}^2
\end{aligned} \tag{2.88}$$

We find a cancellation by paring up these terms $I_{1,2,3,18,1}, I_{1,2,3,18,4}$ in (2.87)

$$I_{1,2,3,18,1} + I_{1,2,3,18,4} = -\sum_{k=1}^{3} \int_{\mathbb{R}^3} b_3\partial_3\partial_k\partial_2 b_3\partial_k b_1 - b_3\partial_k b_1\partial_k\partial_2\partial_3 b_3 dx = 0 \tag{2.89}$$

The strategy is to pair up the two terms $I_{1,2,4}, I_{1,2,10}$ in (2.80) and integrate by parts to deduce

$$I_{1,2,4} + I_{1,2,10} = \sum_{k=1}^{3} \int_{\mathbb{R}^3} b_1 \partial_1 \partial_k \partial_3 b_2 \partial_k b_1 - b_1 \partial_1 \partial_k \partial_3 b_1 \partial_k b_2 dx$$
$$= \sum_{k=1}^{3} \int_{\mathbb{R}^3} b_1 \partial_1 \partial_k \partial_3 b_2 \partial_k b_1 + \partial_1 b_1 \partial_k \partial_3 b_1 \partial_k b_2 + b_1 \partial_k \partial_3 b_1 \partial_1 \partial_k b_2 dx$$
$$= \sum_{k=1}^{3} \int_{\mathbb{R}^3} b_1 \partial_1 \partial_k \partial_3 b_2 \partial_k b_1 + \partial_1 b_1 \partial_k \partial_3 b_1 \partial_k b_2 - \partial_3 b_1 \partial_k b_1 \partial_1 \partial_k b_2 - b_1 \partial_k b_1 \partial_3 \partial_1 \partial_k b_2 dx$$
$$= \sum_{i=1}^{4} I_{1,2,4,10,i} \tag{2.90}$$

Apply Hölder's inequality, Gagliardo-Nirenberg interpolation, and Young's inequalities in order to estimate the terms $I_{1,2,4,10,2}, I_{1,2,4,10,3}$ in (2.90)

$$I_{1,2,4,10,2} + I_{1,2,4,10,3} = \sum_{k=1}^{3} \int_{\mathbb{R}^3} \partial_1 b_1 \partial_k \partial_3 b_1 \partial_k b_2 - \partial_3 b_1 \partial_k b_1 \partial_1 \partial_k b_2 dx$$
$$\lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} \|\nabla b\|_{L^2}^2 + \frac{1}{32} \|\nabla j\|_{L^2}^2 \tag{2.91}$$

We find a cancellation by paring up these terms $I_{1,2,4,10,1}, I_{1,2,4,10,4}$ in (2.87)

$$I_{1,2,4,10,1} + I_{1,2,4,10,4} = \sum_{k=1}^{3} \int_{\mathbb{R}^3} b_1 \partial_1 \partial_k \partial_3 b_2 \partial_k b_1 - b_1 \partial_k b_1 \partial_3 \partial_1 \partial_k b_2 dx = 0 \tag{2.92}$$

The strategy is to pair up the two terms $I_{1,2,5}, I_{1,2,11}$ in (2.80) and integrate by parts to deduce

$$I_{1,2,5} + I_{1,2,11} = \sum_{k=1}^{3} \int_{\mathbb{R}^3} b_2 \partial_2 \partial_k \partial_3 b_2 \partial_k b_1 - b_2 \partial_2 \partial_k \partial_3 b_1 \partial_k b_2 dx$$
$$= \sum_{k=1}^{3} \int_{\mathbb{R}^3} b_2 \partial_2 \partial_k \partial_3 b_2 \partial_k b_1 + \partial_2 b_2 \partial_k \partial_3 b_1 \partial_k b_2 + b_2 \partial_k \partial_3 b_1 \partial_k \partial_2 b_2 dx$$
$$= \sum_{k=1}^{3} \int_{\mathbb{R}^3} b_2 \partial_2 \partial_k \partial_3 b_2 \partial_k b_1 + \partial_2 b_2 \partial_k \partial_3 b_1 \partial_k b_2 - \partial_3 b_2 \partial_k b_1 \partial_k \partial_2 b_2 - b_2 \partial_k b_1 \partial_k \partial_2 \partial_3 b_2 dx$$

$$= \sum_{i=1}^{4} I_{1,2,5,11,i} \tag{2.93}$$

Apply Hölder's inequality, Gagliardo-Nirenberg interpolation, and Young's inequalities in order to estimate the terms $I_{1,2,5,11,2}, I_{1,2,5,11,3}$ in (2.93)

$$I_{1,2,5,11,2} + I_{1,2,5,11,3} \lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} \|\nabla b\|_{L^2}^2 + \frac{1}{32}\|\nabla j\|_{L^2}^2 \tag{2.94}$$

We find a cancellation by paring up these terms $I_{1,2,5,11,1}, I_{1,2,5,11,4}$ in (2.93)

$$I_{1,2,5,11,1} + I_{1,2,5,11,4} = \int_{\mathbb{R}^3} b_2\partial_2\partial_k\partial_3 b_2\partial_k b_1 - b_2\partial_k b_1\partial_k\partial_2\partial_3 b_2 dx = 0 \tag{2.95}$$

The strategy is to pair up the two terms $I_{1,2,6}, I_{1,2,12}$ in (2.80) and integrate by parts to deduce

$$\begin{aligned}
&I_{1,2,6} + I_{1,2,12} = \sum_{k=1}^{3} \int_{\mathbb{R}^3} b_3\partial_3\partial_k\partial_3 b_2\partial_k b_1 - b_3\partial_3\partial_k\partial_3 b_1\partial_k b_2 dx \\
&= \int_{\mathbb{R}^3} b_3\partial_3\partial_k\partial_3 b_2\partial_k b_1 + \partial_3 b_3\partial_k\partial_3 b_1\partial_k b_2 + b_3\partial_k\partial_3 b_1\partial_k\partial_3 b_2 dx \\
&= \sum_{k=1}^{3} \int_{\mathbb{R}^3} b_3\partial_3\partial_k\partial_3 b_2\partial_k b_1 + \partial_3 b_3\partial_k\partial_3 b_1\partial_k b_2 - \partial_3 b_3\partial_k b_1\partial_k\partial_3 b_2 - b_3\partial_k b_1\partial_3\partial_k\partial_3 b_2 dx \\
&= \sum_{i=1}^{4} I_{1,2,6,12,i}
\end{aligned} \tag{2.96}$$

We apply the divergence-free condition, Hölder's inequality, Gagliardo-Nirenberg interpolation, and Young's inequalities to estimate the terms $I_{1,2,6,12,2}, I_{1,2,6,12,3}$ in (2.96)

$$I_{1,2,6,12,2} + I_{1,2,6,12,3} \lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} \|\nabla b\|_{L^2}^2 + \frac{1}{32}\|\nabla j\|_{L^2}^2 \tag{2.97}$$

We find a cancellation by paring up these terms $I_{1,2,6,12,1}, I_{1,2,6,12,4}$ in (2.96)

$$I_{1,2,6,12,1} + I_{1,2,6,12,4} = \sum_{k=1}^{3} \int_{\mathbb{R}^3} b_3\partial_3\partial_k\partial_3 b_2\partial_k b_1 - b_3\partial_k b_1\partial_3\partial_k\partial_3 b_2 dx = 0 \tag{2.98}$$

The strategy is to pair up the two terms $I_{1,2,7}, I_{1,2,13}$ in (2.80) and integrate by parts to deduce

$$I_{1,2,7} + I_{1,2,13} = \sum_{k=1}^{3} \int_{\mathbb{R}^3} b_1 \partial_1 \partial_k \partial_1 b_3 \partial_k b_2 - b_1 \partial_1 \partial_k \partial_1 b_2 \partial_k b_3 dx$$
$$= \sum_{k=1}^{3} \int_{\mathbb{R}^3} b_1 \partial_1 \partial_k \partial_1 b_3 \partial_k b_2 + \partial_1 b_1 \partial_k \partial_1 b_2 \partial_k b_3 + b_1 \partial_k \partial_1 b_2 \partial_k \partial_1 b_3 dx$$
$$= \sum_{k=1}^{3} \int_{\mathbb{R}^3} b_1 \partial_1 \partial_k \partial_1 b_3 \partial_k b_2 + \partial_1 b_1 \partial_k \partial_1 b_2 \partial_k b_3 + b_1 \partial_k \partial_1 b_2 \partial_k \partial_1 b_3 dx$$
$$= \sum_{k=1}^{3} \int_{\mathbb{R}^3} b_1 \partial_1 \partial_k \partial_1 b_3 \partial_k b_2 + \partial_1 b_1 \partial_k \partial_1 b_2 \partial_k b_3 - \partial_1 b_1 \partial_k b_2 \partial_k \partial_1 b_3 - b_1 \partial_k b_2 \partial_1 \partial_k \partial_1 b_3 dx$$
$$= \sum_{i=1}^{4} I_{1,2,7,13,i} \tag{2.99}$$

Apply Hölder's inequality, Gagliardo-Nirenberg interpolation, and Young's inequalities in order to estimate the terms $I_{1,2,7,13,2}, I_{1,2,7,13,3}$ in (2.99)

$$I_{1,2,7,13,2} + I_{1,2,7,13,3} \lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} \|\nabla b\|_{L^2}^2 + \frac{1}{32}\|\nabla j\|_{L^2}^2 \tag{2.100}$$

We find a cancellation by paring up these terms $I_{1,2,7,13,1}, I_{1,2,7,13,4}$ in (2.99)

$$I_{1,2,7,13,1} + I_{1,2,7,13,4} = \sum_{k=1}^{3} \int_{\mathbb{R}^3} b_1 \partial_1 \partial_k \partial_1 b_3 \partial_k b_2 - \int_{\mathbb{R}^3} b_1 \partial_k b_2 \partial_1 \partial_k \partial_1 b_3 dx = 0 \tag{2.101}$$

The strategy is to pair up the two terms $I_{1,2,8}, I_{1,2,14}$ in (2.80) and integrate by parts to deduce

$$I_{1,2,8} + I_{1,2,14} = \sum_{k=1}^{3} \int_{\mathbb{R}^3} b_2 \partial_2 \partial_k \partial_1 b_3 \partial_k b_2 - b_2 \partial_2 \partial_k \partial_1 b_2 \partial_k b_3 dx$$
$$= \sum_{k=1}^{3} \int_{\mathbb{R}^3} b_2 \partial_2 \partial_k \partial_1 b_3 \partial_k b_2 + \partial_2 b_2 \partial_k \partial_1 b_2 \partial_k b_3 + b_2 \partial_k \partial_1 b_2 \partial_k \partial_2 b_3 dx$$
$$= \sum_{k=1}^{3} \int_{\mathbb{R}^3} b_2 \partial_2 \partial_k \partial_1 b_3 \partial_k b_2 + \partial_2 b_2 \partial_k \partial_1 b_2 \partial_k b_3 - \partial_1 b_2 \partial_k b_2 \partial_k \partial_2 b_3 - b_2 \partial_k b_2 \partial_1 \partial_k \partial_2 b_3 dx$$

$$= \sum_{i=1}^{4} I_{1,2,8,14,i} \tag{2.102}$$

Apply Hölder's inequality, Gagliardo-Nirenberg interpolation, and Young's inequalities in order to estimate the terms $I_{1,2,8,14,2}, I_{1,2,8,14,3}$ in (2.102)

$$I_{1,2,8,14,2} + I_{1,2,8,14,3} \lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} \|\nabla b\|_{L^2}^2 + \frac{1}{32}\|\nabla j\|_{L^2}^2 \tag{2.103}$$

We find a cancellation by paring up these terms $I_{1,2,8,14,1}, I_{1,2,8,14,4}$ in (2.99)

$$I_{1,2,8,14,1} + I_{1,2,8,14,4} = \sum_{k=1}^{3} \int_{\mathbb{R}^3} b_2 \partial_2 \partial_k \partial_1 b_3 \partial_k b_2 - \int_{\mathbb{R}^3} b_2 \partial_k b_2 \partial_1 \partial_k \partial_2 b_3 = 0 \tag{2.104}$$

The strategy is to pair up the two terms $I_{1,2,9}, I_{1,2,15}$ in (2.80) and integrate by parts to deduce

$$\begin{aligned}
I_{1,2,9} + I_{1,2,15} =& \sum_{k=1}^{3} \int_{\mathbb{R}^3} b_3 \partial_3 \partial_k \partial_1 b_3 \partial_k b_2 - b_3 \partial_3 \partial_k \partial_1 b_2 \partial_k b_3 dx \\
=& \sum_{k=1}^{3} \int_{\mathbb{R}^3} b_3 \partial_3 \partial_k \partial_1 b_3 \partial_k b_2 + \partial_3 b_3 \partial_k \partial_1 b_2 \partial_k b_3 + b_3 \partial_k \partial_1 b_2 \partial_k \partial_3 b_3 dx \\
=& \sum_{k=1}^{3} \int_{\mathbb{R}^3} b_3 \partial_3 \partial_k \partial_1 b_3 \partial_k b_2 + \partial_3 b_3 \partial_k \partial_1 b_2 \partial_k b_3 - \partial_1 b_3 \partial_k b_2 \partial_k \partial_3 b_3 \\
& - b_3 \partial_k b_2 \partial_k \partial_3 \partial_1 b_3 dx \\
=& \sum_{i=1}^{4} I_{1,2,9,15,i}
\end{aligned} \tag{2.105}$$

We apply the divergence-free condition, Hölder's inequality, Gagliardo-Nirenberg interpolation, and Young's inequalities to estimate the terms $I_{1,2,9,15,2}, I_{1,2,9,15,3}$ in (2.105)

$$I_{1,2,9,15,2} + I_{1,2,9,15,3} \lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} \|\nabla b\|_{L^2}^2 + \frac{1}{32}\|\nabla j\|_{L^2}^2 \tag{2.106}$$

We find a cancellation by paring up these terms $I_{1,2,9,15,1}, I_{1,2,9,15,4}$ in (2.105)

$$I_{1,2,9,15,1} + I_{1,2,9,15,4} = \sum_{k=1}^{3} \int_{\mathbb{R}^3} b_3 \partial_3 \partial_k \partial_1 b_3 \partial_k b_2 - \int_{\mathbb{R}^3} b_3 \partial_k b_2 \partial_k \partial_3 \partial_1 b_3 dx = 0 \tag{2.107}$$

We simplify I_2 in (2.77)

$$\begin{aligned}
I_2 &= -\int_{\mathbb{R}^3} (j \cdot \nabla) b \cdot \Delta b dx \\
&= -\sum_{k=1}^{3} \int_{\mathbb{R}^3} \left((j_1 \partial_1 + j_2 \partial_2 + j_3 \partial_3) \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \right) \cdot \begin{pmatrix} \partial_k^2 b_1 \\ \partial_k^2 b_2 \\ \partial_k^2 b_3 \end{pmatrix} dx \\
&= -\sum_{k=1}^{3} \int_{\mathbb{R}^3} \begin{pmatrix} j_1 \partial_1 b_1 + j_2 \partial_2 b_1 + j_3 \partial_3 b_1 \\ j_1 \partial_1 b_2 + j_2 \partial_2 b_2 + j_3 \partial_3 b_2 \\ j_1 \partial_1 b_3 + j_2 \partial_2 b_3 + j_3 \partial_3 b_3 \end{pmatrix} \cdot \begin{pmatrix} \partial_k^2 b_1 \\ \partial_k^2 b_2 \\ \partial_k^2 b_3 \end{pmatrix} dx \\
&= -\sum_{k=1}^{3} \int_{\mathbb{R}^3} j_1 \partial_1 b_1 \partial_k^2 b_1 + j_2 \partial_2 b_1 \partial_k^2 b_1 + j_3 \partial_3 b_1 \partial_k^2 b_1 + j_1 \partial_1 b_2 \partial_k^2 b_2 + j_2 \partial_2 b_2 \partial_k^2 b_2 \\
&+ j_3 \partial_3 b_2 \partial_k^2 b_2 + j_1 \partial_1 b_3 \partial_k^2 b_3 + j_2 \partial_2 b_3 \partial_k^2 b_3 + j_3 \partial_3 b_3 \partial_k^2 b_3 dx \\
&= -\sum_{k=1}^{3} \int_{\mathbb{R}^3} j_1 \partial_1 b_1 \partial_k^2 b_1 + j_2 \partial_2 b_1 \partial_k^2 b_1 + j_3 \partial_3 b_1 \partial_k^2 b_1 + j_1 \partial_1 b_2 \partial_k^2 b_2 + j_2 \partial_2 b_2 \partial_k^2 b_2 \\
&+ j_3 \partial_3 b_2 \partial_k^2 b_2 + (\partial_2 b_3 - \partial_3 b_2) \partial_1 b_3 \partial_k^2 b_3 + (-\partial_1 b_3 + \partial_3 b_1) \partial_2 b_3 \partial_k^2 b_3 + j_3 \partial_3 b_3 \partial_k^2 b_3 dx \\
&= -\sum_{k=1}^{3} \int_{\mathbb{R}^3} j_1 \partial_1 b_1 \partial_k^2 b_1 + j_2 \partial_2 b_1 \partial_k^2 b_1 + j_3 \partial_3 b_1 \partial_k^2 b_1 + j_1 \partial_1 b_2 \partial_k^2 b_2 + j_2 \partial_2 b_2 \partial_k^2 b_2 \\
&+ j_3 \partial_3 b_2 \partial_k^2 b_2 + \partial_2 b_3 \partial_1 b_3 \partial_k^2 b_3 - \partial_3 b_2 \partial_1 b_3 \partial_k^2 b_3 - \partial_1 b_3 \partial_2 b_3 \partial_k^2 b_3 + \partial_3 b_1 \partial_2 b_3 \partial_k^2 b_3 \\
&+ j_3 \partial_3 b_3 \partial_k^2 b_3 dx \\
&= \sum_{i=1}^{11} I_{2,i}
\end{aligned} \tag{2.108}$$

Apply Hölder's, Gagliardo-Nirenberg interpolation, and Young's inequalities in order to estimate the terms $I_{2,1}, I_{2,2}, I_{2,3}, I_{2,4}, I_{2,5}, I_{2,6}, I_{2,8}, I_{2,10}, I_{2,11}$ in (2.108)

$$I_{2,1}, I_{2,2}, I_{2,3}, I_{2,4}, I_{2,5}, I_{2,6}, I_{2,8}, I_{2,10}, I_{2,11} \lesssim \|\nabla b_h\|_{L^p} \|\nabla b\|_{L^{\frac{2p}{p-2}}} \|\nabla j\|_{L^2}$$

$$\begin{aligned}
&\approx \|\nabla b_h\|_{L^p}\|\nabla b\|_{L^2}^{\frac{p-3}{p}}\|\nabla j\|_{L^2}^{\frac{3+p}{p}} \\
&\lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}}\|\nabla b\|_{L^2}^2 + \frac{1}{32}\|\nabla j\|_{L^2}^2 \qquad (2.109)
\end{aligned}$$

We find a cancellation by paring up these terms $I_{2,7}$ and $I_{2,9}$ in (2.108)

$$I_{2,7} + I_{2,9} = -\int_{\mathbb{R}^3} \partial_2 b_3 \partial_1 b_3 \partial_k^2 b_3 dx + \int_{\mathbb{R}^3} \partial_1 b_3 \partial_2 b_3 \partial_k^2 b_3 dx \qquad (2.110)$$

From (2.77)

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla b\|_{L^2}^2 + \|\Delta b\|_{L^2}^2 &= \int_{\mathbb{R}^3} \nabla \times (j \times b) \cdot \Delta b dx \\
&= \int_{\mathbb{R}^3} (b \cdot \nabla) j \cdot \Delta b dx - \int_{\mathbb{R}^3} (j \cdot \nabla) b \cdot \Delta b dx \\
&\lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}}\|\nabla b\|_{L^2}^2 + \frac{1}{2}\|\nabla j\|_{L^2}^2 \qquad (2.111)
\end{aligned}$$

We apply Δ to (2.45)and take $L^2(\mathbb{R}^3)$-inner products with $(\Delta u, \Delta b)$ to obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta b\|_{L^2}^2 + \|\Delta \nabla b\|_{L^2}^2 = \int_{\mathbb{R}^3} \Delta(b \cdot \nabla) j \cdot \Delta b dx - \int_{\mathbb{R}^3} \Delta(j \cdot \nabla) b \cdot \Delta b dx = II_1 + II_2 \qquad (2.112)$$

