Foundations of calculus for the young advanced student

by

Kristilyn E. Peterson, B.S.

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Chris Monico
Chair of Committee

Carl Seaquist

Mark Sheridan
Dean of the Graduate School

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TABLE OF CONTENTS

ABSTRACT

This thesis provides explanation of the annotation of W.T. Ingram's journal article "Foundations of Calculus." The purpose of this annotation is to provide an inquiry based learning opportunity for young advanced learners. The idea is motivated by the fact that many young learners finish the traditional mathematics courses early and do not find them challenging. If those students could be given an opportunity to take a course such as Ingram's, using inquiry based learning, they would discover and develop a sense of ownership in their own learning process. This would encourage a new level of success and advance their skills to continue through mathematics and other fields. This would create independent learners which will not only inspire students at a young age to follow a path of research, but at the very least create an extremely employable individual. Ideally, this opportunity would have to be provided by an alternative education setting. This paper explains what inquiry based learning is, how the annotation was approached, and how to put this opportunity into practice for young advanced learners.

CHAPTER I
INTRODUCTION

Though the title may contain words that are found as conflicting ideas in public education, there are many young (junior high/ high school students) that are in classrooms or being schooled in some alternative setting such as in a home-school environment who are ready to learn more advanced ideas than those presented in traditional mathematics courses. Boredom is found in many traditional mathematical learning settings. Imagine a student who learned the idea and concept of an axiomatic mathematical system and who has discovered and has perfected or at the very least sharpened their skills in justifying and proving ideas. This would lead the student to be successful in not only mathematics, but also in other subjects. These skills would lead to success in the work place and create a logical thinker.

1.1 Why this is Beneficial to Others and can be Effective

Inquiry based learning is efficient in teaching and helping students recall the knowledge obtained. This is very important in the mathematical field because many times mathematical concepts are like a building that is built on a foundation and each floor follows one after another. A few things are important in this. The correct materials and the correct construction method as well as the appropriate location (audience) must be used as demonstrated in this annotation.

The material used is W.T. Ingram's "Foundations of Calculus: Properties of the Real Numbers, Functions, and Continuity." The construction method was to expand the material in such a way that the audience (location) could use the material by adding opportunities to answer leading questions and provide counterexamples to develop reasoning for Ingram's questions and theorems. The targeted audience in this case is advanced junior high or high school learners. The purpose is to take the mathematical concepts of Ingram's work and annotate and modify it so that a younger learner could use inquiry based learning to build a foundation in reasoning, proof, and mathematical concepts. This is the motivation for adding in questions and examples to help the learner reach their understanding.

As media and education circles continue to note the struggle learners have with math through adolescence and on, there are many who are not being challenged with the materials and instruction method being used. Those same advanced learners can make progress in math given the opportunity to do so. This annotation is one of those opportunities. A secondary level teacher may be skeptical if asked to use IBL to teach their state tested concepts without using the scope and sequence or the assigned curriculum. They may become even more skeptical if asked to teach Ingram's work in a secondary level. This thesis will help the skeptical teacher.

1.2 Approach

I took the work of W.T. Ingram's "Foundations of Calculus" and annotated the article. I first worked through the material through Theorem 43 of Chapter 1 which is the proof that a closed bounded interval is compact. While working through the mathematics, I asked questions that would allow either me or a student to obtain a better understanding of the axioms, questions, problems, or theorems. Through discovering the mathematics in an inquiry based setting, I was able to assess what would need to be asked to another student to be able to accomplish the mathematics while still allowing the material to be rigorous. These inquiries lead to the expanding of the material through adding examples, questions, and opportunities to provide counterexamples.

1.3 "Foundation of Calculus Properties of the Real Numbers, Functions, and Continuity"

W.T. Ingram is the author of "Foundations of Calculus". The article is an axiomatic approach to the concepts of calculus. Ingram's intended audience is university-level students. Ingram starts with two undefined terms which are "point" and "to the left of". He then has Axiom 1 which is his version of Dedekind's Axiom. This axiom assures completeness. After Axiom1, Ingram has five more axioms that guarantee "to the left of" is a strict order, is transitive, that between any two points there exists another point, and that there is not a leftmost point or rightmost point of the space [4].

Ingram uses language and vocabulary that is not common in a calculus or analysis course. This is so that the student cannot seek additional help from online, other students, or teachers. The reason for this is to encourage an environment which students will be allowed to rely only on their own understanding to develop their own reasoning and proofs to the provided material. Some theorems that will be proven by students include but are not limited to the Intermediate Value Theorem and the Extreme Value Theorem [4].

Ingram allows the student to study Chapter 2, which explores functions, and not be limited to the real number line. It is not until Chapter 3 that Ingram introduces Axiom 7 which makes the axioms categorical and limits the model to the real number line. It is interesting that functions are addressed and not limited to the real number line. This allows the student to work with and learn about other models.

1.4 Journal of Inquiry Based Learning in Mathematics

The Journal of Inquiry Based Learning in Mathematics (JIBLM) is an online journal, which "publishes classroom tested course notes designed for inquiry based courses in university-level mathematics" [6]. The journal defines inquiry based learning as "any form of constructivist, discovery-based, Moore-method, problem-based, or Socratic pedagogy that replaces traditional lecture and textbooks with some form of student centered activities" [6]. JIBLM has five stipulations to be able to submit "refereed notes". The material must have content, testing, guidance, introduction, and student and instructor versions. More specifically, the material needs to be tested in a classroom and be proven to be successful, the author must agree to provide guidance to instructors desiring to use the original notes for a course, and the material must have an introduction to explain the realm for which the notes are intended. The intent is that the journal materials can be used as the foundation of developing a course for the needs of the students or for a student to study independently. These courses cover many fields of mathematics including but not limited to analysis, financial mathematics, mathematics education, topology, linear algebra, and calculus. [6]

1.5 Intended Audience

The intended audience for the material developed through annotations is young advanced students as implied in the title. A young advanced student in this case is defined to be a student of adolescence age, possibly thirteen to eighteen years of age. The student would need to be somewhat strong in math or at the very least an extremely motivated learner. The ideal students would have the desire to learn mathematics primarily on their own. The student would desire the challenge and perhaps want to seek the accomplishment of forming their definite reasoning in mathematics. Public education is focused on scalable methods of teaching to produce quantity so this method is not ideal in that setting. Direct teaching methods produce quantity while student centered learning focuses on quality of those who choose to be involved. Although public education does not seem to have the environment to offer a course like this, it could be offered in private, alternative, or home school environments. This creates a secondary audience which would include mathematics teachers, professors, or even a parent.

1.6 Introduction to Inquiry Based Learning

Inquiry based learning is like an apprenticeship. IBL is the training of upcoming mathematicians. “Inquiry-based learning is a student-centered, active learning approach focused on questioning, critical thinking, and problem solving. Inquiry-based learning activities begin with a question followed by investigating solutions, creating new knowledge as information is gathered and understood, discussing discoveries and experiences, and reflecting on new-found knowledge” [9]. On the website for JIBLM they state: “Inquiry-based learning refers to any pedagogy that replaces traditional lectures and textbooks with some form of student-centered activities. Instructors typically supply students with carefully crafted course notes consisting of a sequence of definitions, problems or theorems. Instructors then serve as mentors, by listening to the students, reading their work, and giving them the minimal information they need to understand the defined concepts, solve the problems, or prove the theorems” [6]. This statement frames the goal of JIBLM and the purpose for the annotation of Ingram’s article.

1.7 Purpose

The purpose of annotating W.T. Ingram’s course material notes is to provide the opportunity for young advanced students to obtain a higher level of reasoning than traditional mathematics courses offer. Through this, the goal is to spark a level of interest that is unstoppable. This interest would lead to further exploration of mathematics and the discovery of many fields of mathematics. Though Ingram's material has been used at several universities including the University of Houston, Missouri University of Science and Technology, and Texas Tech, to my knowledge Ingram's material has never been used in a high school setting. Through the goal of JIBLM, the original material was used as a starting point to develop a course for highly motivated high school students by adding annotations to help provide gentler transitions between theorems and new concepts. This will lead to the material being available and usable for high school students. The students that learn the material will be expected to have a better understanding of mathematics, reasoning, and to have the drive to learn more.

1.8 Contents

This paper contains information about Inquiry Based Learning and the advantages of using IBL. Need and reason for annotating will be covered in further detail as well as how this annotated material can be used and applied in an IBL setting to help students achieve a higher understanding of mathematics. How to measure the effectiveness will be explained, as anything that is not effective is not going achieve a high level of learning. In the appendix, my version of the material with my answers to the proofs, theorems, questions, and problems as well as annotations are included.

CHAPTER II
INQUIRY BASED LEARNING

This chapter explains what inquiry based learning is and the characteristics that identify inquiry based learning and the Moore Method. Benefits of this method will be discussed as well as the history and contributions a great mathematician made. Lastly, this chapter will present how inquiry based learning is used today and how it helps students achieve a deeper understanding of mathematics.

2.1 What is Inquiry Based Learning?

Education can be broken down in several ways. These ways depend on methods of teaching, grading policies, age of student, student ability and many more. Education is researched and studied continuously to achieve the most effective education opportunity for the student. Two diverse categories that can be created in education are teaching methods that are either teacher led or student centered. Many classrooms including high school and college level are filled with teacher led methods that utilize educators lecturing students. The educators are viewed as the ones who have the knowledge sending the knowledge to the student. As time has changed, so has the need to alter the methods of teaching. Today there is educational evidence through many studies that the traditional methods are not as effective as student centered learning [5].

A mathematics class that is teacher led is usually marked by characteristics such as lecture, “drill and kill,” a syllabus that starts from day one and is meant to be kept on schedule with a list of topics, assignments and tests. In a teacher led class there will usually be formulas given, practiced, and memorized by students in order to apply to a computational problem. Later the students may be given an opportunity to use the material in a more practical setting. The class will also have the expectation that all students will learn the same concepts at about the same rate. In its traditional form, teacher led classes consist of the teacher pouring information into the students’ brain and the student receiving and retaining this knowledge. Not only do many colleges and universities use the teacher led method, but so do most high school classrooms. Naturally in the teacher led learning environments the teacher has most of the attention and does most of the work. The teacher presents a topic then guides the class through examples and assigns practice problem.

Inquiry based learning (IBL) cannot be placed in the category described as teacher led learning but is placed in student centered learning. In an IBL class, students play the main role in their learning by taking ownership of the learning that takes place. The teacher is solely a facilitator and provider of materials for the learning that is to come. The students go from “passive passengers” to “active participants” [3]. The teacher becomes a secondary source that is not referred to as much and the students’ abilities become the primary source through which the knowledge is obtained. The ultimate goal of inquiry based learning is to create independent learners. Merriam-Webster defines knowledge as: “information, understanding, or skill that you get from experience or education [7].” Through inquiry the student discovers the information, creates their own understanding, and obtains the skill to research and achieve an even deeper understanding of mathematics. IBL is beneficial at generating creativity and critical thinking in students [3]. This ultimately allows students to critically approach ideas with the ability to solve and justify their own ideas instead of being given ideas of someone else.

There can be many models of inquiry based learning due to subject, age, and goal of course. The Moore Method is a specific form of inquiry based learning that will be addressed as it is directly related to the field of Mathematics. In Moore Method based classes, there are identifiable characteristics that most observers can articulate. Due to varying personalities, using the Moore Method to teach cannot be completely defined, but the following is a list of parameters that are found in a Moore Method class. These parameters can be individualized at the teacher’s discretion.

- Classes are generally built in an axiomatic way or a scaffolding way such that students can start with a basic idea and build a more complex understanding.
- Students will present the work (proofs) and will justify responses to class and teacher.

- Teacher can respond to incorrect statements or reasoning possibly by providing counterexamples.
- Student's grade comes primarily from the students original presentations- "The students are mathematicians: they 'publish' their proofs, solutions and answers as presentations at the board.[1]"
- No outside sources are to be used by students including classmates (specific to Moore Method) to force students to rely on their own ability and understanding to create reasoning skills for writing proofs.
- If a student thinks they are close to presenting a solution to a problem and another student is ready to present they can step out of room so as not to disqualify themselves from working on that problem (this is not to be used often) [1].
- Teacher is a source but, not the main source of knowledge.

The goal of IBL and Moore Method is to create independent thinkers that can approach an unknown concept or idea and use what they know to discover or produce a solution to the unknown. The importance is placed on the student's ability to describe their solution in such a way that others can understand their reasoning. It is a conceptual way of learning and not so much a computational way of learning. Inquiry based learning, or the Moore Method is one that helps the student take pride in their own work although they may have done what another did, they achieved the result on their own. In Moore's own words from the website "The Legacy of R. L. Moore": "That student is taught the best who is told the least [2] ."

2.2 History of IBL and the Moore Method

Robert Lee Moore established what is referred to as the Moore Method. R. L. Moore not only taught mathematics but instilled in many the desire to be researchers of mathematics [3]. In 1911 Moore went to University of Pennsylvania which is where he developed the main principles by which he would teach using his new method. Moore's deep-rooted passion for challenging students in mathematics can be attributed to several experiences in his life. Some may credit his desire to reflect upon his own teaching methods to his time teaching high school in Marshall, Texas. Others might argue that this desire came from the time he taught college and discovered ways to motivate students to be driven to learn more in mathematics. He sought a way to generate the desire in students to take ownership of their learning as he had in his own learning of mathematics. He wanted to create students who desired to learn mathematics not only by lectures from professors but by their own discovery. He was later a professor of point set topology and other topics at the University of Texas, where he would teach for 49 years [3].

It was at the University of Texas that he fully integrated the Moore Method (not named by him but after him). His success was so great that his colleagues desired to use his approach to develop their own method of teaching mathematics to reach his level of success. Moore's positive influence was proven not only by students' responses, but the students' desire to go on in mathematics and their success in doing so. Through Moore and his direct colleagues a few thousand students would aspire to seek doctorate degrees in mathematics and research, eventually leading to the discovery of new mathematical fields. Moore's approach to teaching and implementing the principles he believed in would inspire many others to use this method even today. It is said on the "Creativity in Mathematics" video that a student asked their professor why the professor would not just go ahead and tell the class the answer on a problem they had been working on for a while without success and the professor's response was "Why should I limit you to what I know [3]?"

2.3 How is it Used Today?

Many colleges and universities use the Moore Method to teach courses. Though some courses are easier to teach using IBL than others, it remains a successful method of teaching many mathematics courses. In "Creativity in Mathematics" a professor speaks about busing 20 under-performing high risk students to the college and teaching mathematics through IBL [3]. He attests to its success and how it made the at-risk students to seek out mathematics and desire a deeper understanding. Some colleges that use the method and/or have IBL centers are University of Michigan (Ann Arbor), University of California at Santa Barbara, University of Chicago, Texas Tech, and the home of the Moore Method, University of Texas. "The Legacy of R. L. Moore" website provides information about conferences and seminars that are about IBL. They are highly attended and are co-sponsored by the Mathematical Association of America [2].

In the video "Creativity in Mathematics," a high school student speaks of his success in his Moore-based Pre-Calculus class showing again that this method can be used with younger students. With the continued struggle of math scores on secondary standardized state tests, retention remains an issue. Inquiry based learning addresses this issue. "Teachers who make use of inquiry based learning have found that their students are more likely to remember concepts that they discover on their own rather than memorize information recited to students in a traditional lecture setting" [3].

CHAPTER III
THE ANNOTATIONS OF "FOUNDATIONS OF CALCULUS" BY W.T. INGRAM

The goals of the annotation of "Foundations of Calculus: Properties of the Real Numbers, Functions, and Continuity" by W.T. Ingram is to provide a piece of curriculum to challenge young bright independent learners and to allow them to develop a deeper understanding of mathematics while owning their education through inquiry based learning. This curriculum is a resource to create and sharpen their reasoning skills while obtaining a deep understanding of mathematical concepts for very motivated youth. This chapter will explain why this should interest others and how the annotation was approached. Annotation was only made through number 54 (the beginning of Chapter 2 of Ingram's work) because if a learner can make it that far then they have the foundational understanding of not only mathematics but know the power of inquiry based learning.

3.1 How the Annotation was Approached

A teacher should never ask a student to do what they themselves have never tried. This was motivation to work through the material Ingram wrote. With IBL being known to use the Socratic Method to guide students when they have hit a point of no progress, additional questions were added to Ingram's work to allow students to make forward progression. Examples were added to allow students to reach success in possibly a more computational way before making justifications of new concepts. The idea is that an advanced young learner may not have the mathematical background expected by Ingram as his work is targeted towards students typically in college. The young learner does however have the ability to reason and justify arguments as well as to find a counterexample to a false statement. Thus, the additional questions and the examples are to help keep the young learner motivated while relying on their mathematical ability before stretching their ability to a higher level.

The questions and examples were generated as I had questions about meaning or struggled with a concept. When forward progress could not be made, I asked the question that would be asked to a facilitator (or in my case thesis advisor) and then added a question to Ingram's work to lead myself and the young advance learner to better understand the concept. Examples were given to supply a concrete opportunity to the sometimes abstract idea. At times, if the same concept had been attempted for an amount of time my advisor would question my reasoning until I found the hole or would give me an example to try to do prior to the justification I was attempting. Though this was a new and challenging experience as it was my first IBL opportunity, a quote that one of my past Calculus professors referred to several times in class from a classical Greek philosopher named Socrates: "To know, is to know that you know nothing. That is the meaning of true knowledge." This quote helped me continue working and pushing forward through Ingram's challenging work. When I finally grasped a challenging concept, I knew that I would retain the concept and by actual experience that inquiry based learning is very powerful when used to learn mathematics.

CHAPTER IV
HOW THIS ANNOTATION CAN BE USED THROUGH INQUIRY BASED LEARNING

In the field of secondary public education it may not be welcomed as the districts have assigned curriculum and many teachers would not feel comfortable implementing this type of method. This chapter will provide the environment in which this could be applied and where it may be found to be the most successful. This chapter will go into detail of the why, who, and how of the opportunity provided through this annotation [8].

4.1 Why?

This is possibly the most important aspect of this entire process. Without a proven purpose the entire concept is a loss of time and motivation. As previously discussed, it is important to provide all students an opportunity to learn at their highest potential. Recently, our nation has focused so much in public education on leveling the playing field. Though it is important for those who need additional support to learn and to offset outside factors that hinder learning, we have forgotten about those who are bored and not stimulated by the materials being presented to them. For so long education has applied money, people, and research to help all students learn the given standards required. During this time there are many students who have the potential to stretch beyond the traditional mathematics. Once taught the skills of inquiry based learning and given a foundation of creating justifications of a material such as “Foundations of Calculus,” they can excel in many fields of mathematics. This will open many doors for the student including the possibility that university-level mathematics will be more captivating than expected.

4.2 Who?

Due to where the public education is currently, this opportunity would most likely be for very advanced late junior high or very early advanced high school aged students who are in an alternative learning environment such as home-school. This curriculum piece could also be used with students who are in private school for a mathematics credit in school. There are too many students who are ready to provide reasoning for their mathematics work. In a home school environment, many students finish the traditional set of mathematics curriculum and start on a calculus series. This material could better prepare a student for skills needed to be successful in any field that requires reasoning and justification of arguments.

4.3 How?

Since some parents will not be comfortable with the material, there may have to be a math co-op situation where an adult is the facilitator and the group of students meet regularly. The highest potential could be achieved if the students had access several times a week to the facilitator as to provide opportunities to allow students to share their proofs and ideas with the facilitator. The entire group could meet a few times a week or as needed to have the opportunity to provide their completed proof to the class.

It would not take very much time to witness the success that this opportunity could create in the mathematical abilities in students. First, the student and facilitator would see the skills of the student increase once past the initial adjusting to a new method of learning. Next to see the success would be any people that were responsible for their education at that time. The next person to most likely see the success would be the college professors who would have these students in class not only in mathematics but in other subjects. Lastly, whatever field the student chose to contribute would see success.