We simplify II_1 in (2.112)

$$\begin{aligned}
&\int_{\mathbb{R}^3} \Delta(b \cdot \nabla) j \cdot \Delta b dx \\
&= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} \partial_m[(\partial_m b \cdot \nabla) j + (b \cdot \nabla)\partial_m j] \cdot \partial_n^2 b dx \\
&= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} (\partial_m^2 b \cdot \nabla) j + (\partial_m b \cdot \nabla)\partial_m j + (\partial_m b \cdot \nabla)\partial_m j + (\partial_b \cdot \nabla)\partial_m^2 j] \cdot \partial_n^2 b dx \\
&= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} (\partial_m^2 b \cdot \nabla) j + 2(\partial_m b \cdot \nabla)\partial_m j + (b \cdot \nabla)\partial_m^2 j] \cdot \partial_n^2 b dx
\end{aligned}$$

$$= \sum_{i=1}^{3} II_{1,i} \tag{2.113}$$

We simplify $II_{1,1}$ in (2.113)

$$\begin{aligned}
II_{1,1} &= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} (\partial_m^2 b \cdot \nabla) j \cdot \partial_n^2 b dx \\
&= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} \begin{pmatrix} \partial_m^2 b_1 \partial_1 j_1 + \partial_m^2 b_2 \partial_2 j_1 + \partial_m^2 b_3 \partial_3 j_1 \\ \partial_m^2 b_1 \partial_1 j_2 + \partial_m^2 b_2 \partial_2 j_2 + \partial_m^2 b_3 \partial_3 j_2 \\ \partial_m^2 b_1 \partial_1 j_3 + \partial_m^2 b_2 \partial_2 j_3 + \partial_m^2 b_3 \partial_3 j_3 \end{pmatrix} \cdot \begin{pmatrix} \partial_n^2 b_1 \\ \partial_n^2 b_2 \\ \partial_n^2 b_3 \end{pmatrix} dx \\
&= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} \partial_m^2 b_1 \partial_1 j_1 \partial_n^2 b_1 + \partial_m^2 b_2 \partial_2 j_1 \partial_n^2 b_1 + \partial_m^2 b_3 \partial_3 j_1 \partial_n^2 b_1 + \partial_m^2 b_1 \partial_1 j_2 \partial_n^2 b_2 \\
&+ \partial_m^2 b_2 \partial_2 j_2 \partial_n^2 b_2 + \partial_m^2 b_3 \partial_3 j_2 \partial_n^2 b_2 + \partial_m^2 b_1 \partial_1 j_3 \partial_n^2 b_3 + \partial_m^2 b_2 \partial_2 j_3 \partial_n^2 b_3 + \partial_m^2 b_3 \partial_3 j_3 \partial_n^2 b_3 dx \\
&= \sum_{i=1}^{9} II_{1,1,i}
\end{aligned} \tag{2.114}$$

We apply integration by parts, Hölder's inequality, Gagliardo-Nirenberg interpolation, and Young's inequalities to estimate the term $II_{1,1,i}$ in (2.114)

$$\begin{aligned}
II_{1,1,i} &= -\sum_{m,n=1}^{3} \int_{\mathbb{R}^3} \partial_m b_1 \partial_m (\partial_1 j_1 \partial_n^2 b_1) + \partial_m b_2 \partial_m (\partial_2 j_1 \partial_n^2 b_1) + \partial_n (\partial_m^2 b_3 \partial_3 j_1) \partial_n b_1 \\
&+ \partial_n (\partial_m^2 b_1 \partial_1 j_2) \partial_n b_2 + \partial_m b_2 \partial_m (\partial_2 j_2 \partial_n^2 b_2) + \partial_n (\partial_m^2 b_3 \partial_3 j_2) \partial_n b_2 + \partial_m b_1 \partial_m (\partial_1 j_3 \partial_n^2 b_3) \\
&+ \partial_m b_2 \partial_m (\partial_2 j_3 \partial_n^2 b_3) + j_3 \partial_3 (\partial_m^2 b_3 \partial_n^2 b_3) dx \\
&\lesssim \|\nabla b_h\|_{L^p} \|\Delta b\|_{L^{\frac{2p}{p-2}}} \|\Delta j\|_{L^2} \\
&\approx \|\nabla b_h\|_{L^p} \|\Delta b\|_{L^2}^{\frac{p-3}{p}} \|\Delta j\|_{L^2}^{\frac{3+p}{p}} \\
&\lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} \|\Delta b\|_{L^2}^2 + \frac{1}{32} \|\Delta j\|_{L^2}^2
\end{aligned} \tag{2.115}$$

We simplify $II_{1,2}$ in (2.113)

$$II_{1,2} = \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} (\partial_m b \cdot \nabla) \partial_m j \cdot \partial_n^2 b dx$$

$$= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} \begin{pmatrix} \partial_m b_1 \partial_1 \partial_m j_1 + \partial_m b_2 \partial_2 \partial_m j_1 + \partial_m b_3 \partial_3 \partial_m j_1 \\ \partial_m b_1 \partial_1 \partial_m j_2 + \partial_m b_2 \partial_2 \partial_m j_2 + \partial_m b_3 \partial_3 \partial_m j_2 \\ \partial_m b_1 \partial_1 \partial_m j_3 + \partial_m b_2 \partial_2 \partial_m j_3 + \partial_m b_3 \partial_3 \partial_m j_3 \end{pmatrix} \cdot \begin{pmatrix} \partial_n^2 b_1 \\ \partial_n^2 b_2 \\ \partial_n^2 b_3 \end{pmatrix} dx$$

$$= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} (\partial_m b_1 \partial_1 \partial_m j_1 \partial_n^2 b_1 + \partial_m b_2 \partial_2 \partial_m j_1 \partial_n^2 b_1 + \partial_m b_3 \partial_3 \partial_m j_1 \partial_n^2 b_1 + \partial_m b_1 \partial_1 \partial_m j_2 \partial_n^2 b_2$$

$$+ \partial_m b_2 \partial_2 \partial_m j_2 \partial_n^2 b_2 + \partial_m b_3 \partial_3 \partial_m j_2 \partial_n^2 b_2 + \partial_m b_1 \partial_1 \partial_m j_3 \partial_n^2 b_3 + \partial_m b_2 \partial_2 \partial_m j_3 \partial_n^2 b_3$$

$$+ \partial_m b_3 \partial_3 \partial_m j_3 \partial_n^2 b_3) dx$$

$$= \sum_{i=1}^{9} II_{1,2,i} \tag{2.116}$$

Apply Hölder's inequality, Gagliardo-Nirenberg interpolation, and Young's inequalities in order to estimate the terms $II_{1,2,i}$ in (2.116)

$$\begin{aligned} II_{1,2,1}, II_{1,2,2}, II_{1,2,4}, II_{1,2,5}, II_{1,2,7}, II_{1,2,8} &\lesssim \|\nabla b_h\|_{L^p} \|\Delta b\|_{L^{\frac{2p}{p-2}}} \|\Delta j\|_{L^2} \\ &\approx \|\nabla b_h\|_{L^p} \|\Delta b\|_{L^2}^{\frac{p-3}{p}} \|\Delta j\|_{L^2}^{\frac{3+p}{p}} \\ &\lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} \|\Delta b\|_{L^2}^2 + \frac{1}{32} \|\Delta j\|_{L^2}^2 \end{aligned} \tag{2.117}$$

We apply the divergence-free condition, Hölder's inequality, Gagliardo-Nirenberg interpolation, and Young's inequalities to estimate the terms $II_{1,2,3}, II_{1,2,6}, II_{1,2,9}$ in (2.116)

$$\begin{aligned} II_{1,2,3}, II_{1,2,6}, II_{1,2,9} &\lesssim \|\nabla b_h\|_{L^p} \|\Delta b\|_{L^{\frac{2p}{p-2}}} \|\Delta j\|_{L^2} + \|\nabla_h b_3\|_{L^m} \|\Delta b\|_{L^{\frac{2m}{m-2}}} \|\Delta j\|_{L^2} \\ &\approx \|\nabla b_h\|_{L^p} \|\Delta b\|_{L^2}^{\frac{p-3}{p}} \|\Delta j\|_{L^2}^{\frac{3+p}{p}} + \|\nabla_h b_3\|_{L^m} \|\Delta b\|_{L^2}^{\frac{m-3}{m}} \|\Delta j\|_{L^2}^{\frac{3+m}{m}} \\ &\lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} \|\Delta b\|_{L^2}^2 + \frac{1}{32} \|\Delta j\|_{L^2}^2 + \|\nabla_h b_3\|_{L^m}^{\frac{2m}{m-3}} \|\Delta b\|_{L^2}^2 \\ &+ \frac{1}{32} \|\Delta j\|_{L^2} \end{aligned} \tag{2.118}$$

The following computations show why we desperately need $|\nabla_h b_3|$ bound

$$II_{1,2,3}, II_{1,2,6}, II_{1,2,9} = \sum_{m.n=1}^{3} \int_{\mathbb{R}^3} \partial_m b_3 \partial_3 \partial_m j_1 \partial_n^2 b_1 + \partial_m b_3 \partial_3 \partial_m j_2 \partial_n^2 b_2$$

$$+ \partial_m b_3 \partial_3 \partial_m j_3 \partial_n^2 b_3 dx$$

$$= \sum_{m.n=1}^{3} \int_{\mathbb{R}^3} \partial_m b_3 \partial_3 \partial_m (\partial_2 b_3 - \partial_3 b_2) \partial_n^2 b_1 + \partial_m b_3 \partial_3 \partial_m (-\partial_1 b_3 + \partial_3 b_1) \partial_n^2 b_2$$

$$+ \partial_m b_3 \partial_3 \partial_m (\partial_1 b_2 - \partial_2 b_1) \partial_n^2 b_3 dx$$

$$= \sum_{m.n=1}^{3} \int_{\mathbb{R}^3} \partial_m b_3 \partial_3 \partial_m \partial_2 b_3 \partial_n^2 b_1 - \partial_m b_3 \partial_3 \partial_m \partial_3 b_2 \partial_n^2 b_1 - \partial_m b_3 \partial_3 \partial_m \partial_1 b_3 \partial_n^2 b_2$$

$$+ \partial_m b_3 \partial_3 \partial_m \partial_3 b_1 \partial_n^2 b_2 + \partial_m b_3 \partial_3 \partial_m \partial_1 b_2 \partial_n^2 b_3 - \partial_m b_3 \partial_3 \partial_m \partial_2 b_1 \partial_n^2 b_3 dx \qquad (2.119)$$

We encountered difficulties to bound these terms by $|\nabla b_h||\Delta b||\Delta j|$. The best we can do to couple these terms in the following fashion

$$\sum_{m.n=1}^{3} \int_{\mathbb{R}^3} \partial_m b_3 \partial_3 \partial_m \partial_2 b_3 \partial_n^2 b_1 - \partial_m b_3 \partial_3 \partial_m \partial_2 b_1 \partial_n^2 b_3 dx$$

$$\sum_{m.n=1}^{3} \int_{\mathbb{R}^3} -\partial_m b_3 \partial_3 \partial_m \partial_1 b_3 \partial_n^2 b_2 + \partial_m b_3 \partial_3 \partial_m \partial_1 b_2 \partial_n^2 b_3 dx$$

$$\sum_{m.n=1}^{3} \int_{\mathbb{R}^3} -\partial_m b_3 \partial_3 \partial_m \partial_3 b_2 \partial_n^2 b_1 + \partial_m b_3 \partial_3 \partial_m \partial_3 b_1 \partial_n^2 b_2 dx$$

It can be seen that these above couplings don't help us much to find the bound i.e., $|\nabla b_h|$, thus we ended up $|\nabla_h b_3||\Delta b||\Delta j|$.

We simplify $II_{1,3}$ in (2.113)

$$II_{1,3} = \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} (b \cdot \nabla) \partial_m^2 j \cdot \partial_n^2 b dx$$

$$= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} (b \cdot \nabla)(\nabla \times \partial_m^2 b) \cdot \partial_n^2 b dx$$

$$
\begin{aligned}
&= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} \left((b_1\partial_1 + b_2\partial_2 + b_3\partial_3) \begin{pmatrix} \partial_m^2\partial_2 b_3 - \partial_m^2\partial_3 b_2 \\ -\partial_m^2\partial_1 b_3 + \partial_m^2\partial_3 b_1 \\ \partial_m^2\partial_1 b_2 - \partial_m^2\partial_2 b_1 \end{pmatrix} \right) \cdot \begin{pmatrix} \partial_n^2 b_1 \\ \partial_n^2 b_2 \\ \partial_n^2 b_3 \end{pmatrix} dx \\
&= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} \begin{pmatrix} (b_1\partial_1 + b_2\partial_2 + b_3\partial_3)(\partial_m^2\partial_2 b_3 - \partial_m^2\partial_3 b_2) \\ (b_1\partial_1 + b_2\partial_2 + b_3\partial_3)(-\partial_m^2\partial_1 b_3 + \partial_m^2\partial_3 b_1) \\ (b_1\partial_1 + b_2\partial_2 + b_3\partial_3)(\partial_m^2\partial_1 b_2 - \partial_m^2\partial_2 b_1) \end{pmatrix} \cdot \begin{pmatrix} \partial_n^2 b_1 \\ \partial_n^2 b_2 \\ \partial_n^2 b_3 \end{pmatrix} dx \\
&= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} (b_1\partial_1\partial_m^2\partial_2 b_3\partial_n^2 b_1 + b_2\partial_2\partial_m^2\partial_2 b_3\partial_n^2 b_1 + b_3\partial_3\partial_m^2\partial_2 b_3\partial_n^2 b_1 - b_1\partial_1\partial_m^2\partial_3 b_2\partial_n^2 b_1 \\
&- b_2\partial_2\partial_m^2\partial_3 b_2\partial_n^2 b_1 - b_3\partial_3\partial_m^2\partial_3 b_2\partial_n^2 b_1 - b_1\partial_1\partial_m^2\partial_1 b_3\partial_n^2 b_2 - b_2\partial_2\partial_m^2\partial_1 b_3\partial_n^2 b_2 \\
&- b_3\partial_3\partial_m^2\partial_1 b_3\partial_n^2 b_2 + b_1\partial_1\partial_m^2\partial_3 b_1\partial_n^2 b_2 + b_2\partial_2\partial_m^2\partial_3 b_1\partial_n^2 b_2 + b_3\partial_3\partial_m^2\partial_3 b_1\partial_n^2 b_2 \\
&+ b_1\partial_1\partial_m^2\partial_1 b_2\partial_n^2 b_3 + b_2\partial_2\partial_m^2\partial_1 b_2\partial_n^2 b_3 + b_3\partial_3\partial_m^2\partial_1 b_2\partial_n^2 b_3 - b_1\partial_1\partial_m^2\partial_2 b_1\partial_n^2 b_3 \\
&- b_2\partial_2\partial_m^2\partial_2 b_1\partial_n^2 b_3 - b_3\partial_3\partial_m^2\partial_2 b_1\partial_n^2 b_3)dx \\
&= \sum_{i=1}^{18} II_{1,3,i} \qquad (2.120)
\end{aligned}
$$

The strategy is to pair up the two terms $II_{1,3,1}, II_{1,3,16}$ in (2.120) and apply integration by parts to estimate

$$
\begin{aligned}
&II_{1,3,1} + I_{1,3,16} = \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} b_1\partial_1\partial_m^2\partial_2 b_3\partial_n^2 b_1 - b_1\partial_1\partial_m^2\partial_2 b_1\partial_n^2 b_3 dx \\
&= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} b_1\partial_1\partial_m^2\partial_2 b_3\partial_n^2 b_1 + \partial_1 b_1\partial_m^2\partial_2 b_1\partial_n^2 b_3 + b_1\partial_m^2\partial_2 b_1\partial_1\partial_n^2 b_3 dx \\
&= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} b_1\partial_1\partial_m^2\partial_2 b_3\partial_n^2 b_1 + \partial_1 b_1\partial_m^2\partial_2 b_1\partial_n^2 b_3 - \partial_2 b_1\partial_m^2 b_1\partial_1\partial_n^2 b_3 - b_1\partial_m^2 b_1\partial_1\partial_2\partial_n^2 b_3 dx \\
&= \sum_{i=1}^{4} II_{1,3,1,16,i} \qquad (2.121)
\end{aligned}
$$

We apply Hölder's inequality, Gagliardo-Nirenberg interpolation, and Young's inequalities to estimate the terms $II_{1,3,1,16,2}, II_{1,3,1,16,3}$ in (2.121)

$$
II_{1,3,1,16,2}, II_{1,3,1,16,3} \lesssim \|\nabla b_h\|_{L^p} \|\Delta b\|_{L^{\frac{2p}{p-2}}} \|\Delta j\|_{L^2}
$$

$$\approx \|\nabla b_h\|_{L^p} \|\Delta b\|_{L^2}^{\frac{p-3}{p}} \|\Delta j\|_{L^2}^{\frac{3+p}{p}}$$
$$\lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} \|\Delta b\|_{L^2}^2 + \frac{1}{32}\|\Delta j\|_{L^2}^2 \tag{2.122}$$

We apply integration by parts, Hölder's inequality, Gagliardo-Nirenberg interpolation, and Young's inequalities to estimate the terms $II_{1,3,1,16,1}, II_{1,3,1,16,4}$ in (2.121)

$$\begin{aligned} II_{1,3,1,16,1} + II_{1,3,1,16,4} &= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} b_1 \partial_1 \partial_m^2 \partial_2 b_3 \partial_n^2 b_1 - b_1 \partial_m^2 b_1 \partial_1 \partial_2 \partial_n^2 b_3 dx \\ &= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} -\partial_1 b_1 \partial_m^2 \partial_2 b_3 \partial_n^2 b_1 - b_1 \partial_m^2 \partial_2 b_3 \partial_1 \partial_n^2 b_1 \\ &+ \partial_1 b_1 \partial_m^2 b_1 \partial_2 \partial_n^2 b_3 + b_1 \partial_m^2 \partial_1 b_1 \partial_2 \partial_n^2 b_3 dx \\ &\lesssim \|\nabla b_h\|_{L^p} \|\Delta b\|_{L^2}^{\frac{p-3}{p}} \|\Delta j\|_{L^2}^{\frac{3+p}{p}} + 0 \\ &\lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} \|\Delta b\|_{L^2}^2 + \|\Delta j\|_{L^2}^2 \end{aligned} \tag{2.123}$$

We find a cancellation by paring up the terms in (2.123), i.e.,

$$\int_{\mathbb{R}^3} -b_1 \partial_m^2 \partial_2 b_3 \partial_1 \partial_n^2 b_1 + b_1 \partial_m^2 \partial_1 b_1 \partial_2 \partial_n^2 b_3 dx = 0, \tag{2.124}$$

which can be seen by swapping the role of $m \leftrightarrow n$ in the second integrand.

The strategy is to pair up the two terms $II_{1,3,2}, II_{1,3,17}$ in (2.120) and apply integration by parts to deduce

$$\begin{aligned} &II_{1,3,2} + II_{1,3,17} = \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} b_2 \partial_2 \partial_m^2 \partial_2 b_3 \partial_n^2 b_1 - b_2 \partial_2 \partial_m^2 \partial_2 b_1 \partial_n^2 b_3 dx \\ &= \int_{\mathbb{R}^3} b_2 \partial_2 \partial_m^2 \partial_2 b_3 \partial_n^2 b_1 + \partial_2 b_2 \partial_m^2 \partial_2 b_1 \partial_n^2 b_3 + b_2 \partial_m^2 \partial_2 b_1 \partial_2 \partial_n^2 b_3 dx \\ &= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} b_2 \partial_2 \partial_m^2 \partial_2 b_3 \partial_n^2 b_1 + \partial_2 b_2 \partial_m^2 \partial_2 b_1 \partial_n^2 b_3 - \partial_2 b_2 \partial_m^2 b_1 \partial_2 \partial_n^2 b_3 - b_2 \partial_m^2 b_1 \partial_2 \partial_n^2 \partial_2 b_3 dx \\ &= \sum_{i=1}^{3} II_{1,3,2,17,i} \end{aligned} \tag{2.125}$$

We apply Hölder's inequality, Gagliardo-Nirenberg interpolation, and Young's inequalities to estimate the terms $II_{1,3,2,17,2}, II_{1,3,2,17,3}$ in (2.125)

$$\begin{aligned} II_{1,3,2,17,2} + II_{1,3,2,17,3} &= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} \partial_2 b_2 \partial_m^2 \partial_2 b_1 \partial_n^2 b_3 - \partial_2 b_2 \partial_m^2 b_1 \partial_2 \partial_k b_3 dx \\ &\lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} \|\Delta b\|_{L^2}^2 + \frac{1}{32} \|\Delta j\|_{L^2}^2 \end{aligned} \tag{2.126}$$

We apply integration by parts, Hölder's inequality, Gagliardo-Nirenberg interpolation, and Young's inequalities to estimate the terms $II_{1,3,2,17,1} + II_{1,3,2,17,4}$ in (2.125)

$$\begin{aligned} II_{1,3,2,17,1} + II_{1,3,2,17,4} &= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} b_2 \partial_2 \partial_m^2 \partial_2 b_3 \partial_n^2 b_1 - b_2 \partial_m^2 b_1 \partial_2 \partial_n^2 \partial_2 b_3 dx \\ &= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} -\partial_2 b_2 \partial_m^2 \partial_2 b_3 \partial_n^2 b_1 - b_2 \partial_m^2 \partial_2 b_3 \partial_2 \partial_n^2 b_1 \\ &+ \partial_2 b_2 \partial_m^2 b_1 \partial_2 \partial_n^2 b_3 + b_2 \partial_m^2 \partial_2 b_1 \partial_2 \partial_n^2 b_3 dx \\ &\lesssim \|\nabla b_h\|_{L^p} \|\Delta b\|_{L^2}^{\frac{p-3}{p}} \|\Delta j\|_{L^2}^{\frac{3+p}{p}} + 0 \\ &\lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} \|\Delta b\|_{L^2}^2 + \|\Delta j\|_{L^2}^{\frac{3+p}{p}} + 0 \end{aligned} \tag{2.127}$$

We find a cancellation by paring up the terms in (2.127), i.e.,

$$\sum_{m,n=1}^{3} -\int_{\mathbb{R}^3} b_2 \partial_m^2 \partial_2 b_3 \partial_2 \partial_n^2 b_1 + b_2 \partial_m^2 \partial_2 b_1 \partial_2 \partial_n^2 b_3 dx = 0, \tag{2.128}$$

which can be seen by swapping the role of $m \leftrightarrow n$.

The strategy is to pair up the two terms $II_{1,3,3}, II_{1,3,18}$ in (2.120) and apply integration by parts to deduce

$$\begin{aligned} II_{1,3,3} + II_{1,3,18} &= \sum_{k=1}^{3} \int_{\mathbb{R}^3} b_3 \partial_3 \partial_m^2 \partial_2 b_3 \partial_n^2 b_1 - b_3 \partial_3 \partial_m^2 \partial_2 b_1 \partial_n^2 b_3 dx \\ &= \sum_{k=1}^{3} \int_{\mathbb{R}^3} b_3 \partial_3 \partial_m^2 \partial_2 b_3 \partial_n^2 b_1 + \partial_3 b_3 \partial_m^2 \partial_2 b_1 \partial_n^2 b_3 + b_3 \partial_m^2 \partial_2 b_1 \partial_n^2 \partial_3 b_3 dx \end{aligned}$$

$$= \sum_{k=1}^{3} \int_{\mathbb{R}^3} b_3\partial_3\partial_m^2\partial_2 b_3\partial_n^2 b_1 + \partial_3 b_3\partial_m^2\partial_2 b_1\partial_n^2 b_3 - \partial_2 b_3\partial_m^2 b_1\partial_n^2\partial_3 b_3 - b_3\partial_m^2 b_1\partial_n^2\partial_3\partial_2 b_3 dx$$
$$= \sum_{i=1}^{4} II_{1,3,3,3,18,i} \tag{2.129}$$

We apply the divergence-free condition, Hölder's inequality, Gagliardo-Nirenberg interpolation, and Young's inequalities to estimate the terms $II_{1,3,3,3,18,2}, II_{1,3,3,3,18,3}$ in (2.129)

$$II_{1,3,3,3,18,2} + II_{1,3,3,3,18,3} \lesssim \|\nabla b_h\|_{L^p}\|\Delta b\|_{L^{\frac{2p}{p-2}}}\|\Delta j\|_{L^2} + \|\nabla_h b_3\|_{L^m}\|\Delta b\|_{L^{\frac{2m}{m-2}}}\|\Delta j\|_{L^2}$$
$$\approx \|\nabla b_h\|_{L^p}\|\Delta b\|_{L^2}^{\frac{p-3}{p}}\|\Delta j\|_{L^2}^{\frac{3+p}{p}} + \|\nabla_h b_3\|_{L^m}\|\Delta b\|_{L^2}^{\frac{m-3}{m}}\|\Delta j\|_{L^2}^{\frac{3+m}{m}}$$
$$\lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}}\|\Delta b\|_{L^2}^2 + \frac{1}{32}\|\Delta j\|_{L^2}^2 + \|\nabla_h b_3\|_{L^m}^{\frac{2m}{m-3}}\|\Delta b\|_{L^2}^2 + \frac{1}{32}\|\Delta j\|_{L^2}^2 \tag{2.130}$$

We apply integration by parts, divergence-free condition, Hölder's inequality, Gagliardo-Nirenberg interpolation, and Young's inequalities to estimate the terms $II_{1,3,3,3,18,1}$ and $II_{1,3,3,3,18,4}$ in (2.129)

$$\begin{aligned} II_{1,3,3,3,18,1} + II_{1,3,3,3,18,4} &= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} b_3\partial_3\partial_m^2\partial_2 b_3\partial_n^2 b_1 - b_3\partial_m^2 b_1\partial_3\partial_n^2\partial_2 b_3 dx \\ &= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} -\partial_3 b_3\partial_m^2\partial_2 b_3\partial_n^2 b_1 - b_3\partial_m^2\partial_2 b_3\partial_3\partial_n^2 b_1 \\ &+ \partial_3 b_3\partial_m^2 b_1\partial_2\partial_n^2 b_3 + b_3\partial_m^2\partial_3 b_1\partial_2\partial_n^2 b_3 dx \\ &\lesssim \|\nabla b_h\|_{L^p}\|\Delta b\|_{L^2}^{\frac{p-3}{p}}\|\Delta j\|_{L^2}^{\frac{3+p}{p}} + 0 \\ &\lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}}\|\Delta b\|_{L^2}^2 + \|\Delta j\|_{L^2}^2 + 0 \end{aligned} \tag{2.131}$$

We find a cancellation by paring up the terms in (2.131), i.e.,

$$\sum_{m,n=1}^{3} \int_{\mathbb{R}^3} -b_3\partial_m^2\partial_2 b_3\partial_n^2\partial_3 b_1 + b_3\partial_m^2\partial_3 b_1\partial_n^2\partial_2 b_3 dx = 0, \tag{2.132}$$

which can be seen by swapping the role of $m \leftrightarrow n$ in the second integrand.

The strategy is to pair up the two terms $II_{1,3,4}, II_{1,3,10}$ in (2.120) and integrate by parts to deduce

$$\begin{aligned}
&II_{1,3,4} + I_{1,3,10} = \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} -b_1\partial_1\partial_m^2\partial_3 b_2\partial_n^2 b_1 + b_1\partial_1\partial_m^2\partial_3 b_1\partial_n^2 b_2 dx \\
&= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} -b_1\partial_1\partial_m^2\partial_3 b_2\partial_n^2 b_1 - \partial_1 b_1\partial_m^2\partial_3 b_1\partial_n^2 b_2 - b_1\partial_m^2\partial_3 b_1\partial_1\partial_n^2 b_2 dx \\
&= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} -b_1\partial_1\partial_m^2\partial_3 b_2\partial_n^2 b_1 - \partial_1 b_1\partial_m^2\partial_3 b_1\partial_n^2 b_2 + \partial_1 b_1\partial_m^2\partial_3 b_1\partial_n^2 b_2 + b_1\partial_m^2\partial_1\partial_3 b_1\partial_n^2 b_2 dx \\
&= \sum_{i=1}^{4} II_{1,3,4,3,10,i} \qquad (2.133)
\end{aligned}$$

We apply Hölder's inequality, Gagliardo-Nirenberg interpolation, and Young's inequalities to estimate the terms $II_{1,3,4,3,10,2}, II_{1,3,4,3,10,3}$ in (2.133)

$$\begin{aligned}
II_{1,3,4,3,10,2} + II_{1,3,4,3,10,3} &= \sum_{m,n=1}^{3} -\int_{\mathbb{R}^3} \partial_1 b_1\partial_m^2\partial_3 b_1\partial_n^2 b_2 + \partial_1 b_1\partial_m^2\partial_3 b_1\partial_n^2 b_2 dx \\
&\lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} \|\Delta b\|_{L^2}^2 + \frac{1}{32}\|\Delta j\|_{L^2}^2 \qquad (2.134)
\end{aligned}$$

We apply integration by parts, Hölder's inequality, Gagliardo-Nirenberg interpolation, and Young's inequalities to estimate the terms $II_{1,3,4,3,10,1}, II_{1,3,4,3,10,4}$ in (2.133)

$$\begin{aligned}
&II_{1,3,4,3,10,1} + II_{1,3,4,3,10,4} = \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} -b_1\partial_1\partial_m^2\partial_3 b_2\partial_n^2 b_1 + b_1\partial_m^2\partial_1\partial_3 b_1\partial_n^2 b_2 dx \\
&= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} \partial_1 b_1\partial_m^2\partial_3 b_2\partial_n^2 b_1 + b_1\partial_m^2\partial_3 b_2\partial_n^2\partial_1 b_1 - \partial_3 b_1\partial_m^2\partial_1 b_1\partial_n^2 b_2 - b_1\partial_1\partial_m^2 b_1\partial_3\partial_n^2 b_2 dx \\
&\lesssim \|\nabla b_h\|_{L^p}\|\Delta b\|_{L^2}^{\frac{p-3}{p}} \|\Delta j\|_{L^2}^{\frac{3+p}{p}} + 0 \\
&\lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} \|\Delta b\|_{L^2}^2 + \|\Delta j\|_{L^2}^2 \qquad (2.135)
\end{aligned}$$

We find a cancellation by paring up the terms in (2.135), i.e.,

$$\sum_{m,n=1}^{3} \int_{\mathbb{R}^3} b_1 \partial_m^2 \partial_3 b_2 \partial_1 \partial_n^2 b_1 - \int_{\mathbb{R}^3} b_1 \partial_1 \partial_m^2 b_1 \partial_3 \partial_n^2 b_2 dx = 0, \tag{2.136}$$

can be seen by swapping the role of $m \leftrightarrow n$ in the second integrand.

The strategy is to pair up the two terms $II_{1,3,5}, II_{1,3,11}$ in (2.120) and apply integration by parts to deduce

$$\begin{aligned}
&II_{1,3,5} + II_{1,3,11} = \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} -b_2 \partial_2 \partial_m^2 \partial_3 b_2 \partial_n^2 b_1 + b_2 \partial_2 \partial_m^2 \partial_3 b_1 \partial_n^2 b_2 dx \\
&= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} -b_2 \partial_2 \partial_m^2 \partial_3 b_2 \partial_n^2 b_1 - \partial_2 b_2 \partial_m^2 \partial_3 b_1 \partial_n^2 b_2 - b_2 \partial_m^2 \partial_3 b_1 \partial_2 \partial_n^2 b_2 dx \\
&= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} -b_2 \partial_2 \partial_m^2 \partial_3 b_2 \partial_n^2 b_1 - \partial_2 b_2 \partial_m^2 \partial_3 b_1 \partial_n^2 b_2 + \partial_2 b_2 \partial_m^2 \partial_3 b_1 \partial_n^2 b_2 + b_2 \partial_2 \partial_m^2 \partial_3 b_1 \partial_n^2 b_2 dx \\
&= \sum_{i=1}^{4} II_{1,2,5,3,11,i}
\end{aligned} \tag{2.137}$$

Apply Hölder's, Gagliardo-Nirenberg interpolation, and Young's inequalities in order to estimate the terms $II_{1,2,5,3,11,2}, II_{1,2,5,3,11,3}$ in (2.137)

$$II_{1,2,5,3,11,2} + II_{1,2,5,3,11,3} \lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} \|\Delta b\|_{L^2}^2 + \frac{1}{32} \|\Delta j\|_{L^2}^2 \tag{2.138}$$

We apply Integration by parts, Hölder's, Gagliardo-Nirenberg interpolation, and Young's inequalities to estimate the terms $II_{1,2,5,3,11,1}, II_{1,2,5,3,11,4}$ in (2.137)

$$\begin{aligned}
&II_{1,2,5,3,11,1} + II_{1,2,5,3,11,4} = \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} -b_2 \partial_2 \partial_m^2 \partial_3 b_2 \partial_n^2 b_1 + b_2 \partial_2 \partial_m^2 \partial_3 b_1 \partial_n^2 b_2 dx \\
&= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} \partial_2 b_2 \partial_m^2 \partial_3 b_2 \partial_n^2 b_1 + b_2 \partial_m^2 \partial_3 b_2 \partial_n^2 \partial_2 b_1 - \partial_3 b_2 \partial_2 \partial_m^2 b_1 \partial_n^2 b_2 - b_2 \partial_2 \partial_m^2 b_1 \partial_3 \partial_n^2 b_2 dx \\
&\lesssim \|\nabla b_h\|_{L^p} \|\Delta b\|_{L^2}^{\frac{p-3}{p}} \|\Delta j\|_{L^2}^{\frac{3+p}{p}} + 0 \\
&\lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} \|\Delta b\|_{L^2}^2 + \frac{1}{32} \|\Delta j\|_{L^2}^2
\end{aligned} \tag{2.139}$$

We find a cancellation by paring up the terms in (2.139), i.e.,

$$\sum_{m,n=1}^{3}\int_{\mathbb{R}^3} b_2\partial_m^2\partial_3 b_2\partial_2\partial_n^2 b_1 - b_2\partial_2\partial_m^2 b_1\partial_3\partial_n^2 b_2 dx = 0, \tag{2.140}$$

can be seen by swapping the role of $m \leftrightarrow n$ in the second integrand.

The strategy is to pair up the two terms $II_{1,3,6}, II_{1,3,12}$ in (2.120) and apply integration by parts, divergence-free condition, Hölder's, Gagliardo-Nirenberg interpolation, and Young's inequalities to estimate

$$\begin{aligned}
&II_{1,3,6} + II_{1,3,12} = \sum_{m,n=1}^{3}\int_{\mathbb{R}^3} -b_3\partial_3\partial_m^2\partial_3 b_2\partial_n^2 b_1 + b_3\partial_3\partial_m^2\partial_3 b_1\partial_n^2 b_2 dx\\
&= \sum_{m,n=1}^{3}\int_{\mathbb{R}^3} -b_3\partial_3\partial_m^2\partial_3 b_2\partial_n^2 b_1 - \partial_3 b_3\partial_m^2\partial_3 b_1\partial_n^2 - b_3\partial_m^2\partial_3 b_1\partial_n^2\partial_3 b_2 dx\\
&= \sum_{m,n=1}^{3}\int_{\mathbb{R}^3} -b_3\partial_3\partial_m^2\partial_3 b_2\partial_n^2 b_1 - \partial_3 b_3\partial_m^2\partial_3 b_1\partial_n^2 + \partial_3 b_3\partial_m^2\partial_3 b_1\partial_n^2 b_2 + b_3\partial_3\partial_m^2\partial_3 b_1\partial_n^2 b_2 dx\\
&= \sum_{m,n=1}^{3}\int_{\mathbb{R}^3} -b_3\partial_3\partial_m^2\partial_3 b_2\partial_n^2 b_1 + (\partial_1 b_1 + \partial_2 b_2)\partial_m^2\partial_3 b_1\partial_n^2 - (\partial_1 b_1 + \partial_2 b_2)\partial_m^2\partial_3 b_1\partial_n^2 b_2\\
&+ b_3\partial_3\partial_m^2\partial_3 b_1\partial_n^2 b_2 dx\\
&\lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}}\|\Delta b\|_{L^2}^2 + \frac{1}{32}\|\Delta j\|_{L^2}^2 + \sum_{m,n=1}^{3} -\int_{\mathbb{R}^3} b_3\partial_3\partial_m^2\partial_3 b_2\partial_n^2 b_1 + b_3\partial_3\partial_m^2\partial_3 b_1\partial_n^2 b_2 dx\\
&\lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}}\|\Delta b\|_{L^2}^2 + \frac{1}{32}\|\Delta j\|_{L^2}^2 + \sum_{m,n=1}^{3}\int_{\mathbb{R}^3} \partial_3 b_3\partial_m^2\partial_3 b_2\partial_n^2 b_1 + b_3\partial_3\partial_m^2 b_2\partial_n^2\partial_3 b_1\\
&- \partial_3 b_3\partial_m^2\partial_3 b_1\partial_n^2 b_2 - b_3\partial_m^2\partial_3 b_1\partial_n^2\partial_3 b_2 dx\\
&\lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}}\|\Delta b\|_{L^2}^2 + \frac{1}{32}\|\Delta j\|_{L^2}^2 + \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}}\|\Delta b\|_{L^2}^2 + \frac{1}{32}\|\Delta j\|_{L^2}^2 + 0
\end{aligned} \tag{2.141}$$

We find a cancellation by paring up the terms in (2.141), i.e.,

$$\sum_{m,n=1}^{3}\int_{\mathbb{R}^3} b_3\partial_3\partial_m^2 b_2\partial_n^2\partial_3 b_1 - b_3\partial_m^2\partial_3 b_1\partial_n^2\partial_3 b_2 dx = 0, \tag{2.142}$$

which can be seen by swapping the role of $m \leftrightarrow n$ in the second integrand.

The strategy is to pair up the two terms $II_{1,3,7}, II_{1,3,13}$ in (2.120) and apply integration by parts, Hölder's, Gagliardo-Nirenberg interpolation, and Young's inequalities to estimate

$$\begin{aligned}
&II_{1,3,7} + II_{1,3,13} = \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} -b_1\partial_1\partial_m^2\partial_1 b_3\partial_n^2 b_2 + b_1\partial_1\partial_m^2\partial_1 b_2\partial_n^2 b_3 dx \\
&= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} -b_1\partial_1\partial_m^2\partial_1 b_3\partial_n^2 b_2 - \partial_1 b_1\partial_m^2\partial_1 b_2\partial_n^2 b_3 - b_1\partial_m^2\partial_1 b_2\partial_n^2\partial_1 b_3 dx \\
&= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} -b_1\partial_1\partial_m^2\partial_1 b_3\partial_n^2 b_2 - \partial_1 b_1\partial_m^2\partial_1 b_2\partial_n^2 b_3 + \partial_1 b_1\partial_m^2\partial_1 b_2\partial_n^2 b_3 \\
&+ b_1\partial_1\partial_m^2\partial_1 b_2\partial_n^2 b_3 dx \\
&\lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}}\|\Delta b\|_{L^2}^2 + \frac{1}{32}\|\Delta j\|_{L^2}^2 + \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} -b_1\partial_1\partial_m^2\partial_1 b_3\partial_n^2 b_2 + b_1\partial_1\partial_m^2\partial_1 b_2\partial_n^2 b_3 dx \\
&\lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}}\|\Delta b\|_{L^2}^2 + \frac{1}{32}\|\Delta j\|_{L^2}^2 + \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} \partial_1 b_1\partial_m^2\partial_1 b_3\partial_n^2 b_2 + b_1\partial_m^2\partial_1 b_3\partial_1\partial_n^2 b_2 \\
&- \partial_1 b_1\partial_m^2\partial_1 b_2\partial_n^2 b_3 - b_1\partial_m^2\partial_1 b_2\partial_n^2\partial_1 b_3 dx \\
&\lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}}\|\Delta b\|_{L^2}^2 + \frac{1}{32}\|\Delta j\|_{L^2}^2 + \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} b_1\partial_m^2\partial_1 b_3\partial_1\partial_n^2 b_2 - b_1\partial_m^2\partial_1 b_2\partial_n^2\partial_1 b_3 dx
\end{aligned} \tag{2.143}$$

We find a cancellation by paring up the terms in (2.143), i.e.,

$$\sum_{m,n=1}^{3} \int_{\mathbb{R}^3} b_1\partial_m^2\partial_1 b_3\partial_1\partial_n^2 b_2 - b_1\partial_m^2\partial_1 b_2\partial_n^2\partial_1 b_3 dx = 0, \tag{2.144}$$

which can be seen by swapping the role of $m \leftrightarrow n$ on the second integral.

The strategy is to pair up the two terms $II_{1,3,8}, II_{1,3,14}$ in (2.120) and apply integration by parts, Hölder's inequality, Gagliardo-Nirenberg interpolation inequality, and Young's inequality to estimate as follows:

$$
\begin{aligned}
&II_{1,3,8} + II_{1,3,14} = \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} -b_2\partial_2\partial_m^2\partial_1 b_3\partial_n^2 b_2 + b_2\partial_2\partial_m^2\partial_1 b_2\partial_n^2 b_3 dx \\
&= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} -b_2\partial_2\partial_m^2\partial_1 b_3\partial_n^2 b_2 - \partial_2 b_2\partial_m^2\partial_1 b_2\partial_n^2 b_3 - b_2\partial_m^2\partial_1 b_2\partial_n^2\partial_2 b_3 dx \\
&= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} -b_2\partial_2\partial_m^2\partial_1 b_3\partial_n^2 b_2 - \partial_2 b_2\partial_m^2\partial_1 b_2\partial_n^2 b_3 + \partial_1 b_2\partial_m^2 b_2\partial_n^2\partial_2 b_3 + b_2\partial_m^2 b_2\partial_n^2\partial_2\partial_1 b_3 dx \\
&\lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} \|\nabla b\|_{L^2}^2 + \frac{1}{32}\|\nabla j\|_{L^2}^2 + \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} -b_2\partial_2\partial_m^2\partial_1 b_3\partial_n^2 b_2 + b_2\partial_m^2 b_2\partial_n^2\partial_2\partial_1 b_3 dx \\
&\lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} \|\nabla b\|_{L^2}^2 + \frac{1}{32}\|\nabla j\|_{L^2}^2 + \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} \partial_2 b_2\partial_m^2\partial_1 b_3\partial_n^2 b_2 + b_2\partial_m^2\partial_1 b_3\partial_n^2\partial_2 b_2 \\
&- \partial_2 b_2\partial_m^2 b_2\partial_n^2\partial_1 b_3 - b_2\partial_m^2\partial_2 b_2\partial_n^2\partial_1 b_3 dx \\
&\lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} \|\Delta b\|_{L^2}^2 + \frac{1}{32}\|\Delta j\|_{L^2}^2 + \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} b_2\partial_m^2\partial_1 b_3\partial_n^2\partial_2 b_2 - b_2\partial_m^2\partial_2 b_2\partial_n^2\partial_1 b_3 dx
\end{aligned}
\tag{2.145}
$$

We find a cancellation by paring up the terms in (2.145), i.e.,

$$
\sum_{m,n=1}^{3} \int_{\mathbb{R}^3} b_2\partial_m^2\partial_1 b_3\partial_n^2\partial_2 b_2 - \int_{\mathbb{R}^3} b_2\partial_m^2\partial_2 b_2\partial_n^2\partial_1 b_3 dx = 0, \tag{2.146}
$$

which can be seen by swapping the role of $m \leftrightarrow n$ on the second integrand.