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APPENDIX

TEACHER VERSION

All text in black is Ingram's original work [4]

Basic Properties of Point Sets

Undefined terms: *point*, *to the left of*

Definition 1. *Suppose P is a point and M is a point set. The statement that P is a* **leftmost point of M means P is a point of M and if x is a point of M then x is not to the left of P.**

Problem 2. *Formulate a meaning for the statement that P is a rightmost point of M. (Note that the phrase 'to the right of' is not defined. Try to formulate the meaning without using 'to the right of'.)*

Answer: P is the rightmost point of the point set M means that P is an element of M and if X is a point of M then P is not to the left of X.

We now assume that we have a set whose elements are called points and there is a meaning of the phrase 'to the left of' so that all six of the following axioms hold.

AXIOM 1. If S_1 and S_2 are point sets such that (1) if x is a point then x is in S_1 or x is in S_2 and (2) if x is a point of S_1 and y is a point of S_2 then x is to the left of y, then S_1 has a rightmost point or S_2 has a leftmost point.

My Understanding: *The two conditions that have to be met are that every point is in S_1 or S_2 and that if a point is in S_1 then it is to the left of every point of S_2. This is the idea of being bound by a point.*

(Kristilyn's Question A1): *Now we will check for understanding of Axiom 1. You will be given two subsets of the real line (S_1 and S_2) and you will need to determine if they meet both conditions of the hypothesis of Axiom 1. State which part of the hypothesis (1 or 2) is not met and explain. If both parts of hypothesis are* *true state "True for both." Space considered in the first three examples is the set of all reals with "to the left of" interpreted as $<$.*

Example 1: S_1 will be the set of Rational numbers and S_2 will be the set of irrational numbers:

Since 3 is a rational number it is in S_1 and $\sqrt{2}$ is an irrational so it is in S_2 but 3 is not to the left of $\sqrt{2}$. Second part of hypothesis is not met.

Example 2: S_1 is all x in reals such that x is less than 0 and S_2 is all x in reals such that 0 is less than x.

Since 0 is a real number and 0 is not in S_1 and 0 is not in S_2, the first part of the hypothesis is not met.

Example 3: S_1 is all x in reals such that x is less than or equal to 0 and S_2 is all x in all x in reals such that 0 is less than or equal to 0.

Since 0 is in S_1 and 0 is in S_2 but 0 is not to the left of 0, thus the second part of the hypothesis is not met.

Example 4: Space to be considered is the Reals with "to the left of" interpreted as $<$. S_2 will be all positive integers and S_1 will be the set of all non-positive integers:

This fails part 1 because there exist points such as $\frac{1}{2}$ that is not an element in either set.

Example 5: Space considered in this example is Rationals with "to the left of" interpreted as $<$. Let S_2 be $\{x \in \mathbb{Q} \mid x > 0 \text{ and } x^2 > 2\}$ and let S_1 be the compliment of S_2:

True for both.

AXIOM 2. If each of P and Q is a point and P is to the left of Q then there exists a point x such that P is to the left of x and x is to the left of Q.

My Understanding: *This is the idea that between any two distinct points, there exists another point.*

Does Example 5 from above satisfy the conclusion of Axiom 1 by S_1 having a rightmost point or S_2 having a leftmost point?

Answer: *No, the left bound of S_2 is the square root of 2 which is the right bound of S_1. For any $x \in S_2$, $\exists$ $m \in S_2$ between x and the square root of 2 using Axiom 2. The same reasoning holds true for S_1. By using Axiom 2, for any $x \in S_1$ there exists a $m \in S_1$ between x and the square root of 2. Thus S_1 does not have a rightmost point. Thus the space rationals do not satisfy Axiom 1 but the rationals do satisfy the other 5 axioms.*

AXIOM 3. If P and Q are points then P is to the left of Q or Q is to the left of P.

Note that "points" is equivalent to the statement "two distinct points". Thus "points" implies P and Q are distinct.

My Understanding: *Any two points must be related.*

AXIOM 4. If P, Q, and R are points, P is to the left of Q and Q is to the left of R, then P is to the left of R.

My Understanding: *This relates to the transitive property. If "to the left of" is interpreted as $<$ then it states: $P < Q$ and $Q < R$ then $P < R$.*

AXIOM 5. If P is a point then P is not to the left of P.

My Understanding: *If P is a point then P is not to the left of P. This is the idea of a strict order. For example $\leq$ on the reals is not a strict order, however, $<$ is.*

AXIOM 6. If P is a point then there exist points Q and R such that Q is to the left of P and P is to the left of R.

My Understanding: *For every point, there exists a point to its left and to its right so the space has no leftmost point or rightmost point.*

(Kristilyn's Question A2-6): *The following will be examples to work through to make sure we have an understanding of the applicaiton of Axioms 2-6. For each example determine which axioms are not true for the given set. Explain why not. If all axioms hold true put "True for all."*

Example 1: The set of Natural numbers with "to the left of" interpreted as $<$:

Axiom 2 fails because the numbers 1 and 2 do not have a point between them. Axiom 6 fails as well due to if we have 1, there does not exist a Q and R such that one of them is to the left of 1.

Example 2: The set of rational numbers with "to the left of" interpreted as $<$:

True for all Axioms 2-6.

Example 3: The set of irrational numbers:

True for all Axioms 2-6.

Question 3. *Can a point set have two leftmost points?*

Answer: No, I will approach this using proof by contradiction. Suppose that M is a point set that has two leftmost points. Let us call these two leftmost points A and B. By Definition 1, since A is a leftmost point, every $x \in M$ is not to the left of A. Thus since $B \in M$ it is not to the left of A. By Axiom 3, A is to the left of B or B is to the left of A but we have shown that B is not to the left of A. So then A is to the left of B which contradicts that B is a leftmost point. Therefore there do not exist two leftmost points of M.

Problem 4. *Show that, under the hypothesis of Axiom 1, if S_1 has a rightmost point then S_2 does not have a leftmost point.*

Answer: I will approach this using proof by contradiction. Suppose S_1 has a rightmost point and S_2 has a leftmost point and both conditions of Axiom 1 are met. Let's call the rightmost point of S_1, A, and the leftmost point of S_2, B. By definition of leftmost point, B $\in S_2$ and by definition of rightmost, A $\in S_1$. By part 2 of the hypothesis of Axiom 1, every point in S_1 is to the left of every point in S_2. Therefore A is to the left of B. A can not be B because A can not be to the left of itself. By Axiom 2, there exists an x such that A is to the left of x and x is to the left of B. So, there exists an x that is not in S_1 and not in S_2 which contradicts part 1 of Axiom 1 which states that every x is in S_1 or in S_2. This is a contradiction. Therefore, S_1 has a rightmost point OR (exclusively) S_2 has a leftmost point.

(Kristilyn's Question 4.1): *Now we will check for understanding of truth values. In each of the following examples use the following example:* **If it rains, the sidewalks are wet.** *Assume this is true. Determine the truth value of each statement using only the given statement.*

Warm-Up: Compete the truth table: NOTE- p $\implies$ q can be understood as "If p, then q."

p	q	p $\implies$ q
T	T	T
T	F	F
F	T	T
F	F	T

Notice the only situation that is false is in the case where $p \implies q$ is given such that p is true and q is false. In the given example this would be the case that it rains but miraculously the sidewalks did not get wet. This is not possible.

Example 1: It rained and the sidewalks are not wet.

This is false due to the fact that we said if it rains then the sidewalks have to be wet.

Example 2: It did not rain and the sidewalks are not wet.

This does not contradict the implication and so it is true.

Example 3: It did not rain and the sidewalks are wet.

This does not contradict the implication and so it is true.

Example 4: How would you negate the original statement: If it rains, then the sidewalks are wet.

It rains and the sidewalks are not wet.

Example 5: Negate: If A, then B. (A $\implies$ B)

A and not B.

Example 6: Negate: $\forall$ x, P(x)

$\exists$ *x such that* $\neg(P(x))$.

Example 7: Negate: P and if A, then not B.

Not P or (A and B).

Problem 5. *Complete the following statement: The point P is not the leftmost point of the point set M means ....*
(This is called the *bare denial* (or *negation*) of the statement that P is the leftmost point of P.)

Answer: P is not an element of M or there exists an x $\in$ M such that x is to the left of P.

(Kristilyn's Question 5.1): *In each example you will be given a subset of the space of reals with "to the left" interpreted as* $<$*. Determine if the given point is the leftmost point of the set and if not why. If possible identify the leftmost point of the set.*

Example 1: Positive integers, 0

No, 0 is not an element of $\mathbb{Z}^+$*. 1 is the leftmost point.*

Example 2: Positive Integers, 2

No, 1 is the leftmost point.

Example 3: Rationals, 0

0 is not the leftmost point because -1 is a rational that is to the left of 0.

Theorem 6. *If M is a point set and B is a point to the left of every point of M then M has a leftmost point or there is a rightmost of all the points to the left of every point of M.*

Proof: Assume M is a point set and B is a point to the left of every point of M. Let S_1 be the set of all points to the left of every point of M so $B \in S_1$. Let S_2 be the compliment of S_1. Since no point can be to the left of itself, M is a subset of S_2 and S_2 is the compliment of S_1, so we have $x \in S_1$ or $x \in S_2$ thus part 1 of the hypothesis of Axiom 1 is met. Let x $\in S_1$ and y $\in S_2$. Either y is in M and if so x is to the left of y by definition of S_1 or y is not in M. If y is not in M, there is a point m in M such that m is to the left of y. Thus x is to the left of m and m is to the left of y so by Axiom 4, x is to the left of y. By Axiom 1, either S_1 has a rightmost point or S_2 has a leftmost point. So there is a rightmost point of every point to the left of M, or there is a leftmost point of the compliment (S_2). I will proceed to address the instance that S_2 has a leftmost point.

Case 1: S_2 is equal to M: Then we are done as proven above.

Case 2: M is a proper subset of S_2: I will proceed with proof by contradiction. Let L be the leftmost point of S_2. I want to prove that L is in M and thus L is the leftmost point of M. If L is not in M then L is to the left of every point of M and so in S_1 which is a contradiction because L is in S_2. Therefore, L is the leftmost point of M.