The strategy is to pair up the two terms $II_{1,3,9}, II_{1,3,15}$ in (2.120) and apply integration by parts, divergence-free condition, Hölder's inequality, Gagliardo-Nirenberg interpolation inequality, and Young's inequality to estimate as follows:

$$
\begin{aligned}
&II_{1,3,9} + II_{1,3,15} = \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} -b_3\partial_3\partial_m^2\partial_1 b_3\partial_n^2 b_2 + b_3\partial_3\partial_m^2\partial_1 b_2\partial_n^2 b_3 dx \\
&= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} -b_3\partial_3\partial_m^2\partial_1 b_3\partial_n^2 b_2 - \partial_3 b_3\partial_m^2\partial_1 b_2\partial_n^2 b_3 - b_3\partial_m^2\partial_1 b_2\partial_n^2\partial_3 b_3 dx
\end{aligned}
$$

$$= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} \partial_1 b_3 \partial_3 \partial_m^2 b_3 \partial_n^2 b_2 + b_3 \partial_3 \partial_m^2 b_3 \partial_n^2 \partial_1 b_2 - \partial_3 b_3 \partial_m^2 \partial_1 b_2 \partial_n^2 b_3 - b_3 \partial_m^2 \partial_1 b_2 \partial_n^2 \partial_3 b_3 dx$$

$$\lesssim (\|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} + \|\nabla_h b_3\|_{L^m}^{\frac{2m}{m-3}})\|\Delta b\|_{L^2}^2 + \frac{1}{32}\|\Delta j\|_{L^2}^2 \tag{2.147}$$

We find a cancellation by paring up the terms in (2.147), i.e.,

$$\sum_{m,n=1}^{3} \int_{\mathbb{R}^3} b_3 \partial_3 \partial_m^2 b_3 \partial_n^2 \partial_1 b_2 - \int_{\mathbb{R}^3} b_3 \partial_m^2 \partial_1 b_2 \partial_n^2 \partial_3 b_3 dx = 0, \tag{2.148}$$

which can be seen by swapping the role of $m \leftrightarrow n$ on the second integrand.

We simplify II_2 in (2.112), we compute

$$\begin{aligned} II_2 &= \int_{\mathbb{R}^3} \Delta(j \cdot \nabla) b \cdot \Delta b dx \\ &= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} \partial_m[(\partial_m j \cdot \nabla) b + (j \cdot \nabla)\partial_m b] \cdot \partial_n^2 b \\ &= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} (\partial_m^2 j \cdot \nabla) b + (\partial_m j \cdot \nabla)\partial_m b + (\partial_m j \cdot \nabla)\partial_m b + (j \cdot \nabla)\partial_m^2 b] \cdot \partial_n^2 b dx \\ &= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} (\partial_m^2 j \cdot \nabla) b + 2(\partial_m j \cdot \nabla)\partial_m b + (j \cdot \nabla)\partial_m^2 b] \cdot \partial_n^2 b dx \\ &= \sum_{i=1}^{3} II_{2,i} \end{aligned} \tag{2.149}$$

We simplify $II_{2,1}$ in (2.149)

$$II_{2,1} = \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} (\partial_m^2 j \cdot \nabla) b \cdot \partial_n^2 b dx$$

$$= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} \begin{pmatrix} \partial_m^2 j_1 \partial_1 b_1 + \partial_m^2 j_2 \partial_2 b_1 + \partial_m^2 j_3 \partial_3 b_1 \\ \partial_m^2 j_1 \partial_1 b_2 + \partial_m^2 j_2 \partial_2 b_2 + \partial_m^2 j_3 \partial_3 b_2 \\ \partial_m^2 j_1 \partial_1 b_3 + \partial_m^2 j_2 \partial_2 b_3 + \partial_m^2 j_3 \partial_3 b_3 \end{pmatrix} \cdot \begin{pmatrix} \partial_n^2 b_1 \\ \partial_n^2 b_2 \\ \partial_n^2 b_3 \end{pmatrix} dx$$

$$= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} \partial_m^2 j_1 \partial_1 b_1 \partial_n^2 b_1 + \partial_m^2 j_2 \partial_2 b_1 \partial_n^2 b_1 + \partial_m^2 j_3 \partial_3 b_1 \partial_n^2 b_1 + \partial_m^2 j_1 \partial_1 b_2 \partial_n^2 b_2$$

$$+ \partial_m^2 j_2 \partial_2 b_2 \partial_n^2 b_2 + \partial_m^2 j_3 \partial_3 b_2 \partial_n^2 b_2 + \partial_m^2 j_1 \partial_1 b_3 \partial_n^2 b_3 + \partial_m^2 j_2 \partial_2 b_3 \partial_n^2 b_3 + \partial_m^2 j_3 \partial_3 b_3 \partial_n^2 b_3 dx$$

$$= \sum_{i=1}^{9} II_{2,1,i} \tag{2.150}$$

Using Hölder's inequality, Gagliardo-Nirenberg interpolation, and Young's inequalities, we estimate $II_{2,1,1}$

$$II_{2,1,1} = \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} \partial_m^2 j_1 \partial_1 b_1 \partial_n^2 b_1 dx \lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} \|\Delta b\|_{L^2}^2 + \frac{1}{32} \|\Delta j\|_{L^2}^2 \tag{2.151}$$

Using Hölder's inequality, Gagliardo-Nirenberg interpolation, and Young's inequalities, we estimate $II_{2,1,2}$

$$II_{2,1,2} = \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} \partial_m^2 j_2 \partial_2 b_1 \partial_n^2 b_1 dx \lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} \|\Delta b\|_{L^2}^2 + \frac{1}{32} \|\Delta j\|_{L^2}^2 \tag{2.152}$$

Using Hölder's inequality, Gagliardo-Nirenberg interpolation, and Young's inequalities, we estimate $II_{2,1,3}$

$$II_{2,1,3} = \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} \partial_m^2 j_3 \partial_3 b_1 \partial_n^2 b_1 dx \lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} \|\Delta b\|_{L^2}^2 + \frac{1}{32} \|\Delta j\|_{L^2}^2 \tag{2.153}$$

Using Hölder's inequality, Gagliardo-Nirenberg interpolation, and Young's inequalities, we estimate $II_{2,1,4}$

$$II_{2,1,4} = \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} \partial_m^2 j_1 \partial_1 b_2 \partial_n^2 b_2 dx \lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} \|\Delta b\|_{L^2}^2 + \frac{1}{32} \|\Delta j\|_{L^2}^2 \tag{2.154}$$

Using Hölder's inequality, Gagliardo-Nirenberg interpolation, and Young's inequalities, we estimate $II_{2,1,5}$

$$II_{2,1,5} = \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} \partial_m^2 j_2 \partial_2 b_2 \partial_n^2 b_2 dx \lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} \|\Delta b\|_{L^2}^2 + \frac{1}{32} \|\Delta j\|_{L^2}^2 \tag{2.155}$$

Using Hölder's inequality, Gagliardo-Nirenberg interpolation, and Young's inequalities, we estimate $II_{2,1,6}$

$$II_{2,1,6} = \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} \partial_m^2 j_3 \partial_3 b_2 \partial_n^2 b_2 dx \lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} \|\Delta b\|_{L^2}^2 + \frac{1}{32}\|\Delta j\|_{L^2}^2 \tag{2.156}$$

Using Hölder's inequality, Gagliardo-Nirenberg interpolation, and Young's inequalities, we estimate $II_{2,1,7}$

$$II_{2,1,7} = \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} \partial_m^2 j_1 \partial_1 b_3 \partial_n^2 b_3 dx \lesssim \|\nabla_h b_3\|_{L^m}^{\frac{2m}{m-3}} \|\Delta b\|_{L^2}^2 + \frac{1}{32}\|\Delta j\|_{L^2}^2 \tag{2.157}$$

Using Hölder's inequality, Gagliardo-Nirenberg interpolation, and Young's inequalities, we estimate $II_{2,1,8}$

$$II_{2,1,8} = \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} \partial_m^2 j_2 \partial_2 b_3 \partial_n^2 b_3 dx \lesssim \|\nabla_h b_3\|_{L^m}^{\frac{2m}{m-3}} \|\Delta b\|_{L^2}^2 + \frac{1}{32}\|\Delta j\|_{L^2}^2 \tag{2.158}$$

Using Hölder's inequality, Gagliardo-Nirenberg interpolation, and Young's inequalities, we estimate $II_{2,1,9}$

$$\begin{aligned} III_{2,1,9} = \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} \partial_m^2 j_3 \partial_3 b_3 \partial_n^2 b_3 dx &= -\sum_{m,n=1}^{3} \int_{\mathbb{R}^3} \partial_m^2 j_3 (\partial_1 b_1 + \partial_2 b_2) \partial_n^2 b_3 dx \\ &\lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} \|\Delta b\|_{L^2}^2 + \frac{1}{32}\|\Delta j\|_{L^2}^2 \end{aligned} \tag{2.159}$$

We simplify $II_{2,2}$ in (2.149)

$$\begin{aligned} II_{2,2} &= \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} 2(\partial_m j \cdot \nabla)\partial_m b \cdot \partial_n^2 b dx \\ &= 2\sum_{m,n=1}^{3} \int_{\mathbb{R}^3} \begin{pmatrix} \partial_m j_1 \partial_1 \partial_m b_1 + \partial_m j_2 \partial_2 \partial_m b_1 + \partial_m j_3 \partial_3 \partial_m b_1 \\ \partial_m j_1 \partial_1 \partial_m b_2 + \partial_m j_2 \partial_2 \partial_m b_2 + \partial_m j_3 \partial_3 \partial_m b_2 \\ \partial_m j_1 \partial_1 \partial_m b_3 + \partial_m j_2 \partial_2 \partial_m b_3 + \partial_m j_3 \partial_3 \partial_m b_3 \end{pmatrix} \cdot \begin{pmatrix} \partial_n^2 b_1 \\ \partial_n^2 b_2 \\ \partial_n^2 b_3 \end{pmatrix} dx \\ &= 2\sum_{m,n=1}^{3} \int_{\mathbb{R}^3} \partial_m j_1 \partial_1 \partial_m b_1 \partial_n^2 b_1 + \partial_m j_2 \partial_2 \partial_m b_1 \partial_n^2 b_1 + \partial_m j_3 \partial_3 \partial_m b_1 \partial_n^2 b_1 + \partial_m j_1 \partial_1 \partial_m b_2 \partial_n^2 b_2 \end{aligned}$$

$$
\begin{aligned}
&+ \partial_m j_2 \partial_2 \partial_m b_2 \partial_n^2 b_2 + \partial_m j_3 \partial_3 \partial_m b_2 \partial_n^2 b_2 + \partial_m j_1 \partial_1 \partial_m b_3 \partial_n^2 b_3 + \partial_m j_2 \partial_2 \partial_m b_3 \partial_n^2 b_3 \\
&+ \partial_m j_3 \partial_3 \partial_m b_3 \partial_n^2 b_3 dx \\
&= \sum_{i=1}^{9} II_{2,2,i}
\end{aligned} \tag{2.160}
$$

We apply integration by parts, Hölder's inequality, Gagliardo-Nirenberg interpolation inequality, and Young's inequality to estimate $II_{2,2,1}$

$$
2II_{2,2,1} = 2\sum_{m,n=1}^{3} \int_{\mathbb{R}^3} \partial_m j_1 \partial_1 \partial_m b_1 \partial_n^2 b_1 dx \lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} \|\Delta b\|_{L^2}^2 + \frac{1}{32}\|\Delta j\|_{L^2}^2 \tag{2.161}
$$

We apply integration by parts, Hölder's inequality, Gagliardo-Nirenberg interpolation inequality, and Young's inequality to estimate $II_{2,2,2}$

$$
2III_{2,2,2} = 2\sum_{m,n=1}^{3} \int_{\mathbb{R}^3} \partial_m j_2 \partial_2 \partial_m b_1 \partial_n^2 b_1 dx \lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} \|\Delta b\|_{L^2}^2 + \frac{1}{32}\|\Delta j\|_{L^2}^2 \tag{2.162}
$$

Using integration by parts, Hölder's inequality, Gagliardo-Nirenberg interpolation inequality, and Young's inequality, we estimate $II_{2,2,3}$

$$
2II_{2,2,3} = 2\sum_{m,n=1}^{3} \int_{\mathbb{R}^3} \partial_m j_3 \partial_3 \partial_m b_1 \partial_n^2 b_1 dx \lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} \|\Delta b\|_{L^2}^2 + \frac{1}{32}\|\Delta j\|_{L^2}^2 \tag{2.163}
$$

We apply integration by parts, Hölder's inequality, Gagliardo-Nirenberg interpolation inequality, and Young's inequality to estimate $II_{2,2,4}$

$$
2II_{2,2,4} = 2\sum_{m,n=1}^{3} \int_{\mathbb{R}^3} \partial_m j_1 \partial_1 \partial_m b_2 \partial_n^2 b_2 dx \lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} \|\Delta b\|_{L^2}^2 + \frac{1}{32}\|\Delta j\|_{L^2}^2 \tag{2.164}
$$

We apply integration by parts, Hölder's inequality, Gagliardo-Nirenberg interpolation inequality, and Young's inequality to estimate $II_{2,2,5}$

$$
2II_{2,2,5} = 2\sum_{m,n=1}^{3} \int_{\mathbb{R}^3} \partial_m j_2 \partial_2 \partial_m b_2 \partial_n^2 b_2 dx \lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} \|\Delta b\|_{L^2}^2 + \frac{1}{32}\|\Delta j\|_{L^2}^2 \tag{2.165}
$$

We apply integration by parts, Hölder's inequality, Gagliardo-Nirenberg interpolation inequality, and Young's inequality to estimate $II_{2,2,6}$,

$$2II_{2,2,6} = 2\sum_{m,n=1}^{3}\int_{\mathbb{R}^3} \partial_m j_3 \partial_3 \partial_m b_2 \partial_n^2 b_2 dx \lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} \|\Delta b\|_{L^2}^2 + \frac{1}{32}\|\Delta j\|_{L^2}^2 \tag{2.166}$$

The strategy is to pair up the two terms $II_{2,2,7}, II_{2,2,8}$ in (2.160) and integrate by parts, divergence-free condition, Hölder's inequality, Gagliardo-Nirenberg interpolation inequality, and Young's inequality to estimate

$$\begin{aligned}
&2II_{2,2,7} + 2II_{2,2,8} = 2\sum_{m,n=1}^{3}\int_{\mathbb{R}^3} \partial_m j_1 \partial_1 \partial_m b_3 \partial_n^2 b_3 + \partial_m j_2 \partial_2 \partial_m b_3 \partial_n^2 b_3 dx \\
&= 2\sum_{m,n=1}^{3}\int_{\mathbb{R}^3} \partial_m(\partial_2 b_3 - \partial_3 b_2)\partial_1 \partial_m b_3 \partial_n^2 b_3 + \partial_m(-\partial_1 b_3 + \partial_3 b_1)\partial_2 \partial_m b_3 \partial_n^2 b_3 dx \\
&= 2\sum_{m,n=1}^{3}\int_{\mathbb{R}^3} \partial_m \partial_2 b_3 \partial_1 \partial_m b_3 \partial_n^2 b_3 - \partial_m \partial_3 b_2 \partial_1 \partial_m b_3 \partial_n^2 b_3 - \partial_m \partial_1 b_3 \partial_2 \partial_m b_3 \partial_n^2 b_3 \\
&+ \partial_m \partial_3 b_1 \partial_2 \partial_m b_3 \partial_n^2 b_3 dx \\
&\lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} \|\Delta b\|_{L^2}^2 + \frac{1}{32}\|\Delta j\|_{L^2}^2
\end{aligned} \tag{2.167}$$

We find a cancellation by paring up the terms in (2.167), i.e.,

$$2\sum_{m,n=1}^{3}\int_{\mathbb{R}^3} \partial_m \partial_2 b_3 \partial_1 \partial_m b_3 \partial_n^2 b_3 - \partial_m \partial_1 b_3 \partial_2 \partial_m b_3 \partial_n^2 b_3 dx = 0 \tag{2.168}$$

We apply integration by parts, Hölder's inequality, Gagliardo-Nirenberg interpolation inequality, and Young's inequality to estimate $II_{2,2,9}$

$$\begin{aligned}
2II_{2,2,9} = 2\sum_{m,n=1}^{3}\int_{\mathbb{R}^3} \partial_m j_3 \partial_3 \partial_m b_3 \partial_n^2 b_3 dx =& -2\sum_{m,n=1}^{3}\int_{\mathbb{R}^3} \partial_m j_3 \partial_m(\partial_1 b_1 + \partial_2 b_2)\partial_n^2 b_3 dx \\
&\lesssim \|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} \|\Delta b\|_{L^2}^2 + \frac{1}{32}\|\Delta j\|_{L^2}^2
\end{aligned} \tag{2.169}$$

We simplify $II_{2,3}$ in (2.149) by using divergence-free condition

$$II_{2,3} = \sum_{m,n=1}^{3} \int_{\mathbb{R}^3} (j \cdot \nabla)\partial_m^2 b \cdot \partial_n^2 b dx = 0 \tag{2.170}$$

We get from (2.112)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta b\|_{L^2}^2 + \|\Delta\nabla b\|_{L^2}^2 &= \int_{\mathbb{R}^3} \Delta(b\cdot\nabla)j \cdot \Delta b dx - \int_{\mathbb{R}^3} \Delta(j\cdot\nabla)b\cdot\Delta b dx \\ &\lesssim (\|\nabla b_h\|_{L^p}^{\frac{2p}{p-3}} + \|\nabla_h b_3\|_{L^m}^{\frac{2m}{m-3}})\|\Delta b\|_{L^2}^2 + \frac{1}{2}\|\Delta j\|_{L^2}^2 \end{aligned} \tag{2.171}$$

Combining all the estimates, the theorem is completed with Grönwall's inequality. □

CHAPTER 3
KURAMOTO–SIVASHINSKY EQUATION (KSE)

3.1 Motivation from Physics and real-world applications

Many important physical phenomena are involved with KSE. On the one hand, the equation is full of possibilities. However, from another perspective, it does not possess a divergence-free condition, scaling invariant solution which made the equation more challenging and intriguing in higher dimensions. The model has been used to study flame front instabilities and ion plasma instabilities. For the 1-D KSE, global existence, dissipativity, and the global attractor are well-established. In contrast, 2-D and 3-D present a fundamental difficulty because energy conservation is missing.

In the beginning, the KSE was proposed by Kuramoto, Sivashinsky, and Tsuzuki in studies of crystal growth, flame-front instabilities, and studies of inclined planes. One can refer to [76, 77, 104, 105] for details. [57] treated local and small global existence results for the Kuramoto-Sivashinsky model. One can refer to [3, 39, 51, 55, 68, 108, 81] for the variations of the 2-D KSE. Moreover, the variations on its boundary conditions of the 2-D KSE are investigated in [56, 82, 95]. The global existence of solution under the condition $1/CL_1^{3/5}$ is studied in [79]. In [11], the global well-posedness was shown with an assumption on homogeneous Neumann boundary conditions in an annular domain; where initial data is considered radially symmetric. Steady solutions of the KSE are studied in [88]. The global existence and attractors in 2-D KSE for thin domains were investigated in [103, 7, 55]. There are numerous studies of the existence of the attractor, the global attractor, and the bounds in [41, 42, 54, 53, 67, 78, 90, 96, 107, 116, 58]. Very recently, KSE is shown globally-well-posed for a calmed version of the KSE in [47].

The lack of energy conservation inhibits the proving of global well-posedness for higher dimensions such as $N = 2$ or 3; however, what we can investigate is its global regularity criteria from [48, 71, 94, 100].

Let's introduce the N-D KSE in vector form given by

$$\partial_t u + (u \cdot \nabla)u + \lambda \Delta u + \Delta^2 u = 0, \tag{3.1}$$

$\lambda > 0$ is a constant.

If we take curl on (3.1), then it turns out as follows:

$$\partial_t\omega + \lambda\nabla\omega + \Delta^2\omega = (\nabla\cdot u)\omega - (u\cdot\nabla)\omega + (\nabla\cdot\omega)u + (\omega\cdot\nabla)u, \tag{3.2}$$

As there is no divergence-free condition of (3.1), no terms on the right-hand side of (3.1) vanish, whereas $(\nabla\cdot u)\omega$, and $(\nabla\cdot\omega)u$ of (3.1) vanish for the equations with divergence-free properties. The term $(\omega\cdot\nabla)u$ in (3.2) is called the vortex stretching term. Although the vortex stretching term is considered zero in both cases of 2-D NSE, and 2-D Euler, it inhibits the 3-D NSE and 3-D Euler from being globally well-posed due to missing $\|\omega\|_{L^\infty}$ bound.

Lack of $\nabla\cdot u = 0$ implies that $\int_{\mathbb{T}^N}(u\cdot\nabla)u\cdot u \neq 0$ for the N-D, KSE, which is important in energy estimates. However, $\int_{\mathbb{T}^N}(u\cdot\nabla)u\cdot u = 0$ for N-D NSE. The Burgers' equation with $N\geq 2$ possesses a global regularity solution by applying the maximum principle by Kiselev & Ladyzhenskaya, referring to [92, 101] for details. On the other hand, it remains unknown whether KSE enjoys maximum principle or not.

Recall that there are Prodi-Serrin type regularity criteria and regularity results with one and two components as well. One can refer to [126, 16, 72]. In this study, we are focused to investigate the regularity criteria in terms of u, and one or two components of u. These results are on Sobolev Space and Besov Space.

Definition 1. *We call u a strong solution to the KSE* (3.1) *if, for any $\Psi\in C^\infty(\mathbb{T}^N)$,*

$$(\partial_t u,\Psi) + (u\cdot\nabla u,\Psi) + (\Delta u,\Delta\Psi) = \lambda(\nabla u,\nabla\Psi) \tag{3.3}$$

for almost every $t\in[0,T)$, and

$$u\in L^\infty([0,T);H^1(\mathbb{T}^N)), u\in L^2([0,T);H^3(\mathbb{T}^N)). \tag{3.4a}$$

One can find in [81] that the KSE is locally well-posed in $H^1(\mathbb{T}^N)$ for both $N\in\{2,3\}$. This result can be extended to $\mathbb{R}^N$ by using mollifiers.

Theorem 14. *In case $N\in\{2,3\}$, given any initial data $u^{in}\in H^1(\mathbb{T}^N)$, there exists an interval $[0,T)$ where $T=T(u^{in})>0$ over which the KSE* (3.1) *has a unique strong solution starting from u^{in}.*

3.2 Besov Spaces

The $\dot{B}^{\alpha}_{r,s}$ is called homogeneous Besov space, which can be written as follows:

$$\|g\|_{\dot{B}^{\alpha}_{r,s}} = \begin{cases} (\sum_{k=-\infty}^{\infty}(2^{k\alpha}\|\Delta_k g\|_{L^r})^s)^{\frac{1}{s}} & if\ s<\infty \\ sup_{-\infty<k<\infty}2^{k\alpha}\|\Delta_k g\|_{L^r} & if\ s=\infty \end{cases}$$

The $B^{\alpha}_{r,s}$ is called inhomogeneous Besov Spaces, which can be written as

$$\|g\|_{B^{\alpha}_{r,s}} \equiv \begin{cases} \|\Delta_k g\|_{L^r} + (\sum_{k=-\infty}^{\infty}(2^{k\alpha}\|\Delta_k g\|_{L^r})^s)^{1/s} & if\ s<\infty \\ \|\Delta_{-1} g\|_{L^r} + sup_{-\infty<k<\infty}2^{k\alpha}\|\Delta_k g\|_{L^r} & if\ s=\infty \end{cases}$$

H. Kozono, T. Ogawa, and Y. Taniuchi [74] investigated that if $u \in L^2(0,T,\dot{B}^0_{\infty,\infty}(\mathbb{R}^3))$, then the solution u is regular for NSE. A. Cheskidov and R. Shvydkoy [37] showed that the solution of 3-D NSE is smooth if $u \in L^r(0,T,B^{\frac{2}{r}-1}_{\infty,\infty}(\mathbb{R}^3))$ for $2<r<\infty$. For the generalized NSE, Wu in [112] established the global existence and uniqueness of the solutions in the Besov spaces. Yanqing, Baoquan, Jiefeng, and Daoguo in [115], showed that weak solutions to the NSE are smooth if $u \in C((0,T];BMO^{-1})$ or $u \in (C(0,T];\dot{B}^{-1}_{\infty,\infty})$. Montgomery-Smith [87] introduced a similar equation to NSE and showed the ill-posedness in the Besov space $\dot{B}^{-1}_{\infty,\infty}$. Bourgain and Pavlovíc [12] showed that 3-D NSE is ill-posed in $\dot{B}^{-1}_{\infty,\infty}$. Tsuruni [109] proved the well-posedness and ill-posedness for the Stationary NSE in $\mathbb{T}^d$.

3.3 Regularity criteria results

Here $N \in 2,3$. We listed some results as follows:

Theorem 15. *Let $u_0 \in H^1(\mathbb{T}^N)$. Let u be the corresponding local strong solution of ND KSE on $[0,T)$ and $u \in L^r_t W^{m,p}_x$ where $m \in [0,1)$, and*

$$\begin{cases} p \in [1,\frac{N}{m}), r \in (\frac{4}{3},\frac{4}{1+m}] & if N=2, m\in(0,1) \\ p \in [\frac{N}{m+2},\frac{N}{m}), r \in (\frac{4}{3},4] & if N=3, m\in(0,1) \end{cases} \tag{3.5}$$

$$\begin{cases} p \in (\frac{N}{2},\infty], r \in (\frac{4}{3},4)] & if N=2, m=0 \\ p \in [\frac{N}{2},\infty], r \in [\frac{4}{3},4] & if N=3, m=0 \end{cases} \tag{3.6}$$

satisfy $\frac{N}{p} + \frac{2}{r} = \frac{3+m}{2} + \frac{N}{2p}$, *then we can extend solution of* ND *KSE to* $[0, T+\epsilon)$ *for some* ϵ.

Recall one useful inequality:

$$\|f\|_{\dot{H}^2}^2 \leq (\int ||\xi|^3|\hat{f}(\xi)||^2 d\xi)^{\frac{1}{2}} (\int ||\xi||\hat{f}(\xi)||^2 d\xi)^{\frac{1}{2}} = \|f\|_{\dot{H}^3}\|f\|_{\dot{H}^1} \tag{3.7}$$

The proof of this inequality is just a definition of Sobolev Space in $\mathbb{R}^d$. The proof of the Theorem 15 can be found in one of the previous works [80] with Prof. Adam Larios, and Prof. Kazuo Yamazaki. Rather than repeating the proof, it would be worthwhile to extend the approach to a different space, such as Besov Space.

Theorem 16. *Let* $u_0 \in H^1(\mathbb{R}^N)$. *Let* u *be the corresponding local strong solution of* N*-D KSE on* $[0,T)$ *and* $u \in L^{\frac{8}{5}}(0, T, \dot{B}^{-\frac{1}{2}}_{\infty,\infty}(\mathbb{R}^N))$, *then we can extend the solution of* N*-D KSE to* $[0, T+\epsilon)$ *for some* ϵ.

Proof. Taking $L^2(\mathbb{R}^N)$-inner products on (3.1) with u

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 &= -\int (u\cdot\nabla)u\cdot u dx + \lambda\|\nabla u\|_{L^2}^2 \\
&\leq c\|u\|_{L^4}^2\|\nabla u\|_{L^2} + \frac{1}{4}\|\Delta u\|_{L^2}^2 + c\|u\|_{L^2}^2 \\
&\leq c\|u\|_{\dot{B}^{-\frac{1}{2}}_{\infty,\infty}}\|\Lambda^{\frac{1}{2}}u\|_{L^2}\|\nabla u\|_{L^2} + \frac{1}{4}\|\Delta u\|_{L^2}^2 + c\|u\|_{L^2}^2 \\
&\leq c\|u\|_{\dot{B}^{-\frac{1}{2}}_{\infty,\infty}}^{\frac{8}{5}}\|u\|_{L^2}^2 + \frac{1}{4}\|\Delta u\|_{L^2}^2 + \frac{1}{4}\|\Delta u\|_{L^2}^2 + c\|u\|_{L^2}^2,
\end{aligned} \tag{3.8}$$

in which we applied Hölder inequality, Plancherel theorem, Theorem 37, inequality (3.7), Young's inequality, and Grönwall's inequality to deduce $u \in L_t^\infty L_x^2 \cap L_t^2 H_x^2$.

Taking $L^2(\mathbb{R}^N)$-inner products on (3.1) with Δu

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|\nabla\Delta u\|_{L^2}^2 =& \lambda\|\Delta u\|_{L^2}^2 - \int (u\cdot\nabla)u\cdot\Delta u dx \\
&\leq \|\nabla u\|_{L^2}\|u\|_{L^\infty}\|\Delta u\|_{L^2} + \lambda\|\Delta u\|_{L^2}^2 \\
&\leq \frac{1}{8}\|\nabla\Delta u\|_{L^2}^2 + c\|\nabla u\|_{L^2}^2\|\Delta u\|_{L^2}^2 + \frac{1}{8}\|\nabla\Delta u\|_{L^2}^2 + c\|\nabla u\|_{L^2}^2,
\end{aligned} \tag{3.9}$$

in which we applied Hölder inequality, Plancherel theorem, inequality (3.7), $\dot{H}^2(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$, Young's, Gagliardo-Nirenberg interpolation inequality, and we can complete the theorem with Grönwall's inequality. □

Proposition 17. *Let $u_0 \in H^1(\mathbb{T}^2)$. Let u be the corresponding local strong solution of 2-D KSE on $[0,T)$ and $u_1 \in L_t^r W_x^{m,p}$ where $m \in [0,1)$, and*

$$\begin{cases} p \in [1, \frac{2}{m}), r \in (\frac{4}{3}, \frac{4}{1+m}] & if\ m \in (0,1) \\ p \in (1, \infty], r \in (\frac{4}{3}, 4] & if\ m = 0, \end{cases} \tag{3.10}$$

satisfy $\frac{2}{p} + \frac{2}{r} = \frac{3+m}{2} + \frac{2}{2p}$, then $u_2 \in L_t^\infty L_x^2 \cap L_t^2 H_x^2$.

Proposition 18. *Let $u_0 \in H^1(\mathbb{T}^2)$. Let u be the corresponding local strong solution of 2-D KSE on $[0,T)$ and $u_1 \in L_t^r W_x^{m,p}$ where $m \in [0,1)$, and*

$$\begin{cases} p \in [1, \frac{2}{m}), r \in (\frac{4}{3}, \frac{4}{1+m}] & if\ m \in (0,1) \\ p \in (1, \infty], r \in (\frac{4}{3}, 4] & if\ m = 0, \end{cases} \tag{3.11}$$

satisfy $\frac{2}{p} + \frac{2}{r} = \frac{3+m}{2} + \frac{2}{2p}$, then $u_1 \in L_t^\infty L_x^2 \cap L_t^2 H_x^2$.

By applying Proposition 17 and Proposition 18, we can immediately state the following interesting theorem:

Theorem 19. *Let $u_0 \in H^1(\mathbb{T}^2)$. Let u be the corresponding local strong solution of 2-D KSE on $[0,T)$ and $u_1 \in L_t^r W_x^{m,p}$ where $m \in [0,1)$, and*

$$\begin{cases} p \in [1, \frac{N}{m}), r \in (\frac{4}{3}, \frac{4}{1+m}] & if\ m \in (0,1) \\ p \in [\frac{N}{m+2}, \frac{N}{m}), r \in (\frac{4}{3}, 4] & if\ m = 0, \end{cases} \tag{3.12}$$

satisfy $\frac{2}{p} + \frac{2}{r} = \frac{3+m}{2} + \frac{2}{2p}$, then we can extend the solution of 2-D KSE to $[0, T+\epsilon)$ for some ϵ.

The proofs of Proposition 17, Proposition 18, and Theorem 19 can be found in one of the previous works [80] with Prof. Adam Larios, and Prof. Kazuo Yamazaki. The proofs might be more useful if the idea is extended to another space, like Besov Space, rather than being repeated.

Proposition 20. *Let* $u_0 \in H^1(\mathbb{R}^2)$. *Let* u *be the corresponding local strong solution of 2-D KSE on* $[0,T)$ *and* $u_1 \in L^2(0,T;\dot{B}^{-\frac{1}{2}}_{\infty,\infty}(\mathbb{R}^2)) \cap L^2(0,T;\dot{B}^{\frac{1}{2}}_{\infty,\infty}(\mathbb{R}^2))$, *then* $u_2 \in L^\infty_t L^2_x \cap L^2_t H^2_x$.

Proof. Taking $L^2(\mathbb{R}^2)$-inner products with u_2 on the second component of (3.1) gives

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_2\|^2_{L^2} + \|\Delta u_2\|^2_{L^2} &= -\int (u\cdot\nabla)u_2u_2dx + \lambda\|\nabla u_2\|^2_{L^2}\\
&= -\int u_1(\partial_1 u_2)u_2dx - \int \frac{1}{3}\partial_2(u_2)^3dx + \|u_2\|^2_{L^2} + \frac{1}{4}\|\Delta u_2\|^2_{L^2}\\
&\leq \|u_1\|_{L^4}\|\nabla u_2\|_{L^4}\|u_2\|_{L^2} + \|u_2\|^2_{L^2} + \frac{1}{4}\|\Delta u_2\|^2_{L^2}\\
&\leq \|u_1\|^{\frac{1}{2}}_{\dot{B}^{-\frac{1}{2}}_{\infty,\infty}}\|u_1\|^{\frac{1}{2}}_{\dot{B}^{\frac{1}{2}}_{\infty,\infty}}\|u_2\|_{L^2}\|\Lambda^{\frac{3}{2}}u_2\|_{L^2} + \|u_2\|^2_{L^2} + \frac{1}{4}\|\Delta u_2\|^2_{L^2}\\
&\leq \frac{1}{4}\|\Delta u_2\|^2_{L^2} + \frac{1}{4}\|\Delta u_2\|^2_{L^2} + C(\|u_1\|^2_{\dot{B}^{-\frac{1}{2}}_{\infty,\infty}} + \|u_1\|^2_{\dot{B}^{\frac{1}{2}}_{\infty,\infty}} + 1)\|u_2\|^2_{L^2},
\end{aligned} \tag{3.13}$$

in which we applied Hölder's inequality, Theorem 37, Plancherel Theorem, inequality (3.7), and Young inequality. By applying the Grönwall's inequality, (3.13) implies the desired result. □

Proposition 21. *Let* $u_0 \in H^1(\mathbb{R}^2)$. *Let* u *be the corresponding local strong solution of 2-D KSE on* $[0,T)$ *and* $u_1 \in L^2(0,T;\dot{B}^{-\frac{1}{2}}_{\infty,\infty}(\mathbb{R}^2)) \cap L^2(0,T;\dot{B}^{\frac{1}{2}}_{\infty,\infty}(\mathbb{R}^2))$, *then* $u_1 \in L^\infty_t L^2_x \cap L^2_t H^2_x$.

Proof. Taking $L^2(\mathbb{R}^2)$-inner products on the first component of (3.1) with u_1 and compute

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_1\|^2_{L^2} + \|\Delta u_1\|^2_{L^2} &= -\int (u\cdot\nabla)u_1u_1dx + \lambda\|\nabla u_1\|^2_{L^2}\\
&= -\int u_1\partial_1 u_1u_1dx - \int u_2\partial_2u_1u_1dx + \lambda\|\nabla u_1\|^2_{L^2}\\
&\leq \|\nabla u_1\|_{L^2}\|u_2\|_{L^\infty}\|u_1\|_{L^2} + \lambda\|\Delta u_1\|_{L^2}\|u_1\|_{L^2}\\
&\lesssim \|u_2\|_{H^2}\|u_1\|^{\frac{3}{2}}_{L^2}\|\Delta u_1\|^{\frac{1}{2}}_{L^2} + \|u_1\|^2_{L^2} + \frac{1}{4}\|\Delta u_1\|^2_{L^2}\\
&\leq \frac{1}{2}\|\Delta u_1\|^2_{L^2} + C(\|u_2\|^{\frac{4}{3}}_{H^2} + 1)\|u_1\|^2_{L^2},
\end{aligned} \tag{3.14}$$

in which we applied Hölder's inequality, Sobolev embedding $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$, Plancherel Theorem, inequality (3.7), and Young's inequality. This implies that $u_1 \in L^\infty_T L^2_x \cap L^2_T H^2_x$ because we know $\|u_2(t)\|_{H^2} \in L^2_t$ due to Proposition 20. $\square$

Theorem 22. *Let $u_0 \in H^1(\mathbb{R}^2))$. Let u be the corresponding local strong solution of 2-D KSE on $[0,T)$ and $u_1 \in L^2(0,T;\dot{B}^{-\frac{1}{2}}_{\infty,\infty}(\mathbb{R}^2)) \cap L^2(0,T;\dot{B}^{\frac{1}{2}}_{\infty,\infty}(\mathbb{R}^2))$, then we can extend the solution of 2-D KSE to $[0,T+\epsilon)$ for some ϵ.*

Proof. Taking $L^2(\mathbb{R}^2)$-inner products on (3.1) with Δu

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2_{L^2} + \|\nabla\Delta u\|^2_{L^2} &= -\int(u\cdot\nabla)u\cdot\Delta u dx + \lambda\|\Delta u\|^2_{L^2}\\ &= \lambda\|\Delta u\|^2_{L^2} - \sum_{k=1}^{3}\int\begin{pmatrix}u_1\partial_1 u_1 + u_2\partial_2 u_1\\ u_1\partial_1 u_2 + u_2\partial_2 u_2\end{pmatrix}\cdot\begin{pmatrix}\partial_k^2 u_1\\ \partial_k^2 u_2\end{pmatrix}dx\\ &= \lambda\|\Delta u\|^2_{L^2} - \sum_{k=1}^{3}\int u_1\partial_1 u_1\partial_k^2 u_1 + u_2\partial_2 u_1\partial_k^2 u_1 + u_1\partial_1 u_2\partial_k^2 u_2 + u_2\partial_2 u_2\partial_k^2 u_2 dx,\end{aligned} \tag{3.15}$$

by applying Proposition 20 and Proposition 21, we can complete the theorem with Grönwall's inequality. $\square$

Remark 1. *It might be interesting to investigate further whether or not it is possible to depend on only one velocity component rather than two.*

Let's turn our focus to the 3-D case as follows:

Proposition 23. *Let $u_0 \in H^1(\mathbb{T}^3)$. Let u be the corresponding local strong solution of 3-D KSE on $[0,T)$ and $u_1, u_2 \in L^r_t W^{m,p}_x$ where $m \in [0,1)$, and*

$$\begin{cases} p \in [\frac{3}{m+2}, \frac{3}{m}), r \in (\frac{4}{3}, 4] & if\ m \in (0,1)\\ p \in [\frac{3}{2}, \infty], r \in (\frac{4}{3}, 4] & if\ m = 0,\end{cases} \tag{3.16}$$

satisfy $\frac{3}{p} + \frac{2}{r} = \frac{3+m}{2} + \frac{3}{2p}$, then $u_3 \in L^\infty_t L^2_x \cap L^2_t H^2_x$.

Proposition 24. *Let $u_0 \in H^1(\mathbb{T}^3)$. Let u be the corresponding local strong solution of 3-D KSE on $[0,T)$ and $u_1, u_2 \in L_t^r W_x^{m,p}$ where $m \in [0,1)$, and*

$$\begin{cases} p \in [\frac{3}{m+2}, \frac{3}{m}), r \in (\frac{4}{3}, 4] & if\ m \in (0,1) \\ p \in [\frac{3}{2}, \infty], r \in (\frac{4}{3}, 4] & if\ m = 0, \end{cases} \tag{3.17}$$

satisfy $\frac{3}{p} + \frac{2}{r} = \frac{3+m}{2} + \frac{3}{2p}$, then $u_1, u_2 \in L_t^\infty L_x^2 \cap L_t^2 H_x^2$.

The Proposition 23 and Proposition 24 lead to the following theorem:

Theorem 25. *Let $u_0 \in H^1(\mathbb{T}^3)$. Let u be the corresponding local strong solution of 3-D KSE on $[0,T)$ and $u_1, u_2 \in L_t^r W_x^{m,p}$ where $m \in [0,1)$, and*

$$\begin{cases} p \in [\frac{3}{m+2}, \frac{3}{m}), r \in (\frac{4}{3}, 4] & if\ m \in (0,1) \\ p \in [\frac{3}{2}, \infty], r \in (\frac{4}{3}, 4] & if\ m = 0, \end{cases} \tag{3.18}$$

satisfy $\frac{3}{p} + \frac{2}{r} = \frac{3+m}{2} + \frac{3}{2p}$, the solution of 3-D KSE can be extended to $[0, T+\epsilon)$ for some ϵ.

The proofs of Proposition 23, Proposition 24, and Theorem 25 can be found in one of the previous works [80] with Prof. Adam Larios, and Prof. Kazuo Yamazaki. The extended results over Besov Space might be of interest rather than proving the Proposition 23, Proposition 24, and Theorem 25.