Definition 7. *If P and Q are points, the statement that P is* **to the right of** *Q means Q is to the left of P.*

Definition 8. *If P and Q are points and R is a point, the statement that R is* **between** *P and Q means R is to the right of P and to the left of Q or R is to the right of Q and to the left of P.*

Definition 9. *Suppose A and B are two points and A is to the left of B. By the* **segment** *AB, [denoted (A, B)], is meant the point set to which the point x belongs if and only if x is between A and B. The statement that the point set s is a segment means there exist points A and B such that s is (A, B).*

Definition 10. *Suppose P is a point and M is a point set. The statement that P is a* **limit point** *of M means if s is a segment containing P then s contains a point of M distinct from P.*

Problem 11. *Write the bare denial of the statement that the point P is a limit point of the point set M.*

Answer: P is not a limit point of M if $\exists$ a segment s containing P and s does not contain a point of M distinct from P.

(Kristilyn's Question 11.1): *In each example you will be given a set and a point. Determine if the given point is the limit point of the set and if not why.*

Example 1: $\{1\}$, 1

No, The segment (0.5, 1.5) contains 1 but does not contain any other point of the set distinct from 1.

Example 2: $\{1\}$, 0

No, The segment (-.5, .5) contains 0 but does not contain any other point of the set.

Example 3: $\{1\}$, 1.5

No, The segment (1, 2) contains 1.5 but does not contain any other point of the set distinct from 1.5.

Example 4: $\{1, 2, 3\}$, 2.5

No. The segment (2,3) contains 2.5 but does not contain any other point of the set distinct from 2.5.

Question 12. *Suppose A is a point and M is a point set whose only member is A. Is A a limit point of M? If x is to the left of A, is x a limit point of M? If x is to the right of A is x a limit point of M? Does M have a limit point?*

Answer: No, A is not the limit point of M because by Axiom 6 there exists s that does not contain a point of M distinct from A since A is the only point of M.

For x to the left of A: $\exists$ P to the left of x by Axiom 6. So the segment (P,A) contains x but does not contain a point of M distinct from x. Therefore, any point to the left of A is not a limit point.

For x to the right of A: $\exists$ Q to the right of x by Axiom 6. So the segment (A,Q) contains x but does not contain a point of M distinct from x. Therefore, any point to the left of A is not a limit point.

Therefore M does not have a limit point.

Question 13. *Suppose A and B are two points and M is a point set whose only members are A and B. Does M have a limit point?*

Answer: No. Assume A is to the left of B without loss of generality. There are 5 cases to this proof. No other cases exist and I have shown all points possible are not limit points.

Case 1: Any point, x, to the left of A:

$\exists$ P to the left of x So the segment (P,A) contains x but does not contain a point of M distinct from x. Therefore, any point to the left of A is not a limit point.

Case 2: x=A:

$\exists$ P to the left of A Axiom 6. So, (P,B)contains A but does not contain an element of M distinct from A so A is not a limit point of M.

Case 3: Any point, x, between A and B:

(A,B) contains x but does not contain an element of M distinct from x so x is not a limit point of M.

Case 4: x=B:

$\exists$ Q to the right of B by Axiom 6. So, (A,Q) contains B but does not contain an element of M distinct from B so B is not a limit point of M.

Case 5: Any point, x, to the right of B:

$\exists$ Q to the right of x by Axiom 6. So the segment (B,Q) contains x but does not contain a point of M distinct from x. Therefore, any point to the right of B is not a limit point.

(Kristilyn's Question 13.1): *In each example give all limit points.*

Example 1: (1,2)

Limit points: 1, 2, and all points between 1 and 2

Example 2: (-1,1)

Limit points: -1, 1, and all points between -1 and 1

Question 14. *If M is a segment, does M have a limit point?*

Answer: Yes, any segment (A,B) has limit points which include A and B and any points between. This proof will have five cases to prove that A and B and any x ∈ (A,B) are limits point and that there exist no others.

Case 1: Any point, x, to the left of A

∃ P to the left of x by Axiom 6. So the segment (P,A) contains x but does not contain a point of M distinct from x. Therefore, any point to the left of A is not a limit point.

Case 2: x=A:

Let s=(P,Q) so that s contains A. So P is to the left of A and Q is to the right of A. Let L be the leftmost point of B and Q. By Axiom 2, ∃ x between A and L. So the segment (P,L) contains A and x that is in (A,B) but distinct from A. Therefore, A is a limit point.

Case 3: Any point, x, between A and B:

Let s=(P,Q) so that s contains x. So P is to the left of x and Q is to the right of x. Let L be the leftmost point of B and Q and let R be the rightmost point of A and P. By Axiom 2, ∃ C between x and R as well as a D that exists between x and L. So the segment (R,L) contains x and a point (C or D) of (A,B) that is distinct from x. Therefore, x is a limit point.

Case 4: x=B:

Let s=(P,Q) so that s contains B. So P is to the left of B and Q is to the right of B. Let R be the rightmost point of A and P. By Axiom 2, ∃ x between R and B. So the segment (R,B) contains B and x that is in (A,B) but distinct from B. Therefore, B is a limit point.

Case 5: Any point, x, to the right of B:

∃ Q to the right of x by Axiom 6. So the segment (B,Q) contains x but does not contain a point of (A,B) distinct from x. Therefore, any point to the right of B is not a limit point.

Definition 15. *A point set M is called an* **interval** *provided there exist two points A and B with A to the left of B and such that x belongs to M if and only if x is in the segment (A, B) or x is A or x is B. In this case M is called the interval AB and* *is denoted* $[A, B]$.

(Kristilyn's Question 15.1): *In each example give all limit points.*

Example 1: [1,2]

Limit points: 1, 2, and all points between 1 and 2

Example 2: [-1,1]

Limit points: -1, 1, and all points between -1 and 1

(Kristilyn's Lemma 15.2): *Prove the following statement: If N is a subset of M and P is a limit point of N, then P is a limit point of M.*

Answer: Let N be a subset of M and P be a limit point of N. We know that every element in N is also in M. We also know that for every segment s that contains P, s contains a point y of N distinct from P. Since every element of N is also an element of M, then y is also an element of M. Therefore, every segment s that conatins P also conatins a point of M distinct from P. Thus, P is a limit point of M.

Question 16. *If* M *is an interval, does* M *have a limit point?*

Answer: Yes, any segment [A,B] has limit points which include A and B and any points between. This proof will have three cases to prove that A and B and any x ∈ (A,B) are limits point and that there exist no others.

Case 1: Any point, x, in (A,B)

Using Question 14 and Lemma 15.2, (A,B) is a subset of [A,B] and x is a limit point of (A,B) thus x is a limit point of [A,B].

Case 2: A:

Let (P,Q) contain A. By Axiom 2 there exists a point x between A and Q that is in [A,B] distinct from A. Therefore, A is a limit point. **Case 3: B:**

Let (P,Q) contain B. By Axiom 2 there exists a point x between P and B that is in [A,B] distinct from B. Therefore, B is a limit point. **Case 4: Any point, x, to the left of A**

∃ P to the left of x by Axiom 6. So the segment (P,A) contains x but does not contain a point of M distinct from x. Therefore, any point to the left of A is not a limit point.

Case 5: Any point, x, to the right of B:

∃ Q to the right of x by Axiom 6. So the segment (B,Q) contains x but does not contain a point of (A,B) distinct from x. Therefore, any point to the right of B is not a limit point.

Question 17. *Is there a point set with only one limit point?*

Answer: Yes on the space of real number line where "to the left of" is interpreted as $<$. Given point set M = $\left\{\frac{1}{n}|n \in \mathbb{Z}^+\right\}$, 0 is a limit point because for any segment that contains 0, it also contains a point of M distinct from 0. Let s= (A,B) that contains 0. So A is to the left of 0 and B is to the right of 0. Let's look at $\frac{1}{B}$ $<n_0$ where $n_0 \in \mathbb{Z}$. So $\frac{1}{n_0}$ <B. Therefore, $\frac{1}{n_0}$ is to the left of B and $\frac{1}{n_0}$ >0. Therefore $\frac{1}{n_0} \in$ M. So, $\forall$ s that contains 0, the segment also contains a point of M distinct from 0. Does such an example exist for every model that satisfies Axioms 1-6? (I will explore this at a later date.)

Now I will use cases to prove that no other limit points exist for this set:

Case 1: For x to the left of 0:

$\exists$ P to the left of x by Axiom 6. So the segment (P,0) contains x but does not contain a point of M distinct from x. Therefore, any point to the left of 0 is not a limit point.

Case 2: For x beteween 0 and 1:

Sub Case I: x = $\frac{1}{n}$ where n in $\mathbb{Z}^+$

By Axiom 2, there exists a point A between $\frac{1}{n+1}$ and $\frac{1}{n}$ and there exists a point B between $\frac{1}{n}$ and $\frac{1}{n-1}$. (A,B) contains x but contains no point of M distinct from x. Thus, x is not a limit point of M.

Sub Case II: $\frac{1}{n+1}$ <x <$\frac{1}{n}$ where n is in $\mathbb{Z}^+$

Let A=$\frac{1}{n+1}$ and B=$\frac{1}{n}$. (A,B) contains x but does not contain a point of M distinct from x. Thus, x is not a limit point of M.

Case 3: 1 is not a limit point:

Given the point set above, the rightmost point is 1 ($\frac{1}{1}$). Take the segment (.75,1.25). This segment contains 1 but does not contain a point of M that is distinct from 1. Therefore, 1 is not a limit point.

Case 4: For x to the right of 1:

By Axiom 2, there exists a point A between 1 and x and by Axiom 6, there exists a point B to the right of x. Segment (A,B) contains x but no point of M distinct from x. Therefore x is not a limit point of M.