Proposition 26. *Let $u_0 \in H^1(\mathbb{R}^3)$. Let u be the corresponding local strong solution of 3-D KSE on $[0,T)$ and $u_1, u_2 \in L^2(0,T; \dot{B}^{-\frac{1}{2}}_{\infty,\infty}(\mathbb{R}^3)) \cap L^2(0,T; \dot{B}^{\frac{1}{2}}_{\infty,\infty}(\mathbb{R}^3))$, then $u_3 \in L_t^\infty L_x^2 \cap L_t^2 H_x^2$.*

Proof. Taking $L^2(\mathbb{R}^3)$-inner products with u_2 on the second component of (3.1) gives

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_3\|_{L^2}^3 + \|\Delta u_3\|_{L^2}^2 &= -\int (u\cdot\nabla)u_3 u_3 dx + \lambda\|\nabla u_3\|_{L^2}^2 \\ &= -\int u_1\partial_1 u_3 u_3 dx - \int u_2 \partial_2 u_3 u_3 dx + \lambda\|u_3\|_{L^2}\|\Delta u_3\|_{L^2} dx \\ &\leq (\|u_1\|_{L^4} + \|u_2\|_{L^4})\|\nabla u_3\|_{L^4}\|u_3\|_{L^2} + \lambda\|u_3\|_{L^2}\|\Delta u_3\|_{L^2} \\ &\leq (\|u_1\|^{\frac{1}{2}}_{\dot{B}^{\frac{1}{2}}_{\infty,\infty}}\|u_1\|^{\frac{1}{2}}_{\dot{B}^{-\frac{1}{2}}_{\infty,\infty}} + \|u_2\|^{\frac{1}{2}}_{\dot{B}^{\frac{1}{2}}_{\infty,\infty}}\|u_2\|^{\frac{1}{2}}_{\dot{B}^{-\frac{1}{2}}_{\infty,\infty}})\|u_3\|_{L^2}\|\Lambda^{\frac{3}{2}}u_3\|_{L^2} + \|u_3\|_{L^2}^2 \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4}\|\Delta u_3\|_{L^2}^2\\
\leq&\frac{1}{4}\|\Delta u_3\|_{L^2}^2+\frac{1}{4}\|\Delta u_3\|_{L^2}^2\\
&+C(\|u_1\|^2_{\dot{B}^{\frac{1}{2}}_{\infty,\infty}}+\|u_1\|^2_{\dot{B}^{-\frac{1}{2}}_{\infty,\infty}}+\|u_2\|^2_{\dot{B}^{\frac{1}{2}}_{\infty,\infty}}+\|u_2\|^2_{\dot{B}^{-\frac{1}{2}}_{\infty,\infty}}+1)\|u_3\|^2_{L^2}, \quad (3.19)
\end{aligned}
$$

in which we applied Hölder's inequality, Theorem 37, Plancherel Theorem, inequality (3.7), Theorem 37, and Young inequality. By applying the Grönwall's inequality, (3.19) implies the desired result. □

Proposition 27. *Let $u_0 \in H^1\mathbb{R}^3)$. Let u be the corresponding local strong solution of 3-D KSE on $[0,T)$ and $u_1, u_2 \in L^2(0,T;\dot{B}^{-\frac{1}{2}}_{\infty,\infty}(\mathbb{R}^3)) \cap L^2(0,T;L^2(0,T;\dot{B}^{\frac{1}{2}}_{\infty,\infty}(\mathbb{R}^3))$, then $u_1, u_2 \in L^\infty_t L^2_x \cap L^2_t H^2_x$.*

Proof. Fix $k \in \{1,2\}$. Taking $L^2(\mathbb{R}^3)$-inner products on the k-th component of (3.1) with u_k and compute as follows:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u_k\|^2_{L^2}+\|\Delta u_k\|^2_{L^2}=-\int(u\cdot\nabla)u_k u_k+\lambda\|\nabla u_k\|^2_{L^2}\\
=&-\int u_1\partial_1 u_k u_k-\int u_2\partial_2 u_k u_k-\int u_3\partial_3 u_k u_k+\lambda\|\nabla u_k\|^2_{L^2}\\
&\leq(\|u_1\|_{L^4}+\|u_2\|_{L^4})\|\nabla u_k\|_{L^4}\|u_k\|_{L^2}+\|u_3\|_{L^\infty}\|\nabla u_k\|_{L^2}\|u_k\|_{L^2}+\lambda\|u_k\|_{L^2}\|\Delta u_k\|_{L^2}\\
\leq&(\|u_1\|^{\frac{1}{2}}_{\dot{B}^{-\frac{1}{2}}_{\infty,\infty}}\|u_1\|^{\frac{1}{2}}_{\dot{B}^{\frac{1}{2}}_{\infty,\infty}}+\|u_2\|^{\frac{1}{2}}_{\dot{B}^{-\frac{1}{2}}_{\infty,\infty}}\|u_2\|^{\frac{1}{2}}_{\dot{B}^{\frac{1}{2}}_{\infty,\infty}})\|u_k\|_{L^2}\|\Lambda^{\frac{3}{2}}u_k\|_{L^2}+\|u_k\|^2_{L^2}\\
&+\frac{1}{4}\|\Delta u_k\|^2_{L^2}++\|u_3\|^2_{H^2}\|u_k\|^2_{L^2}+\frac{1}{4}\|\nabla u_k\|^2_{L^2}\\
\leq&\frac{1}{4}\|\Delta u_k\|^2_{L^2}+\frac{1}{4}\|\Delta u_k\|^2_{L^2}\\
&+C(\|u_1\|^2_{\dot{B}^{-\frac{1}{2}}_{\infty,\infty}}+\|u_1\|^2_{\dot{B}^{\frac{1}{2}}_{\infty,\infty}}+\|u_2\|^2_{\dot{B}^{-\frac{1}{2}}_{\infty,\infty}}+\|u_2\|^2_{\dot{B}^{\frac{1}{2}}_{\infty,\infty}}+\|u_3\|^2_{H^2}+1)\|u_k\|^2_{L^2}, \quad (3.20)
\end{aligned}
$$

in which we applied Hölder's inequality, Sobolev embedding $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$, Theorem 37, Plancherel Theorem, inequality (3.7), and Young inequality, and Proposition 23. □

The Proposition 26 and Proposition 27 lead to the following theorem:

Theorem 28. *Let $u_0 \in H^1(\mathbb{R}^3)$. Let u be the corresponding local strong solution* *of* 3*-D KSE on* $[0,T)$ *and* $u_1, u_2 \in L^2(0,T;\dot{B}^{-\frac{1}{2}}_{\infty,\infty}(\mathbb{R}^3)) \cap L^2(0,T;L^2(0,T;\dot{B}^{\frac{1}{2}}_{\infty,\infty}(\mathbb{R}^3))$, *then we can extend the solution of* 3*-D KSE to* $[0,T+\epsilon)$ *for some* ϵ*.*

Proof. Taking $L^2(\mathbb{R}^3)$-inner products on (3.1) with Δu

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2_{L^2} + \|\nabla\Delta u\|^2_{L^2} &= -\int (u\cdot\nabla)u\cdot\Delta u dx + \lambda\|\Delta u\|^2_{L^2} \\
&= \lambda\|\Delta u\|^2_{L^2} - \sum_{k=1}^{3}\int \begin{pmatrix} u_1\partial_1 u_1 + u_2\partial_2 u_1 + u_3\partial_3 u_1 \\ u_1\partial_1 u_2 + u_2\partial_2 u_2 + u_3\partial_3 u_2 \\ u_1\partial_1 u_3 + u_2\partial_2 u_3 + u_3\partial_3 u_3 \end{pmatrix} \cdot \begin{pmatrix} \partial_k^2 u_1 \\ \partial_k^2 u_2 \\ \partial_k^2 u_3 \end{pmatrix} dx \\
&= \lambda\|\Delta u\|^2_{L^2} - \sum_{k=1}^{3}\int (u_1\partial_1 u_1\partial_k^2 u_1 + u_2\partial_2 u_1\partial_k^2 u_1 + u_3\partial_3 u_1\partial_k^2 u_1 + \\
&+ u_1\partial_1 u_2\partial_k^2 u_2 + u_2\partial_2 u_2\partial_k^2 u_2 + u_3\partial_3 u_2\partial_k^2 u_2 + u_1\partial_1 u_3\partial_k^2 u_3 \\
&+ u_2\partial_2 u_3\partial_k^2 u_3 + u_3\partial_3 u_3\partial_k^2 u_3)dx, \qquad (3.21)
\end{aligned}
$$

by applying Proposition 26 and Proposition 27, we can complete the theorem with Grönwall's inequality. □

Theorem 29. *Let* $u_0 \in H^1(\mathbb{T}^N)$*. Let* u *be the corresponding local strong solution of* ND *KSE on* $[0,T)$ *and* $\nabla u \in L^r_t L^p_x$ *where* $p \in [1,\infty]$, $r \in [1, \frac{4}{4-N}]$ *satisfy* $\frac{N}{p} + \frac{2}{r} = 2 + \frac{N}{2p}$, *then we can extend the solution of* ND *KSE to* $[0,T+\epsilon)$ *for some* ϵ*.*

One of the previous works [97] with Prof. Adam Larios, and Prof. Kazuo Yamazaki [80] shows that the proof of Theorem 29. From [125], we know that Navier-Stokes equations possess the scaling invariant solution and the largest critical space is $\dot{B}^{-1}_{\infty,\infty}$ in Besov norm. It is known that KSE does not possess scaling invariant solutions. However, as KSE has some resemblances with NSE to some extent, thus it might be interesting to extend the theorem 29 in Besov Space, as follows:

Theorem 30. *Let* $u_0 \in H^1(\mathbb{R}^3)$*, Let* u *be the corresponding local strong solution of* 3*-D KSE on* $[0,T)$ *and* $\nabla u \in L^2(0,T,\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)) \cap L^2(0,T;L^2(\mathbb{R}^3))$*, then we can extend the solution of* 3*-D KSE to* $[0,T+\epsilon)$ *for some* ϵ*.*

Proof. Taking $L^2(\mathbb{R}^3)$-inner products on (3.1) with Δu

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2_{L^2} + \|\nabla\Delta u\|^2_{L^2} = -\int (u\cdot\nabla)u\cdot\Delta u dx + \lambda\|\Delta u\|^2_{L^2}$$

$$
\begin{aligned}
&\leq c\|\nabla u\|_{L^3}^3 + \lambda\|\Delta u\|_{L^2}^2 \\
&\leq c(\|\nabla u\|_{\dot{B}^{-1}_{\infty,\infty}}^{\frac{1}{3}}\|\nabla u\|_{\dot{H}^{\frac{1}{2}}}^{\frac{2}{3}})^3 + \frac{1}{4}\|\nabla\Delta u\|_{L^2}^2 + c\|\nabla u\|_{L^2}^2 \\
&= c\|\nabla u\|_{\dot{B}^{-1}_{\infty,\infty}}\|\nabla u\|_{\dot{H}^{\frac{1}{2}}}^2 + \frac{1}{4}\|\nabla\Delta u\|_{L^2}^2 + c\|\nabla u\|_{L^2}^2 \\
&\leq c\|\nabla u\|_{L^2}\|\nabla u\|_{\dot{B}^{-1}_{\infty,\infty}}\|\Delta u\|_{L^2} + \frac{1}{4}\|\nabla\Delta u\|_{L^2}^2 + c\|\nabla u\|_{L^2}^2 \\
&\leq \frac{1}{8}\|\Delta u\|_{L^2}^2 + c\|\nabla u\|_{\dot{B}^{-1}_{\infty,\infty}}^2\|\nabla u\|_{L^2}^2 + \frac{1}{4}\|\nabla\Delta u\|_{L^2}^2 + c\|\nabla u\|_{L^2}^2,
\end{aligned}
\tag{3.22}
$$

in which we applied integration by parts, Hölder's inequality, Theorem 37, Plancherel theorem, Young's inequality, Gagliardo-Nirenberg interpolation inequality.

We also need an $L^2(\mathbb{R}^3)$ estimate, thus taking $L^2(\mathbb{R}^3)$-inner products on (3.1) with u

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 &= -\int (u\cdot\nabla)u\cdot u dx + \lambda\|\nabla u\|_{L^2}^2 \\
&\leq c\|u\|_{L^6}\|\nabla u\|_{L^3}\|u\|_{L^2} + c\|u\|_{L^2}^2 + \frac{1}{4}\|\Delta u\|_{L^2}^2 \\
&\leq c\|\nabla u\|_{L^2}\|\nabla\Lambda^{\frac{1}{2}}u\|_{L^2}\|u\|_{L^2} + c\|u\|_{L^2}^2 + \frac{1}{4}\|\Delta u\|_{L^2}^2 \\
&\leq c\|\nabla u\|_{L^2}^2\|u\|_{L^2}^2 + \frac{1}{4}\|\nabla\Lambda^{\frac{1}{2}}u\|_{L^2}^2 + c\|u\|_{L^2}^2 + \frac{1}{4}\|\Delta u\|_{L^2}^2,
\end{aligned}
\tag{3.23}
$$

in which we applied Hölder's inequality, Plancherel theorem, and Young's inequality. We can complete the proof applying (3.22) and (3.23). □

The following result is the criterion on $\nabla\cdot u$ and the proof can be found in one of the previous works [80] with Prof. Adam Larios, and Prof. Kazuo Yamazaki:

Theorem 31. *Let $u_0 \in H^1(\mathbb{T}^3)$. Let u be the corresponding local strong solution of ND KSE on $[0,T)$ and $\nabla\cdot u \in L^r_t L^p_x$ where $p \in [1,\infty]$, $r \in [1, \frac{4}{4-N}]$ satisfy $\frac{N}{p} + \frac{2}{r} = 2 + \frac{N}{2p}$, then we can extend the solution of ND KSE to $[0, T+\epsilon)$ for some ϵ.*

Since proof is in [97], thus rather than proving Theorem 31, we are interested to extend the results as follows:

Theorem 32. *Let $u_0 \in H^1(\mathbb{R}^2)$. Let u be the corresponding local strong solution of 2-D KSE on $[0,T)$ and $\nabla \cdot u \in L^2(0,T,\dot{B}^0_{2,2}(\mathbb{R}^2))$, then we can extend the solution of 2-D KSE to $[0,T+\epsilon)$ for some ϵ.*

Proof. We take $L^2(\mathbb{R}^2)$-inner products on (3.1) with u

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 &= -\int (u\cdot\nabla)u\cdot u dx + \lambda\|\nabla u\|_{L^2}^2 \\
&\le c\|u\|_{L^4}^2\|\nabla\cdot u\|_{L^2} + c\|u\|_{L^2}^2 + \frac{1}{4}\|\Delta u\|_{L^2}^2 \\
&\le c\|\nabla\cdot u\|_{\dot{B}^0_{2,2}}\|u\|_{L^2}\|\nabla u\|_{L^2} + c\|u\|_{L^2}^2 + \frac{1}{4}\|\Delta u\|_{L^2}^2 \\
&\le c\|\nabla\cdot u\|_{\dot{B}^0_{2,2}}^2\|u\|_{L^2}^2 + \frac{1}{4}\|\nabla u\|_{L^2}^2 + c\|u\|_{L^2}^2 + \frac{1}{4}\|\Delta u\|_{L^2}^2, \quad (3.24)
\end{aligned}$$

in which we applied integration by parts, Hölder inequality, Plancherel theorem,Theorem 37, Young's inequality, and Grönwall's inequality to deduce $u \in L^\infty_t L^2_x \cap L^2_t H^2_x$.

Taking $L^2(\mathbb{R}^2)$-inner products on (3.1) with Δu

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|\nabla\Delta u\|_{L^2}^2 &= -\int (u\cdot\nabla)u\cdot\Delta u dx + \lambda\|\Delta u\|_{L^2}^2 \\
&\le \|\Delta u\|_{L^2}\|u\|_{L^\infty}\|\nabla u\|_{L^2} + c\|\nabla u\|_{L^2}^2 + \frac{1}{4}\|\nabla\Delta u\|_{L^2}^2 \\
&\le \frac{1}{8}\|\Delta u\|_{L^2}^2 + c\|\Delta u\|_{L^2}^2\|\nabla u\|_{L^2}^2 + c\|\nabla u\|_{L^2}^2 + \frac{1}{4}\|\nabla\Delta u\|_{L^2}^2, \quad (3.25)
\end{aligned}$$

in which we applied Hölder's inequality, Plancherel theorem, Young's inequality, $u \in L^\infty_t L^2_x \cap L^2_t H^2_x$ from (3.24), Gagliardo-Nirenberg interpolation inequality. Consequently, we can finish the proof with Grönwall's inequality.

□

CHAPTER 4
MAGNETOHYDRODYNAMICS SYSTEM

4.1 Motivations for Magnetohydrodynamics system from Physics and real-world applications.

Magnetohydrodynamics (MHD) system has an abundance of relevance to the study of geophysics, astrophysics, and cosmology; to control the motion of electrically conducting fluids in a magnetic field (see [8, 44, 91]).

Recently, there has been a lot of interest in the global regularity problem of the MHD system, and several significant findings have been made about it and related problems (see ([38], [20], [21], [23], [24], [25], [26], [27], [28], [40]).

The 3-D MHD system with standard Laplacian dissipation remains an open problem concerning whether the 3-D MHD system is globally well-posed or not. As is well-known (see [110],[111],[122]), the generalized 3-D MHD system:

$$\partial_t u + (u\cdot\nabla)u + \nabla\pi + \nu(-\Delta)^{\alpha}u = (b\cdot\nabla)b \tag{4.1a}$$

$$\partial_t b + (u\cdot\nabla)b + \eta(-\Delta)^{\beta}b = (b\cdot\nabla)u \tag{4.1b}$$

$$\nabla\cdot u = 0; \tag{4.1c}$$

with α and β satisfying the following inequality

$$\alpha \geq \frac{5}{4},\ \beta \geq \frac{5}{4}$$

always possesses a global regularity solution. For $\alpha = \frac{5}{4},\ \beta = \frac{5}{4}$ we can recover a global regularity result. In which the note is made that if $\nabla\cdot b_0 = 0$, then $\nabla\cdot b = 0$ remains true for all the $t > 0$. For a better understanding, we included the sketch of proof as follows:

The following 3-D MHD system has global regularity, which can be written as follows:

$$\partial_t u + (u\cdot\nabla)u + \nabla\pi + \nu(-\Delta)^{\frac{5}{4}}u = (b\cdot\nabla)b \tag{4.2a}$$

$$\partial_t b + (u\cdot\nabla)b + \eta(-\Delta)^{\frac{5}{4}}b = (b\cdot\nabla)u \tag{4.2b}$$

$$\nabla \cdot u = 0; \tag{4.2c}$$

where the note is made that if $\nabla \cdot b_0 = 0$, then $\nabla \cdot b = 0$ remains true for all the $t > 0$. For convenience, throughout the entire discussion, we assume $\nu = \eta = 1$. The energy equality is as follows:

$$\frac{1}{2}\frac{d}{dt}(\|u\|_{L^2}^2 + \|b\|_{L^2}^2) + \|\Lambda^{\frac{5}{4}}u\|_{L^2}^2 + \|\Lambda^{\frac{5}{4}}b\|_{L^2}^2 = \int_{\mathbb{R}^3}(b \cdot \nabla)(b \cdot u)dx = 0 \tag{4.3}$$

We are looking for H^1-bound which implies the following identity as follows:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \|\Lambda^{\frac{9}{4}}u\|_{L^2}^2 + \|\Lambda^{\frac{9}{4}}b\|_{L^2}^2 &= -\int_{\mathbb{R}^3}(u \cdot \nabla)u \cdot \Delta u dx \\
&+ \int_{\mathbb{R}^3}(b \cdot \nabla)b \cdot \Delta u dx - \int_{\mathbb{R}^3}(u \cdot \nabla)b \cdot \Delta b dx + \int_{\mathbb{R}^3}(b \cdot \nabla)u \cdot \Delta b dx \\
&= I_1 + I_2 + I_3 + I_4
\end{aligned} \tag{4.4}$$

Using the Hölder's, Gagliardo-Nirenberg interpolation, and Young's inequalities, we can estimate I_1 as follows:

$$\begin{aligned}
I_1 &= -\int_{\mathbb{R}^3}(u \cdot \nabla)u \cdot \Delta u dx \\
&\lesssim \|\nabla u\|_{L^4}^2 \|\nabla u\|_{L^2} \\
&\lesssim \|\Lambda^{\frac{5}{4}}u\|_{L^2}\|\Lambda^{\frac{9}{4}}u\|_{L^2}\|\nabla u\|_{L^2} \\
&\lesssim \|\Lambda^{\frac{5}{4}}u\|_{L^2}^2\|\nabla u\|_{L^2}^2 + \frac{1}{4}\|\Lambda^{\frac{9}{4}}u\|_{L^2}^2
\end{aligned}$$

Using the Hölder's, Gagliardo-Nirenberg interpolation, and Young's inequalities, we can estimate $I_2, I_3,$ and I_4 as follows:

$$\begin{aligned}
I_2, I_3, I_4 &= \int_{\mathbb{R}^3}(b \cdot \nabla)b \cdot \Delta u dx \\
&\lesssim \|\nabla b\|_{L^4}^2 \|\nabla u\|_{L^2} \\
&\lesssim \|\Lambda^{\frac{5}{4}}b\|_{L^2}\|\Lambda^{\frac{9}{4}}b\|_{L^2}\|\nabla u\|_{L^2} \\
&\lesssim \|\Lambda^{\frac{5}{4}}b\|_{L^2}^2\|\nabla u\|_{L^2}^2 + \frac{1}{4}\|\Lambda^{\frac{9}{4}}b\|_{L^2}^2,
\end{aligned}$$

Applying the estimates, we have

$$\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \|\Lambda^{\frac{9}{4}} u\|_{L^2}^2 + \|\Lambda^{\frac{9}{4}} b\|_2^2 \lesssim (\|\Lambda^{\frac{5}{4}} u\|_{L^2}^2 + \|\Lambda^{\frac{5}{4}} b\|_{L^2}^2)(1 + \|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \frac{1}{4}\|\Lambda^{\frac{9}{4}} u\|_{L^2}^2 + \frac{1}{4}\|\Lambda^{\frac{9}{4}} b\|_{L^2}^2 \tag{4.5}$$

Since $u \in L_t^2 \dot{H}_x^{\frac{5}{4}}$ and $b \in L_t^2 \dot{H}_x^{\frac{5}{4}}$ because of (4.3), we can close this estimate. For higher regularity, we can continue the process inductively.

In addition, we know that the 2-D MHD system is globally well-posed with standard Laplacian dissipation and standard Laplacian diffusion. The question is whether or not the global regularity still holds if we consider zero viscous dissipation. Thus, we focus on the 2-D MHD system with zero viscous dissipation and consider magnetic dissipation in the critical space level, which can be written as follows:

$$\partial_t u + (u \cdot \nabla)u + \nabla \pi = -\nu u + (b \cdot \nabla)b \tag{4.6a}$$

$$\partial_t b + (u \cdot \nabla)b = -\eta \Lambda^2 b + (b \cdot \nabla)u \tag{4.6b}$$

$$\nabla \cdot u = 0; \tag{4.6c}$$

where $u = u(x,t)$ refers velocity, $b = b(x,t)$ refers magnetic field, $\pi = \pi(x,t)$ the pressure, ν is viscosity, and η is magnetic resistivity. Where the note is made that if $\nabla \cdot b_0 = 0$, then $\nabla \cdot b = 0$ remains true for all the $t > 0$. The question of global well-posedness issue on $(1a)$-$(1c)$ has drawn a substantial amount of interest. Many progresses have been made in this direction.

The fundamental difficulty in proving the global well-posedness on (4.6a)-(4.6c) is lack of L^∞-bound for ω.

4.2 Statement of the Theorem 33 and its proof

In order to gain a better understanding of the system (4.6a)-(4.6c), we took an attempt in proving the global well-posedness result; however, due to the difficulties, we ended up with a global regularity criterion result which is stated as follows:

Theorem 33. *Let* $(u_0, b_0) \in H^\beta, \beta > 2$ *and* (u, b) *be a local strong solution to* (4.6a)-

(4.6c) *on [0,T) and if*

$$\int_0^t \|\omega\|_{\dot{H}^1} d\tau < \infty, \tag{4.7}$$

then the solution (u, b) *can be continued to the strong solution on* $[0, T^*)$ *for some* $T^* > T$.

Proof. We assume $\nu = \eta = 1$.

Taking the $L^2(\mathbb{R}^2)$-inner product of (4.6a) with u implies that

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|u\|_{L^2}^2 = -\int_{\mathbb{R}^2}(u\cdot\nabla)u\cdot u dx + \int_{\mathbb{R}^2}(b\cdot\nabla)b\cdot u dx = 0 + \int_{\mathbb{R}^2}(b\cdot\nabla)b\cdot u dx \tag{4.8}$$

Taking the $L^2(\mathbb{R}^2)$-inner product of (4.6b) with b implies that

$$\frac{1}{2}\frac{d}{dt}\|b\|_{L^2}^2 + \|\Lambda b\|_{L^2}^2 = -\int_{\mathbb{R}^2}(u\cdot\nabla)b\cdot b dx + \int_{\mathbb{R}^2}(b\cdot\nabla)u\cdot b dx = 0 + \int_{\mathbb{R}^2}(b\cdot\nabla)u\cdot b dx \tag{4.9}$$

Summing up (4.8) and (4.9) implies the energy identity

$$\frac{1}{2}\frac{d}{dt}(\|u\|_{L^2}^2 + \|b\|_{L^2}^2) + \|u\|_{L^2}^2 + \|\Lambda b\|_{L^2}^2 = \int_{\mathbb{R}^2}(b\cdot\nabla)(b\cdot u)dx = 0 \tag{4.10}$$

Taking the curl on (4.6a) and (4.6b) respectively implies that

$$\partial_t\omega + (u\cdot\nabla)\omega = -\omega + (b\cdot\nabla)j \tag{4.11a}$$

$$\partial_t j + (u\cdot\nabla)j = -\Lambda^2 j + (b\cdot\nabla)\omega + 2[-\partial_1 u_1(\partial_1 b_2 + \partial_2 b_1) + \partial_1 b_1(\partial_1 u_2 + \partial_2 u_1)]; \tag{4.11b}$$

Taking the $L^2(\mathbb{R}^2)$-inner product of (4.11a) with ω implies that

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2}^2 + \|\omega\|_{L^2}^2 = \int_{\mathbb{R}^2}(b\cdot\nabla)j\cdot\omega dx \tag{4.12}$$

Taking the $L^2(\mathbb{R}^2)$-inner product of (4.11b) with j implies that

$$\frac{1}{2}\frac{d}{dt}\|j\|_{L^2}^2 + \|\Lambda j\|_{L^2}^2 = \int_{\mathbb{R}^2}(b\cdot\nabla)\omega\cdot j dx + 2\int_{\mathbb{R}^2} j[-\partial_1 u_1(\partial_1 b_2 + \partial_2 b_1) + \partial_1 b_1(\partial_1 u_2 + \partial_2 u_1)]dx \tag{4.13}$$

Summing up (4.12) and (4.13) implies the following identity

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2) + \|\omega\|_{L^2}^2 + \|\Lambda j\|_{L^2}^2 &= \int_{\mathbb{R}^2}(b\cdot\nabla)(\omega\cdot j)dx \\ &+ 2\int_{\mathbb{R}^2} j[-\partial_1 u_1(\partial_1 b_2 + \partial_2 b_1) + \partial_1 b_1(\partial_1 u_2 + \partial_2 u_1)]dx \\ &\lesssim 0 + 2\int_{\mathbb{R}^2} j[-\partial_1 u_1(\partial_1 b_2 + \partial_2 b_1) + \partial_1 b_1(\partial_1 u_2 + \partial_2 u_1)]dx \\ &\lesssim \|\nabla b\|_{L^4}\|\nabla u\|_{L^2}\|j\|_{L^4} \\ &\lesssim \|\nabla b\|_{L^2}^{\frac{1}{2}}\|\Delta b\|_{L^2}^{\frac{1}{2}}\|\omega\|_{L^2}\|j\|_{L^2}^{\frac{1}{2}}\|\nabla j\|_{L^2}^{\frac{1}{2}} \\ &\lesssim \|j\|_{L^2}\|\omega\|_{L^2}\|\nabla j\|_{L^2} \\ &\lesssim \frac{1}{4}\|\nabla j\|_{L^2}^2 + \|\omega\|_{L^2}^2\|j\|_{L^2}^2 \\ &\lesssim \frac{1}{4}\|\nabla j\|_{L^2}^2 + (\|j\|_{L^2}^2 + \|\omega\|_{L^2}^2)\|j\|_{L^2}^2,\end{aligned} \tag{4.14}$$

in which we applied Hölder's inequality, Gagliardo-Nirenberg interpolation inequality, and Young's inequality. By Gronwall's inequality,

$$(\|\omega(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2) \lesssim e^{\int_0^t \|j(\tau)\|_{L^2}^2 d\tau}(\|\omega(0)\|_{L^2} + \|j(0)\|_{L^2}) \tag{4.15}$$

Since $b \in L_t^2\dot{H}_x^1$, thus $\omega \in L_t^\infty L_x^2$, $j \in L_t^\infty L_x^2$, $\omega \in L_t^2 L_x^2$, and $j \in L_t^2\dot{H}_x^1$.
Recall that

$$\begin{aligned}&\int_{\mathbb{R}^2}(b\cdot\nabla)\Lambda^\beta b\cdot\Lambda^\beta u dx \neq 0. \\ &\int_{\mathbb{R}^2}(b\cdot\nabla)\Lambda^\beta u\cdot\Lambda^\beta b dx \neq 0.\end{aligned} \tag{4.16}$$

However,

$$\int_{\mathbb{R}^2} (b\cdot\nabla)\Lambda^\beta b\cdot\Lambda^\beta u dx + \int_{\mathbb{R}^2} (b\cdot\nabla)\Lambda^\beta u\cdot\Lambda^\beta b dx = \int_{\mathbb{R}^2} (b\cdot\nabla)(\Lambda^\beta u\cdot\Lambda^\beta b) dx = 0. \tag{4.17}$$

Applying Λ^β (with $\beta > 1$) on the (4.6b) and taking the $L^2(\mathbb{R}^2)$-inner products with $\Lambda^\beta b$

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\Lambda^\beta b\|_{L^2} + \|\Lambda^{\beta+1} b\|_{L^2}^2 &= -\int_{\mathbb{R}^2} \Lambda^\beta[(u\cdot\nabla)]b\cdot\Lambda^\beta b dx + \int_{\mathbb{R}^2} \Lambda^\beta[(b\cdot\nabla)]u\cdot\Lambda^\beta b dx \\
&\lesssim \|\Lambda^\beta u\|_{L^2}\|\nabla b\|_{L^4}\|\Lambda^\beta b\|_{L^4} + \|\Lambda^\beta b\|_{L^2}\|\nabla u\|_{L^\infty}\|\Lambda^\beta b\|_{L^2} \\
&+ \|\Lambda^\beta b\|_{L^4}\|\nabla u\|_{L^2}\|\Lambda^\beta b\|_{L^4} + \|\Lambda^\beta u\|_{L^2}\|\nabla b\|_{L^\infty}\|\Lambda^\beta b\|_{L^2} \\
&\lesssim \|\Lambda^\beta u\|_{L^2}\|\Lambda b\|_{L^2}^{\frac{2\beta-1}{2\beta}}\|\Lambda^{\beta+1} b\|_{L^2}^{\frac{1}{2\beta}}\|\Lambda b\|_{L^2}^{\frac{1}{2\beta}}\|\Lambda^{\beta+1} b\|_{L^2}^{\frac{2\beta-1}{2\beta}} \\
&+ \|\Lambda^\beta b\|_{L^2}^{\frac{1}{2}}\|\Lambda^{\beta+1} b\|_{L^2}^{\frac{1}{2}}\|w\|_{L^2}\|\Lambda^\beta b\|_{L^2}^{\frac{1}{2}}\|\Lambda^{\beta+1} b\|_{L^2}^{\frac{1}{2}} \\
&+ (\|\nabla u\|_{L^\infty} + \|\nabla b\|_{L^\infty})(\|\Lambda^\beta u\|_{L^2}^2 + (\|\Lambda^\beta b\|_{L^2}^2) \\
&\lesssim \|\Lambda^\beta u\|_{L^2}\|\Lambda b\|_{L^2}\|\Lambda^{\beta+1} b\|_{L^2} + \|\Lambda^\beta b\|_{L^2}\|\omega\|_{L^2}\|\Lambda^{\beta+1} b\|_{L^2} \\
&+ (\|\nabla u\|_{L^\infty} + \|\nabla b\|_{L^\infty})(\|\Lambda^\beta u\|_{L^2}^2 + (\|\Lambda^\beta b\|_{L^2}^2) \\
&\lesssim \|\Lambda^\beta u\|_{L^2}^2\|\Lambda b\|_{L^2}^2 + \frac{1}{4}\|\Lambda^{\beta+1} b\|_{L^2}^2 + \|\omega\|_{L^2}^2\|\Lambda^\beta b\|_{L^2}^2 \\
&+ \frac{1}{4}\|\Lambda^{\beta+1} b\|_{L^2}^2 \\
&+ (\|\nabla u\|_{L^\infty} + \|\nabla b\|_{L^\infty})(\|\Lambda^\beta u\|_{L^2}^2 + (\|\Lambda^\beta b\|_{L^2}^2),
\end{aligned} \tag{4.18}$$

in which we applied integration by parts, Hölder's inequality, commutator estimate, Gagliardo-Nirenberg interpolation inequality, and Young's inequality. For simplicity, the details computations of the interpolation parts are as follows:

$$\begin{aligned}
&\|\nabla b\|_{L^4} \lesssim \|\Lambda b\|_{L^2}^{\epsilon}\|\Lambda^{\beta+1} b\|_{L^2}^{1-\epsilon} \\
&\frac{1}{4} - \frac{1}{2} = (1-\epsilon)(\frac{1}{2} - \frac{\beta+1}{2}) \\
&\frac{-1}{4} = (1-\epsilon)(\frac{-\beta}{2}) \\
&\frac{-1}{4} + \frac{\beta}{2} = \frac{\epsilon\beta}{2}
\end{aligned}$$

$$\epsilon = \frac{2\beta - 1}{2\beta}$$
$$1 - \epsilon = 1 - \frac{2\beta - 1}{2\beta} = \frac{1}{2\beta}$$

By G-N

$$\|\Lambda^\beta b\|_{L^4} \lesssim \|\nabla b\|_{L^2}^{\epsilon} \|\Lambda^{\beta+1} b\|_{L^2}^{1-\epsilon}$$
$$\frac{1}{4} - \frac{\beta}{2} = (1-\epsilon)(\frac{1}{2} - \frac{\beta+1}{2})$$
$$\frac{-2\beta+1}{4} = (1-\epsilon)(\frac{-\beta}{2})$$
$$\frac{-2\beta+1}{4} + \frac{\beta}{2} = \frac{\epsilon\beta}{2}$$
$$\epsilon = \frac{1}{2\beta}$$
$$1 - \epsilon = 1 - \frac{2\beta - 1}{2\beta} = \frac{1}{2\beta}$$

Applying Λ^β (with $\beta > 1$) on (4.6a) and taking the $L^2(\mathbb{R}^2)$-inner products with $\Lambda^\beta u$ implies that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Lambda^\beta u\|_{L^2}^2 + \|\Lambda^\beta u\|_{L^2}^2 = -\int_{\mathbb{R}^3} \Lambda^\beta[(u\cdot\nabla)]u \cdot \Lambda^\beta u dx + \int_{\mathbb{R}^3} \Lambda^\beta[(b\cdot\nabla)]b \cdot \Lambda^\beta u dx \\
&\lesssim \|\Lambda^\beta u\|_{L^2}\|\nabla u\|_{L^\infty}\|\Lambda^\beta u\|_{L^2} + \|\nabla b\|_{L^4}\|\Lambda^\beta u\|_{L^2}\|\Lambda^\beta b\|_{L^4} \\
&\lesssim \|\nabla u\|_{L^\infty}(\|\Lambda^\beta u\|_{L^2}^2 + \|\Lambda^\beta u\|_{L^2}^2) + \|\Lambda^\beta u\|_{L^2}\|\Lambda b\|_{L^2}^{\frac{2\beta-1}{2\beta}}\|\Lambda^{\beta+1} b\|_{L^2}^{\frac{1}{2\beta}}\|\Lambda b\|_{L^2}^{\frac{1}{2\beta}}\|\Lambda^{\beta+1} b\|_{L^2}^{\frac{2\beta-1}{2\beta}} \\
&\lesssim \|\nabla u\|_{L^\infty}(\|\Lambda^\beta u\|_{L^2}^2 + \|\Lambda^\beta u\|_{L^2}^2) + \|\Lambda^\beta u\|_{L^2}^2\|\Lambda b\|_{L^2}^2 + \frac{1}{4}\|\Lambda^{\beta+1} b\|_{L^2}^2, \qquad (4.19)
\end{aligned}$$

in which we applied integration by parts, Hölder's inequality, commutator estimate, Gagliardo-Nirenberg interpolation inequality, and Young's inequality.

Summing up (4.18) and(4.19),

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\Lambda^\beta u\|_{L^2}^2 + \|\Lambda^\beta b\|_{L^2}) + \|\Lambda^\beta u\|_{L^2}^2 + \|\Lambda^{\beta+1} b\|_{L^2}^2 \lesssim \|\Lambda^\beta u\|_{L^2}^2\|\Lambda b\|_{L^2}^2 + \frac{1}{4}\|\Lambda^{\beta+1} b\|_{L^2}^2 \\
&+ \|\omega\|_{L^2}^2\|\Lambda^\beta b\|_{L^2}^2 + \frac{1}{4}\|\Lambda^{\beta+1} b\|_{L^2}^2 + \|\Lambda^\beta u\|_{L^2}^2\|\Lambda b\|_{L^2}^2 + \frac{1}{4}\|\Lambda^{\beta+1} b\|_{L^2}^2 \\
&+ (\|\nabla u\|_{L^\infty} + \|\nabla b\|_{L^\infty})(\|\Lambda^\beta u\|_{L^2}^2 + \|\Lambda^\beta b\|_{L^2}^2)
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \|\Lambda^{\beta} u\|_{L^2}^2 \|\Lambda b\|_{L^2}^2 + \frac{1}{4}\|\Lambda^{\beta+1} b\|_{L^2}^2 + \|\omega\|_{L^2}^2 \|\Lambda^{\beta} b\|_{L^2}^2 + \frac{1}{4}\|\Lambda^{\beta+1} b\|_{L^2}^2 + \|\Lambda^{\beta} u\|_{L^2}^2 \|\Lambda b\|_{L^2}^2 \\
&+ \frac{1}{4}\|\Lambda^{\beta+1} b\|_{L^2}^2 + ((\|\omega\|_{BMO} log(1 + \|\Lambda^{\beta} u\|_{L^2}^2)) \\
&+ (\|j\|_{BMO} log(1 + \|\Lambda^{\beta} b\|_{L^2}^2)))(\|\Lambda^{\beta} u\|_{L^2}^2 + (\|\Lambda^{\beta} b\|_{L^2}^2) \\
&\lesssim \|\Lambda^{\beta} u\|_{L^2}^2 \|\Lambda b\|_{L^2}^2 + \frac{1}{4}\|\Lambda^{\beta+1} b\|_{L^2}^2 + \|\omega\|_{L^2}^2 \|\Lambda^{\beta} b\|_{L^2}^2 + \frac{1}{4}\|\Lambda^{\beta+1} b\|_{L^2}^2 + \|\Lambda^{\beta} u\|_{L^2}^2 \|\Lambda b\|_{L^2}^2 \\
&+ \frac{1}{4}\|\Lambda^{\beta+1} b\|_{L^2}^2 \\
&+ ((\|\omega\|_{\dot{H}^1} log(1 + \|\Lambda^{\beta} u\|_{L^2}^2)) + (\|j\|_{\dot{H}^1} log(1 + \|\Lambda^{\beta} b\|_{L^2}^2)))(\|\Lambda^{\beta} u\|_{L^2}^2 + (\|\Lambda^{\beta} b\|_{L^2}^2)
\end{aligned} \tag{4.20}
$$

Since $\int_0^t \|\omega(\tau) d\tau\|_{L^2}^2 d\tau < \infty$, $\int_0^t \|\nabla j(\tau) d\tau\|_{L^2}^2 d\tau < \infty$, and moreover if $\int_0^t \|\omega\|_{\dot{H}^1}^2 d\tau < \infty$ then by Gronwall's inequality, we can conclude from (4.20) that $u \in L^\infty(0, T, H^\beta) \cap L^2(0, T, H^\beta)$ and $b \in L^\infty(0, T, H^\beta) \cap L^2(0, T, H^{\beta+1})$. Thus, we can conclude the theorem. □

As we know that the system of $2\frac{1}{2}$-D NSE:

$$\partial_t u + (u \cdot \nabla) u + \nabla p = \Delta u_h + \Delta u_v \tag{4.21a}$$

$$\nabla \cdot u = 0 \tag{4.21b}$$

always has global regularity solutions with standard Laplacian dissipation. However, we wanted to examine how much dissipation is necessary to ensure global regularity. Thus one of our previous investigations, we went through the following $\frac{1}{2}$-D NSE:

$$\partial_t u + (u \cdot \nabla) u + \nabla p + \Lambda u_h = \Delta u_v \tag{4.22a}$$

$$\nabla \cdot u = 0, \tag{4.22b}$$

in which we have global regularity solutions. The immediate question is whether we can reduce dissipation more than what we have now in (4.22a)-(4.22b). The answer of this question is Yes, because $2\frac{1}{2}$-D Euler equations:

$$\partial_t u + (u \cdot \nabla) u + \nabla p = 0 \tag{4.23a}$$

$$\nabla \cdot u = 0, \tag{4.23b}$$

are considered to be globally well-posed.

Inspiration from the previous studies, we focus on the global well-posedness for the $2\frac{1}{2}$-D MHD system that can be written as follows:

$$\partial_t u + (u \cdot \nabla)u + \nabla\pi + \Lambda u_h = -\Lambda^2 u_v + (b \cdot \nabla)b \tag{4.24a}$$

$$\partial_t b + (u \cdot \nabla)b = -\Lambda^2 b_h - \Lambda^2 b_v + (b \cdot \nabla)u \tag{4.24b}$$

$$\nabla \cdot u = 0; \tag{4.24c}$$

where the note is made that if $\nabla \cdot b_0 = 0$, then $\nabla \cdot b = 0$ remains true for all the $t > 0$.