Problem 18. *Show that if each of S_1 and S_2 is a segment containing the point P,* *then there is a segment s containing P such that s is a subset of S_1 and s is a subset of S_2.*

Answer: Let S_1 and S_2 each be a segment that contains point P. I need to show that there exist a segment s that is a subset of S_1 and s containing P is a subset of S_2. Let S_1 = (A,B) and let S_2 = (X,Y). Let R be the rightmost point of {A,X} and let L be the leftmost point of {B,Y}. Let s = (R,L) so s is the points that S_1 and S_2 have in common, then P is in s. Therefore s is a subset of S_1 and a subset of S_2 and P $\in$ s.

(Kristilyn's Question 18.1:) *Prove: If P is not the limit point of M and P is not the limit point of N, then P is not limit point of M union N.*

Answer: Assume P is not the limit point of M and P is not the limit point of N. Let S_1=(A,B) that contains P and no point of S_1 distinct from P and let S_2=(C,D) that contains P and no point of S_2 distinct from P. By Theorem 18, there exists a s that is a subset of S_1 and s be a subset of S_2 such that s contains P. Segment s does not contain any points of either M or N distinct from P (since P is not limit of M and P is not limit point of N). So P is not a limit point of M union N.

Theorem 19. *If M is a point set without a leftmost point and there is a point to the left of every point of M, then M has a limit point.*

Answer: Assume M is a point set without a leftmost point and there is a point to the left of every point of M. Let B be the point to the left of every point of M and let S_1 be the set of all points to the left of every point of M. By Theorem 6, M has a leftmost point or S_1 has a rightmost point. As stated above, M does not have a leftmost point so S_1 has a rightmost point. Let R be the rightmost point of S_1 which means that R is in S_1 and every other point of S_1 is not to the right of R. Let s=(C,D) that contains R. So C is to the left of R and D is to the right of R. From here there exist two cases that I will proceed with.

Case 1: D is in M: Since D is in M and D is to the right of R, then there is a point of M to the left of D. So we have (C,D) that contains R and a point of M distinct from R. R is a limit point of M.

Case 2: D is not in M: If D is to the right of R and D is not in M and D is not in S_1, then there must be a point of M to the left of D. So we have (C,D) that contains R and a point of M distinct from R thus R is a limit point of M.

Definition 20. *The statement that a set M is* **finite** *means there is a positive integer n such that M contains only n elements.*

We will use without proof the following property of the set of positive integers:

If K is a set of positive integers, K has a least element.

(Kristilyn's Question 20.1): *Prove $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$ for all n in $\mathbb{Z}^+$ using proof by induction. So prove that the sum: $\frac{1}{1(1+1)} + \frac{1}{2(2+1)} + \frac{1}{3(3+1)} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$.*

Answer:

Let P(n) $\equiv \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$.
Notice P(1) $\equiv \sum_{i=1}^{1} \frac{1}{1(1+1)} = \frac{1}{2} = \frac{1}{1+1}$.
Suppose P(k') is false for some k' in $\mathbb{Z}^+$.
Let K = $\{k \in \mathbb{Z}^+ | P(k) \text{ is false}\}$
So: K $\subseteq \mathbb{Z}^+$, k' $\in$ K and 1 $\notin$ K. Let k_0 be the least element of K. So $k_0 > 1$.
Then P(k_0-1) is true since k_0 is the least element of K and $k_0 - 1 < k_0$.
So, P(k_0-1) $\equiv \frac{1}{1(1+1)} + \frac{1}{2(2+1)} + \frac{1}{3(3+1)} + \ldots \frac{1}{(k_0-1)(k_0-1+1)} = \frac{k_0-1}{(k_0-1)+1}$ is true.
Then by adding $\frac{1}{(k_0)(k_0+1)}$ to both sides of the equality we obtain:
$\frac{1}{1(1+1)} + \frac{1}{2(2+1)} + \ldots \frac{1}{(k_0-1)(k_0-1+1)} + \frac{1}{(k_0)(k_0+1)} = \frac{k_0-1}{k_0-1+1} + \frac{1}{(k_0)(k_0+1)} = \frac{k_0^2-1+1}{k_0(k_0+1)} = \frac{k_0}{k_0+1}$,
which is P(k_0).
So P(k_0) is true and $k_0 \notin K$.
This contradicts our original statement that $k_0 \in K$. Thus K $\nsubseteq \mathbb{Z}^+$ and P(n) is true for all n $\in \mathbb{Z}^+$

Theorem 21. *If M is a finite point set then M does not have a limit point.*

Answer: Let P(n) be the statement: "every point set with exactly n elements does not have a limit point."

Notice P(1) is true because a point set with one point does not have a limit point (proven in Question 12).

Suppose P(k') is false for some k' in $\mathbb{Z}^+$.

Let K=$\{k \in \mathbb{Z}^+ | P(k)$ is false$\}$.

K $\subseteq \mathbb{Z}^+$, k' $\in$ K and 1 $\notin$ K.

Let k_0 be the least element of K. So k_0 >1, and P(k_0)is false which implies that there is a finite point set M with exactly k_0 points and M has a limit point. Since k_0 > 1 and $k_0 - 1 < k_0$ then P($k_0 - 1$) is true, we can pick a point Q in M and define N to be all the points of M that are different from Q thus N has $k_0 - 1$ elements and so it has no limit point.

By problem 18.1 if P is not the limit point of set N and P is also not the limit point of set {Q}, then we know that P is not the limit point of N∪{Q}=M. Therefore, the set that contains (k_0) does not have a limit point thus contradicting that P(k_0) is false. So P(k_0) is true so $k_0 \notin K$.

This contradicts our original statement that $k_0 \in K$ and P(k_0) is false. Thus K $\nsubseteq$ $\mathbb{Z}^+$ and P(n) is true for all n $\in \mathbb{Z}^+$. So, every point set that is finite does not have a limit point.

Question 22. *Is there an infinite point set that does not have a limit point?*

Answer:

Yes, let the space be $\mathbb{R}$ with "to the left of" interpreted as <, then $\mathbb{Z}^+$ does not have a limit point. I will show this by cases.

Case 1 : any point, Y, between any two consecutive x in the positive integers

Let Y be between two consecutive positive integers and lets call these integers A and B. (A,B) contains Y but no other positive integer. Therefore any Y between two consecutive integers is not a limit point of the positive integers.

Case 2 : any x that is a positive integer

For all x in the set of positive integer there exists s = (x-a, x+a) where a is in $\mathbb{R}$ and a <1 such that s contains x but does not contain a point of the positive integers that is distinct from x.

I suspect that this in not true for all models that satisfy Axioms 1-6.

Theorem 23. *If M is an infinite subset of a segment then M has a limit point.*

Answer: Assume M is an infinite point set that is a subset of a segment (A,B). Let S_2 be the point set that contains points P such that there are infinitely many points of M to the left of point P. Let S_1 be the complement of S_2. Notice that A is in S_1 and B is S_2. If x is in (A,B) then x is in S_1 or S_2 since x has to be in S_2 or its compliment and no point can be to the left of itself. Every point in S_1 is to the left of every point of S_2 since S_2 is the set of points such that there are an infinite number of points of M to the left. By Axiom 1, S_1 has a rightmost point or S_2 has a leftmost point. I will proceed by cases.

Case 1: X is the rightmost point of S_1

Notice X is in S_1. Let s=(C,D) which contains X. Therefore, C is to the left of X and D is to the right of X. Now since X is an element of S_1 there are only finitely many points of M to the left of X. However, since X is the rightmost point of S_1 and D is to the right of X, then D is an element of S_2 and there infinitely many points of M to the left of D. Thus, there are infinitely many points of M between X and D. So there is a point of M distinct from X in (C,D) thus X is a limit point of M.

Case 2: Y is the leftmost of S_2

Notice Y is in S_2 since it is the leftmost point. Let s=(C',D') which contains Y. So, C' is to the left of Y and D' is to the right of Y. Now since Y is am element of S_2, there are infinitely many points of M to the left of Y. However, since Y is the leftmost point of S_2 and C' is to the left of Y, then C' is in S_1 and there are finitely many points of M to the left of C'. Thus there are infinitely many points of M between C' and Y. So there is a point of M distinct from Y in (C',D'). Therefore Y is a limit point of M.

Therefore, M has a limit point.

Question 24. *Does each finite point set have a leftmost point?*

Answer: Yes. Let P(n) be the statement: "every point set with exactly n elements has a leftmost point."

Notice P(1) is true because a point set with one point satisfies that is x is in M then it it not to the left of itself (Axiom 5).

Suppose P(k') is false for some k' in $\mathbb{Z}^+$.

Let K=$\{k \in \mathbb{Z}^+ | P(k)$ is false$\}$.

K $\subseteq \mathbb{Z}^+$, k' $\in$ K and 1 $\notin$ K.

Let k_0 be the least element of K. So k_0 >1. There exists a set M with k_0 elements that does not have leftmost point.

Since $k_0 > 1$ and $k_0 - 1 < k_0$ then P($k_0 - 1$) is true, we can pick a point Q in M and define N = M - {Q}, thus N has $k_0 - 1$ elements.

N has a leftmost point, call it L. Assume when Q is added to N, Q is to the left of L. So Q is the leftmost point of M. Otherwise, assume that Q is added and is not to the left of L, then L is the leftmost point of M. Therefore, the set that contains (k_0) has a leftmost point thus contradicting that P(k_0) is false. So P(k_0) is true and $k_0 \notin K$.

This contradicts our original statement that $k_0 \in K$ and P(k_0) is false. Thus K $\not\subseteq$ $\mathbb{Z}^+$ and P(n) is true for all n $\in \mathbb{Z}^+$. So, every point set that is finite has a leftmost point.

Question 25. *Suppose s_1, s_2, $s_3, \ldots$ is a sequence of segments such that s_2 is a subset of s_1, s_3 is a subset of $s_2, \ldots$. Is there a point that belongs to every term of the sequence s_1, s_2, $s_3, \ldots$?*

Answer: No, assuming the space is $\mathbb{R}$ and "to the left of" is interpreted as <. Let S=$\{s_1, s_2, s_3, \ldots\}$ such that $s_n = (0,\frac{1}{n})$ where n is positive integer. Suppose that x is a point that belongs to every term of the sequence. Let's look at $\frac{1}{x}$ <n_c such that n_c is in $\mathbb{Z}^+$. So $\frac{1}{n_c}$ <x. Therefore, $\frac{1}{n_c}$ is to the left of x and $\frac{1}{n_c}$ >0. Therefore, x is not in s_n = (o, $\frac{1}{n_c}$) thus contradicting that x is in every term of S. Therefore there exists a sequence of segments such that there is not a point that belongs to every term of the sequence of segments.