4.3 The statement of Theorem 34 and its proof

Theorem 34. *Suppose that* $(u_0, b_0) \in H^\beta(\mathbb{R}^2) \times H^\beta(\mathbb{R}^2)$ *where* $\beta > 2$ *and* $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. *Then there exists a unique solution to* (4.24a)-(4.24c) *such that*

$$\begin{aligned} &u \in L^\infty((0,\infty); H^\beta(\mathbb{R}^2)), \quad u_h \in L^2((0,\infty); H^{\beta+\frac{1}{2}}(\mathbb{R}^2)), \\ &u_v \in L^2((0,\infty); H^{\beta+1}(\mathbb{R}^2)), \quad b \in L^\infty((0,\infty); H^\beta(\mathbb{R}^2)), \\ &b_h, b_v \in L^2((0,\infty); H^{\beta+1}(\mathbb{R}^2)). \end{aligned} \tag{4.25}$$

Proof. Taking the $L^2(\mathbb{R}^2)$-inner products of (4.24a) with u implies that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}} u_h\|_{L^2}^2 + \|\Lambda u_v\|_{L^2}^2 &= -\int_{\mathbb{R}^3} (u \cdot \nabla)\frac{|u|^2}{2} dx + \int_{\mathbb{R}^3} (b \cdot \nabla)b \cdot u dx \\ &= 0 + \int_{\mathbb{R}^3} (b \cdot \nabla)b \cdot u dx \end{aligned} \tag{4.26}$$

Taking the $L^2(\mathbb{R}^2)$-inner products of (4.24b) with b implies that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|b\|_{L^2}^2 + \|\Lambda b_h\|_{L^2}^2 + \|\Lambda b_v\|_{L^2}^2 &= -\int_{\mathbb{R}^3} (u \cdot \nabla)\frac{|b|^2}{2} dx + \int_{\mathbb{R}^3} (b \cdot \nabla)u \cdot b dx \\ &= 0 + \int_{\mathbb{R}^3} (b \cdot \nabla)u \cdot b dx \end{aligned} \tag{4.27}$$

Summing up (4.26) and (4.27) implies the energy identity

$$\frac{1}{2}\frac{d}{dt}(\|u\|_{L^2}^2 + \|b\|_{L^2}^2) + \|\Lambda^{\frac{1}{2}} u_h\|_{L^2}^2 + \|\Lambda u_v\|_{L^2}^2 + \|\Lambda b_h\|_{L^2}^2 + \|\Lambda b_v\|_{L^2}^2 = 0 \tag{4.28}$$

Taking the curl on (4.24a) and (4.24b) respectively implies that

$$\partial_t\omega - (\omega\cdot\nabla)u + (u\cdot\nabla)\omega - \Lambda\omega_v = \Lambda^2\omega_h - (j\cdot\nabla)b + (b\cdot\nabla)j \tag{4.29a}$$

$$\begin{aligned}&\partial_t j + (u\cdot\nabla)j + (\omega\cdot\nabla)b = \Lambda^2 j_h + \Lambda^2 j_v + (b\cdot\nabla)\omega + (j\cdot\nabla)u\\ &+ 2(0,0,\partial_1 b\cdot\partial_2 u - \partial_2 b\cdot\partial_1 u);\end{aligned} \tag{4.29b}$$

$$\nabla\cdot\omega = 0 \tag{4.29c}$$

Taking the $L^2(\mathbb{R}^2)$-inner products of (4.29a) with ω implies that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2}^2 + \|\Lambda\omega_h\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}}\omega_v\|_{L^2}^2 &= \int_{\mathbb{R}^2}(\omega\cdot\nabla)u\cdot\omega dx + \int_{\mathbb{R}^2}(b\cdot\nabla)j\cdot\omega dx\\ &- \int_{\mathbb{R}^2}(j\cdot\nabla)b\cdot\omega dx\end{aligned} \tag{4.30}$$

Taking the $L^2(\mathbb{R}^2)$-inner products of (4.29b) with j implies that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|j\|_{L^2}^2 + \|\Lambda j_h\|_{L^2}^2 + \|\Lambda j_v\|_{L^2}^2 &= -\int_{\mathbb{R}^2}(\omega\cdot\nabla)b\cdot j dx + \int_{\mathbb{R}^2}(b\cdot\nabla)\omega\cdot j dx\\ &+ \int_{\mathbb{R}^2}(j\cdot\nabla)u\cdot j dx + 2\int_{\mathbb{R}^2}2(0,0,\partial_1 b\cdot\partial_2 u - \partial_2 b\cdot\partial_1 u)\cdot j dx\end{aligned} \tag{4.31}$$

Summing up (4.30) and (4.31) implies the following identity

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2) + \|\Lambda\omega_h\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}}\omega_v\|_{L^2}^2 + \|\Lambda j_h\|_{L^2}^2 + \|\Lambda j_v\|_{L^2}^2 &= \int_{\mathbb{R}^2}(\omega\cdot\nabla)u\cdot\omega dx\\ &- \int_{\mathbb{R}^2}(j\cdot\nabla)b\cdot\omega dx - \int_{\mathbb{R}^2}(\omega\cdot\nabla)b\cdot j dx\\ &+ \int_{\mathbb{R}^2}(j\cdot\nabla)u\cdot j dx + 2\int_{\mathbb{R}^2}(0,0,\partial_1 b\cdot\partial_2 u - \partial_2 b\cdot\partial_1 u)\cdot j dx\\ &= I_1 + I_2 + I_3 + I_4 + I_5\end{aligned} \tag{4.32}$$

Holder's inequality, Gagliardo-Nirenberg interpolation, and Young's inequalities are applied to estimate I_1 as follows:

$$I_1 = \int_{\mathbb{R}^2}(\omega\cdot\nabla)u\cdot\omega dx = \sum_{i=1}^{2}\sum_{k=1}^{3}\int_{\mathbb{R}^2}\omega_i\partial_i u_k\omega_k dx$$

$$
\begin{aligned}
&= \sum_{i=1}^{2}\sum_{k=1}^{2}\int_{\mathbb{R}^2} \omega_i \partial_i u_k \omega_k dx + \sum_{i=1}^{2}\int_{\mathbb{R}^2} \omega_i \partial_i u_3 \omega_3 dx \\
&\leq \|\omega_h\|_{L^4}^2 \|\omega\|_{L^2} + 0 \\
&\lesssim \|\omega_h\|_{L^2} \|\Lambda\omega_h\|_{L^2} \|\omega\|_{L^2} \\
&\lesssim \|\omega_h\|_{L^2}^2 \|\omega\|_{L^2}^2 + \frac{1}{4}\|\Lambda\omega_h\|_{L^2}^2 \qquad (4.33)
\end{aligned}
$$

Holder's inequality, Gagliardo-Nirenberg interpolation, and Young's inequalities are applied to estimate I_2, as follows:

$$
\begin{aligned}
I_2 &= -\int_{\mathbb{R}^2} (j\cdot\nabla) b \cdot \omega = -\sum_{i=1}^{2}\sum_{k=1}^{3}\int_{\mathbb{R}^2} j_i \partial_i b_k \omega_k dx \\
&= -\sum_{i=1}^{2}\int_{\mathbb{R}^2} j_i \partial_i b_3 \omega_3 dx - \sum_{i=1}^{2}\sum_{k=1}^{2}\int_{\mathbb{R}^2} j_i \partial_i b_k \omega_k dx \\
&\leq 0 + \|j_h\|_{L^4} \|\omega_h\|_{L^2} \|\nabla b\|_{L^4} \\
&\lesssim \|\nabla b\|_{L^4} \|\omega\|_{L^2} \|\nabla b\|_{L^4} \\
&\approx \|\nabla j\|_{L^2} \|j\|_{L^2} \|\omega\|_{L^2} \\
&\lesssim \|j\|_{L^2}^2 \|\omega\|_{L^2}^2 + \frac{1}{4}\|\nabla j\|_{L^2}^2 \\
&\approx (\|j_h\|_{L^2}^2 + \|j_v\|_{L^2}^2) \|\omega\|_{L^2} + \frac{1}{4}(\|\nabla j_h\|_{L^2}^2 + \|\nabla j_v\|_{L^2}^2). \qquad (4.34)
\end{aligned}
$$

Holder's inequality, Gagliardo-Nirenberg interpolation, and Young's inequalities are applied to estimate I_3, as follows:

$$
\begin{aligned}
I_3 &= -\int_{\mathbb{R}^2} (\omega\cdot\nabla) b \cdot j dx = -\sum_{i=1}^{2}\sum_{k=1}^{3}\int_{\mathbb{R}^2} \omega_i \partial_i b_k j_k dx \\
&= -\sum_{i=1}^{2}\int_{\mathbb{R}^2} \omega_i \partial_i b_3 j_3 dx + \sum_{i=1}^{2}\sum_{k=1}^{2}\int_{\mathbb{R}^2} \omega_i \partial_i b_k j_k dx \\
&\leq \|\omega_h\|_{L^2} \|\nabla b_v\|_{L^4} \|j_3\|_{L^4} + \|\omega_h\|_{L^2} \|\nabla b_h\|_{L^4} \|j_h\|_{L^4} \\
&\lesssim \|\nabla b\|_{L^4} \|\omega\|_{L^2} \|\nabla b\|_{L^4} \\
&\approx \|\omega\|_{L^2} \|j\|_{L^2} \|\nabla j\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \|\omega\|_{L^2}^2\|j\|_{L^2}^2 + \frac{1}{4}\|\nabla j\|_{L^2}^2 \\
&\approx (\|j_h\|_{L^2}^2 + \|j_v\|_{L^2}^2)\|\nabla u\|_{L^2} + \frac{1}{4}(\|\nabla j_h\|_{L^2}^2 + \|\nabla j_v\|_{L^2}^2). \qquad (4.35)
\end{aligned}
$$

Holder's inequality, Gagliardo-Nirenberg interpolation, and Young's inequalities are applied to estimate I_4, as follows:

$$
\begin{aligned}
I_4 &= \int_{\mathbb{R}^2} (j\cdot\nabla)u\cdot j dx = \sum_{i=1}^{2}\sum_{k=1}^{3}\int_{\mathbb{R}^2} j_i\partial_i u_k j_k dx \\
&= \sum_{i=1}^{2}\int_{\mathbb{R}^2} j_i\partial_i u_3 j_3 dx + \sum_{i,k=1}^{2}\int_{\mathbb{R}^2} j_i\partial_i u_k j_k dx \\
&\leq \|\nabla b\|_{L^4}\|\nabla u\|_{L^2}\|\nabla b\|_{L^4} \\
&\approx \|j\|_{L^2}\|\nabla u\|_{L^2}\|\nabla j\|_{L^2} \\
&\lesssim \|j\|_{L^2}^2\|\nabla u\|_{L^2}^2 + \frac{1}{4}\|\nabla j\|_{L^2}^2 \\
&\approx (\|j_h\|_{L^2}^2 + \|j_v\|_{L^2}^2)\|\nabla u\|_{L^2} + \frac{1}{4}(\|\nabla j_h\|_{L^2}^2 + \|\nabla j_v\|_{L^2}^2). \qquad (4.36)
\end{aligned}
$$

Holder's inequality, Gagliardo-Nirenberg interpolation, and Young's inequalities are applied to estimate I_5, as follows:

$$
\begin{aligned}
I_5 =& 2\int_{\mathbb{R}^2} (0,0,\partial_1 b\cdot\partial_2 u - \partial_2 b\cdot\partial_1 u)\cdot j dx \\
&= \int_{\mathbb{R}^2} (\partial_1 b\cdot\partial_2 u)j_3 - (\partial_2 b\cdot\partial_1 u)j_3 dx \\
&\leq \|\nabla b\|_{L^4}\|\nabla u\|_{L^2}\|\nabla b\|_{L^4} \\
&\approx \|j\|_{L^2}\|\nabla u\|_{L^2}\|\nabla j\|_{L^2} \\
&\lesssim \|j\|_{L^2}^2\|\nabla u\|_{L^2}^2 + \frac{1}{4}\|\nabla j\|_{L^2}^2 \\
&\approx (\|j_h\|_{L^2}^2 + \|j_v\|_{L^2}^2)\|\nabla u\|_{L^2} + \frac{1}{4}(\|\nabla j_h\|_{L^2}^2 + \|\nabla j_v\|_{L^2}^2). \qquad (4.37)
\end{aligned}
$$

Combining all these estimates (4.33),(4.34), (4.35), (4.36), and (4.37), we have from

(4.32)

$$\frac{1}{2}\frac{d}{dt}(\|\omega\|_{L^2}^2+\|j\|_{L^2}^2)+\|\Lambda\omega_h\|_{L^2}^2+\|\Lambda^{\frac{1}{2}}\omega_v\|_{L^2}^2+\|\Lambda j_h\|_{L^2}^2+\|\Lambda j_v\|_{L^2}^2 \lesssim \|\omega_h\|_{L^2}^2\|\omega\|_{L^2}^2$$
$$+\frac{1}{4}\|\Lambda\omega_h\|_{L^2}^2+(\|j_h\|_{L^2}^2+\|j_v\|_{L^2}^2)\|\nabla u\|_{L^2}+\frac{1}{4}(\|\nabla j_h\|_{L^2}^2+\|\nabla j_v\|_{L^2}^2). \tag{4.38}$$

Recall that

$$\int_{\mathbb{R}^2}(b\cdot\nabla)\Lambda^\beta b\cdot\Lambda^\beta b dx \neq 0.$$
$$\int_{\mathbb{R}^2}(b\cdot\nabla)\Lambda^\beta u\cdot\Lambda^\beta b dx \neq 0. \tag{4.39}$$

However,

$$\int_{\mathbb{R}^2}(b\cdot\nabla)\Lambda^\beta b\cdot\Lambda^\beta u dx+\int_{\mathbb{R}^2}(b\cdot\nabla)\Lambda^\beta u\cdot\Lambda^\beta b dx=\int_{\mathbb{R}^2}(b\cdot\nabla)(\Lambda^\beta u\cdot\Lambda^\beta b)dx=0. \tag{4.40}$$

Applying Λ^β (with $\beta > 1$) on the (4.24b) and taking the $L^2(\mathbb{R}^2)$-inner products with $\Lambda^\beta b$

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^\beta b\|_{L^2}+\|\Lambda^{\beta+1}b_h\|_{L^2}^2+\|\Lambda^{\beta+1}b_v\|_{L^2}^2=-\int_{\mathbb{R}^2}\Lambda^\beta[(u\cdot\nabla)b]\cdot\Lambda^\beta b-[(u\cdot\nabla)\Lambda^\beta b]\cdot\Lambda^\beta b dx$$
$$+\int_{\mathbb{R}^2}\Lambda^\beta[(b\cdot\nabla)u]\cdot\Lambda^\beta b-[(b\cdot\nabla)\Lambda^\beta u]\cdot\Lambda^\beta b dx$$
$$\lesssim \|\Lambda^\beta u\|_{L^2}\|\nabla b\|_{L^3}\|\Lambda^\beta b\|_{L^6}+\|\nabla u\|_{L^2}\|\Lambda^\beta b\|_{L^4}^2$$
$$\lesssim \|\Lambda^\beta u\|_{L^2}\|\Lambda^{\frac{4}{3}}b\|_{L^2}\|\Lambda^{\beta+\frac{2}{3}}b\|_{L^2}+\|\omega\|_{L^2}\|\Lambda^\beta b\|_{L^2}\|\Lambda^{\beta+1}b\|_{L^2}$$
$$\lesssim (\|\Lambda^{\frac{4}{3}}b_h\|_{L^2}^2+\|\Lambda^{\frac{4}{3}}b_v\|_{L^2}^2)\|\Lambda^\beta u\|_{L^2}^2+\frac{1}{4}(\|\Lambda^{\beta+\frac{2}{3}}b_h\|_{L^2}^2+\|\Lambda^{\beta+\frac{2}{3}}b_v\|_{L^2}^2)+(\|\omega_h\|_{L^2}^2$$
$$+\|\omega_v\|_{L^2}^2)\|\Lambda^\beta b\|_{L^2}^2+\frac{1}{4}(\|\Lambda^{\beta+1}b_h\|_{L^2}^2+\|\Lambda^{\beta+1}b_v\|_{L^2}^2), \tag{4.41}$$

in which we applied integration by parts, Hölder's inequality, commutator estimate, Gagliardo-Nirenberg interpolation inequality, and Young's inequality.

Applying Λ^β (with $\beta > 1$) on (4.24a) and taking the $L^2(\mathbb{R}^2)$-inner products with

$\Lambda^\beta u$ implies that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Lambda^\beta u\|_{L^2}^2 + \|\Lambda^{\beta+\frac{1}{2}}u_h\|_{L^2}^2 + \|\frac{1}{2}\Lambda^{\beta+1}u_v\|_{L^2}^2 = -\int_{\mathbb{R}^2}\Lambda^\beta[(u\cdot\nabla)u]\cdot\Lambda^\beta u \\
&-[(u\cdot\nabla)\Lambda^\beta u]\cdot\Lambda^\beta u dx + \int_{\mathbb{R}^2}\Lambda^\beta[(b\cdot\nabla)b]\cdot\Lambda^\beta u - [(b\cdot\nabla)\Lambda^\beta b]\cdot\Lambda^\beta u dx \\
&\lesssim \|\nabla u\|_{L^4}\|\Lambda^\beta u\|_{L^4}\|\Lambda^\beta u\|_{L^2} + \|\Lambda^\beta u\|_{L^2}\|\nabla b\|_{L^3}\|\Lambda^\beta b\|_{L^6} \\
&\lesssim \|\Lambda^\beta u\|_{L^2}\|\Lambda^{\frac{3}{2}}u\|_{L^2}\|\Lambda^{\beta+\frac{1}{2}}u\|_{L^2} + \|\Lambda^\beta u\|_{L^2}\|\Lambda^{\frac{4}{3}}b\|_{L^2}\|\Lambda^{\beta+\frac{2}{3}}b\|_{L^2} \\
&\lesssim (\|\Lambda^{\frac{3}{2}}u_h\|_{L^2}^2 + \|\Lambda^{\frac{3}{2}}u_v\|_{L^2}^2)\|\Lambda^\beta u\|_{L^2}^2 + \frac{1}{4}(\|\Lambda^{\beta+\frac{1}{2}}u_h\|_{L^2}^2 + \|\Lambda^{\beta+\frac{1}{2}}u_v\|_{L^2}^2) \\
&+ (\|\Lambda^{\frac{4}{3}}b_h\|_{L^2}^2 + \Lambda^{\frac{4}{3}}b_v\|_{L^2}^2)\|\Lambda^\beta u\|_{L^2}^2 + \frac{1}{4}(\|\Lambda^{\beta+\frac{2}{3}}b_h\|_{L^2}^2 + \|\Lambda^{\beta+\frac{2}{3}}b_v\|_{L^2}^2),
\end{aligned}
\tag{4.42}
$$

in which we applied integration by parts, Hölder's inequality, commutator estimate, Gagliardo-Nirenberg interpolation inequality, and Young's inequality.

Summing up (4.41) and (4.42)

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\Lambda^\beta u\|_{L^2}^2 + \|\Lambda^\beta b\|_{L^2}) + \|\Lambda^{\beta+\frac{1}{2}}u_h\|_{L^2}^2 + \|\Lambda^{\beta+\frac{3}{2}}u_v\|_{L^2}^2 + \|\Lambda^{\beta+1}b_h\|_{L^2}^2 + \|\Lambda^{\beta+1}b_v\|_{L^2}^2 \\
&\qquad\lesssim (\|\Lambda^{\frac{3}{2}}u_h\|_{L^2}^2 + \|\Lambda^{\frac{3}{2}}u_v\|_{L^2}^2)\|\Lambda^\beta u\|_{L^2}^2 + \frac{1}{4}(\|\Lambda^{\beta+\frac{1}{2}}u_h\|_{L^2}^2 + \|\Lambda^{\beta+\frac{1}{2}}u_v\|_{L^2}^2) \\
&\qquad+ (\|\Lambda^{\frac{4}{3}}b_h\|_{L^2}^2 + \Lambda^{\frac{4}{3}}b_v\|_{L^2}^2)\|\Lambda^\beta u\|_{L^2}^2 + \frac{1}{4}(\|\Lambda^{\beta+\frac{2}{3}}b_h\|_{L^2}^2 + \|\Lambda^{\beta+\frac{2}{3}}b_v\|_{L^2}^2) \\
&\qquad+ (\|\Lambda^{\frac{4}{3}}b_h\|_{L^2}^2 + \|\Lambda^{\frac{4}{3}}b_v\|_{L^2}^2)\|\Lambda^\beta u\|_{L^2}^2 + \frac{1}{4}(\|\Lambda^{\beta+\frac{2}{3}}b_h\|_{L^2}^2 + \|\Lambda^{\beta+\frac{2}{3}}b_v\|_{L^2}^2) \\
&\qquad+ (\|\omega_h\|_{L^2}^2 + \|\omega_v\|_{L^2}^2)\|\Lambda^\beta b\|_{L^2}^2 + \frac{1}{4}(\|\Lambda^{\beta+1}b_h\|_{L^2}^2 + \|\Lambda^{\beta+1}b_v\|_{L^2}^2)
\end{aligned}
\tag{4.43}
$$

We can complete the proof with Gronwall's inequality and (4.38). □

APPENDIX

One can find the following Lemma in [75]

Lemma 35. *Let* $\alpha > 0$, $s, s_1, s_3 \in (1, \infty)$, $s_2, s_4 \in [1, \infty]$ *satisfying*

$$\frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{s_3} + \frac{1}{s_4}$$

Then for two constants C_1*, and* C_2

$$\|\Lambda^\alpha (fg)\|_{L_s} \leq C_1 \|\Lambda^\alpha f\|_{L_{s_1}} \|g\|_{L_{s_2}} + \|f\|_{L_{s_3}} \|\Lambda^\alpha g\|_{L_{s_4}} \tag{4.44}$$

$$\|\Lambda^\alpha (fg) - f\Lambda^\alpha g\|_{L_s} \leq C_1 \|\Lambda^\alpha f\|_{L_{s_1}} \|g\|_{L_{s_2}} + \|\nabla f\|_{L_{s_3}} \|\Lambda^{\alpha-1} g\|_{L_{s_4}} \tag{4.45}$$

There is a proof of the following theorem in [9]

Theorem 36. *The space* $L^1_{loc}(\mathbb{R}^n) \cap \dot{H}^{\frac{n}{2}}(\mathbb{R}^n)$ *is included in* $BMO(\mathbb{R}^n)$*. Moreover,* $\exists$ *a constant* C *such that* $\|u\|_{BMO} \leq C\|u\|_{\dot{H}^{\frac{n}{2}}}$ $\forall u \in L^1_{loc}(\mathbb{R}^n) \cap \dot{H}^{\frac{n}{2}}(\mathbb{R}^n)$*.*

Theorem 37. *Let* $1 \leq s < r < \infty$ *and* $\beta > 0$ *. Then* $\exists$ c *such that* $\|g\|_{L^r} \leq c\|g\|^{1-\alpha}_{\dot{B}^{-\beta}_{\infty,\infty}}$ $\|g\|^{\alpha}_{\dot{B}^{\gamma}_{s,s}}$ *with* $\gamma = \beta(\frac{r}{s} - 1)$*, and* $\alpha = \frac{s}{r}$*.*

Lemma 38. *Let* $2 \leq q \leq \infty$*. Let* $\alpha > d(\frac{1}{2} - \frac{1}{q})$*. Then* $\exists$ *a constant* $C = C(d, q, \alpha)$ *such that for any d-dimensional functions* $g \in H^\alpha(\mathbb{R}^d)$*,*

$$\|g\|_{L^q(\mathbb{R}^d)} \leq C\|g\|^{1-\frac{d}{\alpha}(\frac{1}{2}-\frac{1}{q})}_{L^2(\mathbb{R}^d)} |\Lambda^\alpha g\|^{\frac{d}{\alpha}(\frac{1}{2}-\frac{1}{q})}_{L^2(\mathbb{R}^d)}, \tag{4.46}$$

when $q \neq \infty$*, (4.46) also holds for* $\alpha = d(\frac{1}{2} - \frac{1}{q})$*.*

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