Does there always exist such a nested sequence in every model that satisfies Axioms 1-6.

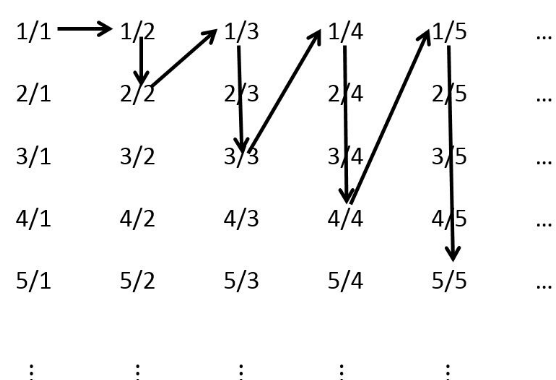
Theorem 26. *Suppose I_1, I_2, $I_3, \ldots$ is a sequence of intervals such that I_{n+1} is a subset of I_n for each positive integer n. Then there is a point that belongs to every interval in the sequence I_1, I_2, $I_3, \ldots$. Further, if the common part of all the intervals in the sequence I_1, I_2, $I_3, \ldots$ contains two points, the common part is an interval. **Moreover, if C is the common part and s is a segment containing C, then there is a positive integer n such that I_n is a subset of s.***

** I will address the last statement at a later time.** **Answer:**Let $I_i = [A_i, B_i]$. Let M = $\{A_i$ | i is in $\mathbb{Z}^+\}$ and let N = $\{B_i$ | i is in $\mathbb{Z}^+\}$.

Let A be either the rightmost point of M or the leftmost point of all points to the right of M, which ever exists. Similarly, let B be either the leftmost point of N or the rightmost point of all the points to the left of N, which ever exists. So [A,B] is a subset of I_i for every i. [A,B] contains all points that belong to every I_i and is an interval.

Question 27. *Is there a sequence r_1, r_2, $r_3, \ldots$ of numbers such that $\{r_1, r_2, r_3, \ldots\}$ is the set of rational numbers in $[0, 1]$?*

Answer: Yes, [0,1] as shown below.



1/1, 1/2, 2/2, 1/3, 2/3, 3/3, 1/4, 2/4, 3/4, 4/4, 1/5, 2/5 3/5, 4/5, 5/5,...

Question 28. *Is there a sequence x_1, x_2, $x_3, \ldots$ of points of the interval $[A, B]$ such that $\{x_1, x_2, x_3, \ldots\}$ is $[A, B]$?*

Answer: No. Assume there is a sequence of points x_1, x_2, $x_3, \dots$ of [A,B] such that $\{x_1, x_2, x_3, \dots\}$ is $[A, B]$. Let $I_i = [A_i,\ B_i]$ where i is in $\mathbb{Z}^+$ such that I_{i+1} excludes x_i and I_{i+1} is a subset of I_i. We know there exists a point x in the intersection of all I_i by Theorem 26 which contradicts that I_i excludes x_i. Therefore, there does not exist a sequence of points such that $\{x_1, x_2, x_3, \dots\}$ is $[A, B]$.

Definition 29. *The statement that the point sets H and K are* **mutually exclusive** *means no point belongs to both H and K.*

(Kristilyn's Question 29.1:) *For each pair of sets below, determine if they are mutually exclusive.*

Pair 1: Rationals, Irrationals

Yes

Pair 2: Reals, Integers

No, integers are points that belong to the reals thus there are points that belong to both.

Pair 3: Naturals, Non-positive Integers

Yes

Definition 30. *The statement that H and K are* **mutually separated** *means H and K are mutually exclusive and neither contains a limit point of the other.*

(Kristilyn's Question 30.1:) *For each pair of sets below, determine if they are mutually seperated.*

Pair 1: Rationals, Irrationals

No, these sets are mutally exclusive but for every segment that contains a rational, that same segment contains an irrational. Also, for every segment that contains an irrational, the same segment contains a rational. Thus rationals contain limit points of irrationals and irrationals contain limit points of rationals. Therefore, they are not mutually seperated.

Pair 2: [0,1], (1,2)

No, they are mutually exclusive, but 1 is a limit point point for both sets and [0,1] contains 1 therefore they are not mutually seperated.

Pair 3: (0,1), (1,2)

Yes, these sets are mutually exclusive. Each set does not contain a limit point of the other. Therefore these sets are mutually separated.

Question 31. *Is the interval $[A, B]$ the union of two mutually separated point sets?*

Answer: See Theorem 42.

Notation: If M is a point set, $\overline{M}$ denotes the set to which the point P belongs if and only if P is a point of M or P is a limit point of M.

(Kristilyn's Question 31.1:) *Below will be given set A. Determine $\overline{A}$.*

1. A: Rationals

$\overline{A}$ = *Reals*

2. A: (1,2)

$\overline{A}$ = *[1,2]*

Definition 32. *The statement that the point set M is* **closed** *means if x is a limit point of M then x is a point of M.*

(Kristilyn's Question 32.1:) *Determine if the given set is closed.*

1: Rationals

No, they do not contain all their limit points.

2: [1,2]

Yes, [1,2] contains all the limit points of [1,2].

3: (1,2]

No, 1 is a limit point but is not contained in (1,2].

4: Reals

Yes, the reals contains all the limit points of the reals. **5:** $\mathbb{Z}$

Yes, $\mathbb{Z}$ is closed because it contains all its limit points (it has not limit points).

Problem 33. *If the point set A is a subset of the point set B then $\overline{A}$ is a subset of $\overline{B}$.*

Answer: Assume A is a subset of B. We need to show that if x is in $\overline{A}$ then x is in $\overline{B}$
If x is in A then x is in B. Let x be in $\overline{A}$ then x is in A or x is a limit point of A. If x is in A then x is in B therefore x is in $\overline{B}$. Let x be a limit point of A. Then for every S_1 = (C,D) such that S_1 contains x, there exists a y that is in S_1 and y is in A where x does not equal y. Since y is in A then y is also in B Thus x is a limit point of B because (C,D) contains x and a point of B distinct from x. So x is in $\overline{B}$.

Problem 34. *If M is a point set, $\overline{(\overline{M})} = \overline{M}$.*

Answer:

Assume x is in $\overline{(\overline{M})}$, we need to show x is in $\overline{M}$ so need to show that x is in M or x is a limit point of M.

Assume x is in $\overline{M}$ or x is a limit point of $\overline{M}$. If x is in $\overline{M}$ then we are finished. So let x be a limit point of $\overline{M}$. Then for every S_1 = (C,D) that contains x such that there exists a y in $\overline{M}$ (so y is in M or is a limit point of M) such that x does not equal y and y is in S_1. Now either x is to the left of y or y is to the left of x.

Case 1: x is to the left of y

If x is to the left of y, then x is in the segment (x,D) and since y is a limit point of M there is a point m $\in$ M in (x,D) distinct from y. Now m is distinct from x and in (C,D). So x is a limit point of M.

Case 2: y is to the left of x

If y is to the left of x, then y is in the segment (C,x). There is m$\in$ M in (C,D). So x is a limit point of M.

Assume x is in $\overline{M}$, we need to show x is in $\overline{(\overline{M})}$ so need to show x is in $\overline{M}$ or x is a limit point of $\overline{M}$. So we are done since x is in $\overline{M}$.

Problem 35. *If M is a point set, $\overline{M}$ is closed.*

Answer: Since $\overline{(\overline{M})} = \overline{M}$, every limit point of $\overline{M}$ is in $\overline{M}$ and $\overline{M}$ is closed.

Problem 36. *If each of H and K is a point set, $\overline{H \cup K} = \overline{H} \cup \overline{K}$.*

Answer:

I will show that if x is in $\overline{H \cup K}$ and x is not in $\overline{H}$ then x is in $\overline{K}$. Assume x is in $\overline{H \cup K}$ and x is not in $\overline{H}$. Then x is either in $H \cup K$ and not in $\overline{H}$ or x is a limit point of $H \cup K$ and not in $\overline{H}$.

Let x be in $H \cup K$ and x is not in $\overline{H}$ which means x is in H or x is in K but since x is not in $\overline{H}$ then x is not in H thus x is in K which is a subset of $\overline{K}$

Let x be a limit point of $H \cup K$ and x not be in $\overline{H}$. I need to show x is a limit point of K. Let x be in (C,D). Since x is not a limit point of H, then there exist (E,F) such that x $\in$ (E,F) and so no points of H are in (E,F) distinct from x. By Theorem 18 there is a segment s that is a subset of (C,D) and s is a subset of (E,F) such that s contains x. Segment s has no points of H distinct from x but since x is a limit point of H $\cup$ K, there exists some point y $\in$ K that is in s where y is not equal to x. Since s is a subset of (C,D), then y is in (C,D). Therefore for every (C,D) that contains x, (C,D) also contains y in K distinct from x. Thus, x is a limit point of K.

Assume x is in $\overline{H}$ or x is in $\overline{K}$, need to show x is in $H \cup K$(x is in H or x is in K) or x is a limit point of $H \cup K$.

Assume x is $\overline{H}$. Then x is in H or is a limit point of H. If x is in H then x is in $H \cup K$ so x is in $\overline{H \cup K}$. Let x be a limit point of H. Then for any S_1 = (C,D) containing x, there exists a y in S_1 such that x does not equal y and y is in H so y is in H $\cup$ K. Then, (C,D) contains x and also contains a point y of $H \cup K$ that is distinct from x. So x is a limit point of $H \cup K$. Now suppose x is in $\overline{K}$. So x is in K or x is in is a limit point of K. If x is in K, then x is in $H \cup K$ which means x is in $\overline{H \cup K}$. Let x be a limit point of K. Then for any S_1 = (C,D) containing x, there exists a y in S_1 such that x does not equal y and y is in K thus y is in H $\cup$ K. Then, (C,D) contains x and also contains a point y of $H \cup K$ that is distinct from x. So x is a limit point of $H \cup K$.

Question 37. *Is every finite point set closed?*

Answer: Yes, Let A be a finite set. Point x is in $\overline{A}$ if and only if x is in A or x is in $\overline{A}$. By Theorem 21, a finite set does not have a limit point, therefore A = $\overline{A}$. We know that $\overline{A}$ is closed so A is closed since they are equal sets.

Question 38. *Can "$\cup$" in Problem 36 be replaced by "$\cap$"?*

Answer: $\overline{H \cup K} = \overline{H} \cap \overline{K}$

Assume x is in $\overline{H \cap K}$, need to show x is in $\overline{H}$ and x is in $\overline{K}$ so I need to show that (x is in H or a limit point of H) and (x is in K or x is a limit point of K).

Assume is in $H \cap K$ or x is a limit point of $H \cap K$. If x is in $H \cap K$ then x is in H and x is in K. If x is a limit point of $H \cap K$ then for any S_1 = (C,D) containing x, there exists a y in S_1 such that y is in $H \cap K$ and x does not equal y. So (C,D) contains x and a point in H and a point in K that is distinct from x so x is a limit point of H and x is a limit point of K.

$\overline{H} \cap \overline{K} = \overline{H \cup K}$ **(not true)**Assume x is in $\overline{H}$ and x is in $\overline{K}$, need to show x is in $H \cap K$ or x is a limit point of $H \cap K$.

Let H =(0,1) and K =(1,2). $\overline{H}$ = [0,1] and $\overline{K}$ = [1,2] the intersection of these sets = [1] but $\overline{H \cap K}$ is an empty set.

Definition 39. *The statement that the collection $\mathcal{G}$ of sets* **covers** *the set M means if x is in M then some element of $\mathcal{G}$ contains x.*

(Kristilyn's Question 39.1:) *Determine if the given $\mathcal{G}$ covers the set M.*

1: $\mathcal{G}$ = (0,1), (1,2) M=(0,2)

No, M contains 1 but no element of $\mathcal{G}$ contains 1.

2: $\mathcal{G}$ = (.5,1.5), (1,2) M=(1,2)

Yes, every x in M is in some element of $\mathcal{G}$.

3: $\mathcal{G} = \{(\frac{1}{n+1} \mid n \in \mathbb{Z}^+, 1)\}$ M=(0,1)

Yes, every x in M is in some element of $\mathcal{G}$.

4: $\mathcal{G} = \{(\frac{1}{n+1} \mid n \in \mathbb{Z}^+, 1)\}$ M=[0,1]

No, O and 1 are in M but not in any element of $\mathcal{G}$.

Question 40. (Four Questions)

a *Does there exist a collection $\mathcal{G}$ of segments covering the segment (A, B) such that if $\mathcal{H}$ is a finite subcollection of $\mathcal{G}$ then $\mathcal{H}$ does not cover (A, B)?*

Answer:

Yes, look at example 3 in Kristilyn's 39.1. If we take a finite subset $\mathcal{H}$ of $\mathcal{G}$, not every point of M will be contained in some element of $\mathcal{H}$.

b *Does there exist a collection $\mathcal{G}$ of intervals covering the segment (A, B) such that if $\mathcal{H}$ is a finite subcollection of $\mathcal{G}$ then $\mathcal{H}$ does not cover (A, B)?*

Answer:

Yes, let $\mathcal{G} = \{[\frac{1}{n+1} \mid n \in \mathbb{Z}^+, 1]\}$ M=(0,1). If we take a finite subset $\mathcal{H}$ of $\mathcal{G}$, not every point of M will be contained in some element $\mathcal{H}$.

c *Does there exist a collection $\mathcal{G}$ of segments covering the interval $[A, B]$ such that if $\mathcal{H}$ is a finite subcollection of $\mathcal{G}$ then $\mathcal{H}$ does not cover $[A, B]$?*

Answer:

No, if $\mathcal{G}$ is a set of segments that cover [A,B], every finite subset of $\mathcal{G}$ covers [A,B]. (Theorem 43)

d *Does there exist a collection $\mathcal{G}$ of intervals covering the interval $[A, B]$ such that if $\mathcal{H}$ is a finite subcollection of $\mathcal{G}$ then $\mathcal{H}$ does not cover $[A, B]$?*

Answer:

Yes, let $\mathcal{G} = \{[\frac{1}{n+1}, 1] \mid n \in \mathbb{Z}^+\} \cup \{[-1, 0]\}$. No finite subset of $\mathcal{G}$ will cover M.

Question 41. *Does there exist a closed point set M such that every point of M is a limit point of M and M contains no interval?*

Answer:

Yes. Take the Cantor Middle Third Set of [0,1]. Let I_0 = [0,1] and $I_1 = [0,\frac{1}{3}] \cup [\frac{2}{3}, 1]$. We proceed by removing the open middle third of each interval I_m to create I_{m+1}. Take the intersection of all I_m and let that be M. M is a closed point set because it contains all the limit points of M as well as every x in M is a limit point and M contains no intervals.

Theorem 42. *The interval $[A, B]$ is not the union of two mutually separated point sets.*

Answer: Let [A,B]=H∪K and $\overline{H}$ ∩K contain no points and H∩ $\overline{K}$ contain no points without loss of generality let A ∈ H.Let S_1= { x ∈ [A,B] | x is to the left of every point of K} and S_2 is the compliment of S_1.

Using Axiom 1, we know S_1 has a rightmost point or S_2 has a leftmost point. Let R be the rightmost point of S_1, so R is in H (if R is in K, it can not be in K and be to the left of itself). [A,R] contains no point of K. Let R be in (C,D) such that C is to the left of R and D is to the right of R. There exists a point y in (R,D) and y is in K (if such a y does not exist then D would be the rightmost point of S_1). Therefore, R is a limit point of K which is a contradiction since R is in H∩$\overline{K}$.

Let L be the leftmost point of S_2. Let (C,D) be a segment containing L. If L is an element of H then L is the rightmost point of S_1 and there must be points of K in (L,D). Thus, L will be a limit point of K which implies that L is in H∩$\overline{K}$. This is a contradiction. If L is an element of K, then (C,L) must contain points of H so L is a limit point of H which implies that L is in $\overline{H}$∩K. This is a contradiction. So [A,B] = H union K is not the union of two mutually separated sets.

Theorem 43. *If $\mathcal{G}$ is a collection of segments covering the interval $[A, B]$, then there is a finite subcollection $\mathcal{H}$ of $\mathcal{G}$ that covers $[A, B]$.*

Answer: Let M = {x ∈ [A,B] a finite sub-cover of [A,x] of $\mathcal{G}$ exists.}. Let S_2={x| x is to the right of every point of M}. We know A is in M because a finite sub-cover will cover of $\mathcal{G}$ A. Let S_1 be the compliment of S_2. Let R be the rightmost point of S_1. Let R be in (C,D). There exists a point y in (R,D). With the addition of (C,D), y is covered by finitely many members of $\mathcal{G}$ so [A,y] is in M which contradicts that R is the rightmost point of S_1 because y is to the right of R. Let L be the leftmost point of S_2. Let L be in (C,D). There exists point x in (C,L). We know that [A,L) covers x but with the addition of (C,D) L is also covered by finitely many members of $\mathcal{G}$ so [A,L] is in M therefore L is in M. L can not both be to the right of every point of M and be in M. This is a contradiction. Thus, if $\mathcal{G}$ is a collection of segments covering the interval $[A, B]$, then there is a finite sub-collection $\mathcal{H}$ of $\mathcal{G}$ that covers $[A, B]$.

Question 44. *Does Theorem 43 remain true if the interval $[A, B]$ is replaced by a closed subset of $[A, B]$?*

Question 45. *Let M be a point set with the property that if $\mathcal{G}$ is a collection of segments covering M then some finite subcollection of $\mathcal{G}$ covers M. Is it true that M is closed?*

Functions

Definition 46. *A* **function** *is a set of ordered pairs such that no two pairs in the set have the same first term. If f is a function and M is the set of first terms of pairs in f, then f is said to be a* **function defined on** *M.*

Notation: If f is a function and the pair (x, y) is in f, we sometimes write $y = f(x)$.

Some Examples

1: f_1 is the set to which the pair (x, y) of points belongs if and only if x is y.

2: Suppose P is a point and Q_1 and Q_2 are two points. Denote by f_2 the set to which the pair (x, y) of points belongs if and only if it is true that if x is to the left of P then y is Q_1 and y is Q_2 otherwise.

3: Suppose P is a point. Denote by f_3 the set to which the pair (x, y) of points belongs if and only if y is P.

Problem 47. *Show that each of f_1, f_2, and f_3 is a function.*

Definition 48. *Suppose f is a function defined on the point set M, P is a point of M and the set of second terms of pairs in f is a point set. The statement that f is* **continuous at** *P means if t is a segment containing $f(P)$ then there is a segment s containing P such that if x is a point of M in s then $f(x)$ is in t.*

Problem 49. *Check the definition of f is continuous at P for each of the examples f_1, f_2, and f_3 at several points.*

Problem 50. *Write the bare denial of the definition of f is continuous at P.*

Theorem 51. *Suppose M is a point set, P is a point of M and P is not a limit point of M. If f is a function defined on M, then f is continuous at P.*

Theorem 52. *Suppose M is a point set, P is a point of M and f is a function defined on M. If f is continuous at P and Q is a point to the left of $f(P)$, then there is a segment s containing P such that if x is a point of M in s then Q is to the left of $f(x)$.*

Problem 53. *Suppose f is a function defined on the interval $[A, B]$ such that if P is a point of $[A, B]$ then f is continuous at P. If Q is a point, let M_Q denote the set to which the point x of $[A, B]$ belongs if and only if $f(x)$ is not to the left of Q. Show that for each point Q such that Q is to the left of $f(t)$ for some point t of $[A, B]$, the set M_Q is a closed point set.*

Theorem 54. *Suppose f is a function defined on the interval $[A, B]$, $f(A)$ is not $f(B)$, and if P is a point of $[A, B]$ then f is continuous at P. If Q is a point between $f(A)$ and $f(B)$ then there is a point C between A and B such that $f(C)$ is Q.*

Theorem 55. *Suppose f is a function defined on the interval $[A, B]$ such that if P is a point of $[A, B]$ then f is continuous at P. Then there exist points C and D such that if x is a point of $[A, B]$, then $f(x)$ is in the segment (C, D).*

Definition 56. *Suppose f is a function defined on the point set M, P is a limit point of M and L is a point. The statement that f* **has limit** *L* **at** *P means if t is a segment containing L then there is a segment s containing P such that if x is a point of M in s and x is distinct from P then $f(x)$ is in t.*

Theorem 57. *Suppose M is a point set, f is a function defined on M and P is a point of M that is a limit point of M. Then f is continuous at P if and only if f has limit $f(P)$ at P.*

Theorem 58. *Suppose M is a point set, f is a function defined on M and P is a limit point of M and K and L are points. If f has limit L at P then f does not have limit K at P.*

Definition 59. *A collection $\mathcal{G}$ of point sets is called* **monotonic** *provided it is true that if g and h are in $\mathcal{G}$ then h is a subset of g or g is a subset of h.*

Problem 60. *Show that if $\mathcal{G}$ is a monotonic collection of closed subsets of the interval $[A, B]$, then there is a point that belongs to every set in $\mathcal{G}$.*

Definition 61. *The statement that the point set M has the* **finite covering property** *means if $\mathcal{G}$ is a collection of segments covering M then there is a finite subcollection $\mathcal{H}$ of $\mathcal{G}$ that covers M.*

Problem 62. *Show that if the point set M has the finite covering property then M is closed.*

Problem 63. *Show that if M is a closed subset of the interval $[A, B]$ then M has the finite covering property.*

Definition 64. *If f is a function defined on the point set M, denote by $f[M]$ the set of all second terms of pairs in f.*

Definition 65. *If f is a function defined on the point set M and f is continuous at each point of M, we say f is* **continuous** *on M.*

Theorem 66. *If f is continuous on the interval $M = [A, B]$ and $f[M]$ contains two points, then $f[M]$ is an interval.*

Problem 67. *Show that if P and Q are points, there exist mutually exclusive segments s_P and s_Q containing P and Q, respectively.*

Definition 68. *A set is said to be* **open** *if it is the union of a collection of segments.*

Problem 69. *Show that if P is a point and K is a closed set not containing P, then there exist mutually exclusive open sets O_P and O_K containing P and K, respectively.*

Problem 70. *Show that if H and K are mutually exclusive closed points sets, then there exist mutually exclusive open sets O_H and O_K containing H and K, respectively.*

Problem 71. *Show that if H and K are mutually separated point sets, then there exist mutually exclusive open sets O_H and O_K containing H and K, respectively.*

Axiom 7 and Its Consequences

Definition 72. *A* **sequence** *is a function defined on the set of positive integers.*

If r is a sequence and i is a positive integer, we often denote $r(i)$ by r_i (as we have been doing).

AXIOM 7 There exists a sequence of points $r_1, r_2, r_3, \ldots$ such that if A and B are points then there is a positive integer i such that r_i is between A and B.

Problem 73. *Show that if A and B are points there is a point x between A and B such that x does not belong to $\{r_1, r_2, r_3, \ldots\}$.*

Theorem 74. *If x is a point then x is a limit point of $\{r_1, r_2, r_3, \ldots\}$.*

Theorem 75. *If P is a point, there is a sequence $y_1, y_2, y_3, \ldots$ of points of $\{r_1, r_2, r_3, \ldots\}$ such that y_1 is to the left of y_2, y_2 is to the left of $y_3, \ldots$ and P is a limit point of $\{y_1, y_2, y_3, \ldots\}$.*

Theorem 76. *If P is a point, there is a sequence $s_1, s_2, s_3, \ldots$ of segments containing P such that s_2 is a subset of s_1, s_3 is a subset of $s_2, \ldots$ and P is the only point common to $s_1, s_2, s_3, \ldots$.*

Theorem 77. *There exists a sequence $s_1, s_2, s_3, \ldots$ of segments such that if s is a segment and P is a point of s then there is a segment s_i in the sequence $s_1, s_2, s_3, \ldots$ that contains P and is a subset of s.*

Theorem 78. *Suppose P is a point, M is a point set and $s_1, s_2, s_3, \ldots$ is a sequence of segments as in Theorem 77. Then, P is a limit point of M if and only if it is true that if s_i is a segment in the sequence $s_1, s_2, s_3, \ldots$ and s_i contains P then s_i contains a point of M distinct from P.*

Definition 79. *The statement that a set X is* **countable** *means there is a sequence $x_1, x_2, x_3, \ldots$ such that $X = \{x_1, x_2, x_3, \ldots\}$. If a set is not countable we say that it is* **uncountable**.

Theorem 80. *If M is an uncountable point set then some point of M is a limit point of M.*

Definition 81. *Suppose $x_1, x_2, x_3, \ldots$ is a sequence of points and x is a point. The statement that $x_1, x_2, x_3, \ldots$* **converges to** *x means if s is a segment containing x there is a positive integer N such that if $n \geq N$ then x_n is in s.*

Problem 82. *Guess what. Yes, write a bare denial for Definition 81.*

Theorem 83. *Suppose $x_1, x_2, x_3, \ldots$ is a sequence of points that converges to the point x and y is a point different from x. Then, $x_1, x_2, x_3, \ldots$ does not converge to y.*

Theorem 84. *Suppose the sequence $x_1, x_2, x_3, \ldots$ converges to the point x and the sequence $y_1, y_2, y_3, \ldots$ converges to x. If $z_1, z_2, z_3, \ldots$ is a sequence of points such that, for each positive integer i, z_i is between x_i and y_i, then $z_1, z_2, z_3, \ldots$ converges to x.*

Definition 85. *If $x_1, x_2, x_3, \dot{s}$ is a sequence, the statement that $x_1, x_2, x_3, \ldots$* **converges** *means there is a point x such that $x_1, x_2, x_3, \ldots$ converges to x.*

Question 86. *If x is a point and $x_1, x_2, x_3, \ldots$ is a sequence of points such that, for each positive integer i, $x_i = x$, does the sequence $x_1, x_2, x_3, \ldots$ converge?*

Theorem 87. *Suppose $x_1, x_2, x_3, \ldots$ is a sequence such that, for each positive integer i, x_i is to the left of x_{i+1}. If there is a point B such that, for each positive integer i, x_i is to the left of B then $x_1, x_2, x_3, \ldots$ converges.*

Theorem 88. *If the point P is a limit point of the point set M then there is a sequence of points of M that converges to P.*

Theorem 89. *If M is a point set, the point P belongs to $\overline{M}$ if and only if there is a sequence of points of M that converges to P.*

Theorem 90. *Suppose M is a point set, f is a function defined on M and P is a point of M. Then, f is continuous at P if and only if it is true that if $x_1, x_2, x_3, \ldots$ is a sequence of points of M that converges to P then $f(x_1), f(x_2), f(x_3), \ldots$ converges to $f(P)$.*

Theorem 91. *Suppose $[A, B]$ is an interval and f is a function defined on $[A, B]$. Then, f is continuous on $[A, B]$ if and only if for each subset M of $[A, B]$, $f[\overline{M}]$ is a subset of $\overline{f[M]}$.*

Simple Graphs

For the remainder of the course, the over-riding assumption is that S is the set of real numbers. We will use the word "point" in two different senses. Most of the time, point will mean "ordered number pair". However, at times (and it should be clear from context) point and real number may be used synonymously.

Definition 92. *A* **simple graph** f *is a set of points such that if* h *is a vertical line that intersects* f *then* h *contains only one point of* f*.*

Definition 93. *The statement that the simple graph* f *has* **property** U *at the point* P *of* f *means if* l *is a horizontal line with* P *below it, there exist vertical lines* h *and* k *with* P *between them such that if* Q *is a point of* f *between* h *and* k *then* Q *is below* l*.*

Definition 94. *If* f *is a simple graph, by the* x**-projection** *of* f *is meant the set of first terms of pairs in* f*.*

Theorem 95. *Suppose* f *is a simple graph with* x*-projection an interval and* f *has property* U *at each point. If* l *is a horizontal line and* $M = \{z \mid z$ *is the* x*-projection of a point of* f *that is on or above* $l\}$ *then* M *is closed.*

Definition 96. *If* f *is a simple graph, the statement that the point* P *of* f *is a* **high point** *of* f *means if* Q *is a point of* f *then* Q *is not above the horizontal line passing through* P*.*

Theorem 97. *If* f *is a simple graph with* x*-projection an interval and* f *has property* U *at each point then* f *has a high point.*

Definition 98. *The statement that the simple graph* f *has* **property** L *at* P *means if* l *is a horizontal line with* P *above it, then there exist vertical lines* h *and* k *with* P *between them such that if* Q *is a point of* f *between* h *and* k *then* Q *is above* l*.*

Problem 99. *State and prove theorems analogous to the two stated above for simple graphs with property* L*.*

Theorem 100. *Suppose f is a simple graph and let P be a point of f. Then f has both property U and property L at P if and only if f is continuous at the x-projection of P.*

Definition 101. *Suppose f is a simple graph and P is a point of f such that the x-projection of P is a limit point of the x-projection of f. The statement that f has* **slope** *m at P means if A is a line containing P with slope greater than m and B is a line containing P with slope less than m then there exist vertical lines h and k with P between them such that if Q is a point of f distinct from P between h and k then Q is between A and B.*

Problem 102. *Show that if f has slope m_1 at P and f has slope m_2 at P then $m_1 = m_2$.*

Definition 103. *If f has slope m at P and l is the line through P with slope m, then l is called the* **tangent line** *to f at P.*

Theorem 104. *Suppose f is a simple graph with slope m at P. If l is a line containing P such that no point of f is above l then l is the tangent line to f at P.*

Theorem 105. *If the simple graph f has slope m at P then f has property U and property L at P.*